The nonlinear inhomogeneous Galbrun-Equation: Derivation and possible Ways to solve numerically

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ABSTRACT

In recent years, the field of aeroacoustics has gained more attention in engineering developments like low noise ventilators, HVAC systems in electric vehicles, passenger cabins of commercial aircraft as well as novel engine designs of airplanes in general, to fulfill regulations of low level noise exposure. In order to close the gap between expensive prototypes with experimental testing and short development periods under high cost pressure, numerical simulations are state of the art for virtual prototyping. Numerous theories and acoustic analogies have been developed to investigate the effect of flow combined with acoustic radiation and propagation. Important contributions are known as the Linearized Euler Equations (LEE) or the Acoustic Perturbation Equations (APE). A different approach is pursued following the work of Galbrun. His displacement based linear formulation of aeroacoustics is extended to account for nonlinear effects as well as acoustic sources in turbulent flow. Difficulties arise when solving the Galbrun equation numerically. Therefore some already established numerical techniques that can be used to cope with these problems such as the Finite-Element-Method and the Discontinuous-Galerkin-Method are proposed.

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1. INTRODUCTION

In the field of aeroacoustics many theories and algorithms have been developed in order to give a confident estimation of sound generated and convected by moving medium such as air or water. While this engineering field originally enlivened by the famous acoustic analogy of Lighthill (1) and his followers like Curle (2) and Ffowcs Williams and Hawkings (3), many other attempts to cope with flow induced noise have been published such as the Linearized Euler Equations (LEE) (4), the Acoustic Perturbation Equations (APE) (5) or the Galbrun Equation for sound propagation in flow. While most of the theories utilize an Eulerian description of continuum mechanics, Galbrun (6) used a mixed Eulerian/Lagrange formulation in order to reduce all desired physical quantities to one - the perturbation of the particle displacement. With this approach the amount of unknowns decrease to the particle displacement perturbation vector field. All other quantities such as the fluid pressure or the particle velocity can be calculated once the particle displacement is known (7, 8). Besides this reduction of unknowns the coupling of flow acoustics with structural vibration of adjacent parts can be accomplished in a straight forward manner since in structural mechanics the unknown vector field is the particle displacement, too cf. (9). The disadvantage of this approach is that the equations become more complicated resulting in numerous difficulties when trying to solve numerically, cf. (9, 10, 11). Several publications such as (12, 13, 14) deal with these problems and give more or less attempts for a solution to the numerical difficulties. Up to this point the authors have no evidence of a successful and convincing solution when solving the Galbrun equation in its original displacement based formulation. Besides these difficulties, utilizing Galbrun's approach seems to be suitable in order to take into account the influence of the acoustics on the flow as a back reaction, as pointed out by Bonnet-Ben Dhia (15). One can think of such a situation when dealing with Helmholtz resonators which are excited by the flow and create acoustic sound pressure levels that are in the order of the flow pressure.

In this paper, following the work of Brazier and Minotti et al. (7, 8), the Galbrun equation is extended to a nonlinear formulation with source terms by means of a fluctuating volume force on the right hand side. In addition this formulation shows that neither gradients of the reference flow pressure nor the divergence of the friction related stress tensor can be a source of sound. Furthermore because of the nonlinear character of the governing equations, the influence of the sound pressure waves on the flow can be taken into account.

2. THEORY

The following section presents the governing equations to the problem and some principles of continuum mechanics that will help to build up the final equations for the inhomogeneous nonlinear Galbrun equation. To ensure a maximum of readability the nabla calculus as well as the coordinate notation combined with the Einstein summation convention is presented wherever appropriate. Furthermore, a Cartesian coordinate system is assumed for all following expressions, where all indices take $1,2,3$ for the Cartesian coordinates.

2.1 Governing Equations

In the work at hand a compressible fluid flow is considered which is expressed by the well-known Navier-Stokes equations:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0, & (1)\\ \frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v v) &= \nabla \cdot \sigma + f, & (2)\\ \frac{\partial \rho e_t}{\partial t} + \nabla \cdot (\rho e_t v) &= \nabla \cdot (\sigma \cdot v) - q + f \cdot v + \dot{\vartheta}. & (3) \end{aligned}$$

The term ρ denotes the fluid density, t the independent time, v the fluid velocity, σ the stress tensor, f the external forces per unit volume, $e_t = e + \frac{1}{2}v^2$ the specific total energy, q the heat flux vector and $\dot{\vartheta}$ heat sources, e.g. radiation etc. The ∇-symbol corresponds to the space derivative while the $(\cdot)$ expresses the contradicting vector product. Moreover, the stress tensor can be separated in the following way:

$$\sigma = -p + \tau \quad \text{and} \quad \sigma_{ij} = -p_{ij} + \tau_{ij}. \tag{4}$$

The negative sign in front of the pressure tensor corresponds to a negative stress whenever a positive pressure is present and is related to the stress tensor using

$$p = \frac{1}{3}\mathrm{tr}(\sigma) I \quad \text{and} \quad p_{ij} = \frac{1}{3}\sigma_{ll}\delta_{ij} \tag{5}$$

where $\mathrm{tr}()$ corresponds to the trace of the second order tensor, I represents the second order unit tensor and δ_{ij} is the Kronecker symbol with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Typically, for a Newtonian fluid under Stoke's hypothesis the friction tensor τ is a function of the velocity and the dynamic viscosity $\mu(T)$ which itself is temperature dependent:

$$\tau(v,\mu) = \mu(\nabla v + \nabla^T v - \frac{2}{3} I \nabla \cdot v) \quad \text{and} \quad \tau_{ij}(v_l,\mu) = \mu(v_{i,j} + v_{j,i} - \frac{2}{3} v_{k,k}\delta_{ij}). \tag{6}$$

Using Fourier's law, the heat flux vector is given by:

$$q = -k\nabla T \quad \text{and} \quad q_i = -kT_{,i}. \tag{7}$$

The value k represents the heat conductivity and is itself temperature dependent $k = k(T)$.
The fluid is set to be in thermodynamical equilibrium, i.e. the density ρ and the internal energy e can be expressed by a potential depending only on two other variables:

$$\text{thermal: } \rho = \rho(T,p) \quad \text{calorical: } e = e(T,p). \tag{8}$$

Hence it is possible to derive a total differential

$$\delta\rho = \left(\frac{\partial \rho}{\partial T}\right)_p \delta T + \left(\frac{\partial \rho}{\partial p}\right)_T \delta p \quad \text{and} \quad \delta e = \left(\frac{\partial e}{\partial T}\right)_p \delta T + \left(\frac{\partial e}{\partial p}\right)_T \delta p. \tag{9}$$

The subscripts denote the quantities which are set constant during partial differentiation. Introducing Gibbs law:

$$T\delta s = \delta e - \left(\frac{p}{\rho^2}\right)\delta\rho = \delta h - \frac{1}{\rho}\delta p \tag{10}$$

with s being the entropy, and h being the enthalpy, an additional potential $\rho = \rho(p,s)$ can be stated. By eliminating the internal energy e

$$\delta\rho = \left(\frac{\partial \rho}{\partial p}\right)_s \delta p + \left(\frac{\partial \rho}{\partial s}\right)_p \delta s \quad \text{and} \quad \delta\rho = \left(\frac{1}{c^2}\right)\delta p - \rho\phi\,\delta s \tag{11}$$

follow. Because it is assumed that the thermodynamical equilibrium is only valid inside a fluid element, the spatial and temporal variation couple, i.e. the total differential must be understood as a material change and becomes $\delta() = \frac{\mathrm{D}()}{\mathrm{D}t}\mathrm{d}t$ where $\frac{\mathrm{D}()}{\mathrm{D}t} := \frac{\partial()}{\partial t} + v \cdot \nabla()$ and is called the material time derivative.
For an ideal gas the following relations are known:

$$\left(\frac{\partial \rho}{\partial p}\right)_s = \frac{1}{c^2}, \quad C_p = R + C_v, \quad \frac{C_p}{C_v} = \gamma, \quad e = C_v T, \quad s = C_v \ln\left(\frac{p}{\rho^\gamma}\right), \quad p = \rho R T. \tag{12}$$

c being the speed of sound, C_p and C_v the specific heat capacity for constant pressure and constant volume, respectively. R denotes the specific gas constant and γ the isentropic coefficient. Furthermore, the Navier-Stokes equations can be reformulated using the material time derivative. The mass, momentum and energy balance can be rewritten as follows:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot v \qquad \text{and} \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\, v_{l,l}, \tag{13}$$

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = \nabla \cdot \sigma + f \qquad \text{and} \qquad \rho \frac{\mathrm{D}v_j}{\mathrm{D}t} = \sigma_{ij,i} + f_j, \tag{14}$$

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = \sigma : v - \nabla \cdot q + \dot{\vartheta} \qquad \text{and} \qquad \rho \frac{\mathrm{D}e}{\mathrm{D}t} = \sigma_{kl} v_{l,k} - q_{j,j} + \dot{\vartheta}. \tag{15}$$

Using Gibbs equation (10) the energy is substituted by the entropy:

$$\rho \frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T}\left[\tau : v - \nabla \cdot q + \dot{\vartheta}\right] \quad \text{and} \quad \rho \frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{T}\left[\tau_{kl} v_{l,k} - q_{j,j} + \dot{\vartheta}\right]. \tag{16}$$

These equations along with the constitutive equation build the fundamental set of partial differential equations of fluid dynamics. Together with boundary conditions, which have to be defined, the complete problem is stated.

2.2 Principle description

The following outline is used to explain the difference between the reference flow and the perturbed flow. Considering two arbitrary trajectories of any particle, cf. Fig. 1, where trajectory 1 is a reference state (dashed line) and trajectory 2 is a perturbed state (solid line).

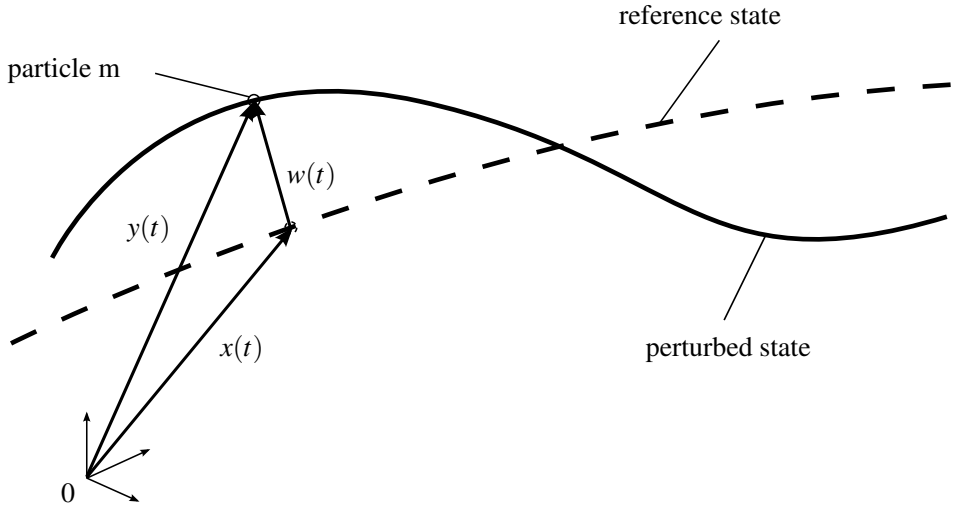

Figure 1 – Fundamental vector description

The basic relations of continuum mechanics will be presented and reviewed in order to give a fundamental understanding of Eulerian and Lagrangian variables.

2.2.1 Eulerian description

In the sense of an Eulerian description the perturbed state is given for a defined position rather than a certain particle. Considering any physical variable Φ the Eulerian perturbation Φ' is defined as:

$$\Phi'(y,t) = \Phi(y,t) - \Phi_0(y,t). \tag{17}$$

For a clear understanding one can consider a mesh fixed in space where the nodes of the mesh are the points of interest. In an Eulerian description the flow or any given particle m moves relative to the fixed mesh and one is not interested in the physical properties of the particle itself but the properties at the points of interest, which are fixed in space. It can be seen as a comparison of any quantity in space with respect to the reference state and the perturbed state.

2.2.2 Lagrangian description

The Lagrangian point of view considers the physical properties of a particle m and its path rather than the field quantities of a given point in space. In contrast with the previous example, the mesh of interest is not fixed in space but associated to the path of the particles trajectory. Therefore, a Lagrangian perturbation is defined as:

$$\tilde{\Phi}(m,t) = \Phi(m,t) - \Phi_0(m,t). \tag{18}$$

This expression compares the physical properties of a given particle in the reference state and the perturbed state.

2.2.3 Mixed Eulerian/Lagrangian description

In the mixed formulation both Eulerian and Lagrangian point of views are combined. The perturbed state is expressed as a function of the reference state at a given position $x(t)$ and a Lagrangian perturbation at the same point, cf. Fig. 1. In this way equation (18) becomes

$$\tilde{\Phi}(x,t) = \Phi(y,t) - \Phi_0(x,t). \tag{19}$$

It is now possible to combine the Eulerian perturbation with the mixed Eulerian/Lagrange perturbation using equation (17) and (19):

$$\Phi'(y,t) = \tilde{\Phi}(x,t) - \left(\Phi_0(y,t) - \Phi_0(x,t)\right). \tag{20}$$

Therefore, it is possible to calculate the Eulerian perturbation once the Lagrangian perturbation is known and how to move from x to y for all times t, or know $w(t)$ respectively.

2.3 Nonlinear formulation of the Eulerian-Lagrangian description

In order to account for any magnitude of perturbation a nonlinear formulation is necessary. The following expressions are known since they represent the general approach of continuum mechanics and can be found in many textbooks.

2.3.1 Description of displacement and coordinate transformation

From Figure 1 it can be clearly seen that:

$$w(t) = y(t) - x(t) \quad \text{and} \quad w_i(t) = y_i(t) - x_i(t) \tag{21}$$

where the Latin index accounts for the vector components $1,2,3$. Differentiation of equation (21) leads to

$$\mathrm{d}y = F \cdot \mathrm{d}x \quad \text{and} \quad \mathrm{d}y_i = F_{ij}\mathrm{d}x_j, \tag{22}$$

where F is the deformation gradient due to the perturbation and is defined as:

$$F = I + \nabla w \quad \text{and} \quad F_{ij} = \delta_{ij} + w_{i,j}. \tag{23}$$

The determinant of F is called the Jacobian and is given by:

$$J = \det F \quad \text{and} \quad J = \frac{1}{6} e_{lmn} e_{pqr} F_{lp} F_{mq} F_{nr} \tag{24}$$

with

$$e_{lmn} = \begin{cases} 1 & \text{if} \quad l,m,n = 1,2,3 \quad \text{cyclic} \\ -1 & \text{if} \quad l,m,n = 3,2,1 \quad \text{cyclic} \\ 0 & \text{else} \end{cases} \tag{25}$$

or in terms of the invariants of F

$$J = 1 + I_1(\nabla w) + I_2(\nabla w) + I_3(\nabla w) \tag{26}$$

where

$$\begin{aligned} I_1(\nabla w) &= \nabla \cdot w \quad \text{and} \quad I_1(\nabla w) = w_{j,j}, \\ I_2(\nabla w) &= \frac{1}{2}\left[(\nabla \cdot w)^2 - \nabla w : \nabla w\right] \quad \text{and} \quad I_2(\nabla w) = \frac{1}{2}\left[(w_{j,j})^2 - w_{k,l}w_{l,k}\right], \\ I_3(\nabla w) &= \det(\nabla w) \quad \text{and} \quad I_3(\nabla w) = \frac{1}{6}e_{lmn}e_{pqr}w_{l,p}w_{m,q}w_{n,r}. \end{aligned} \tag{27}$$

The inverse of F is given by:

$$G = F^{-1} = \frac{1}{J}T^T \quad \text{and} \quad G_{ij} = F_{ij}^{-1} = \frac{1}{J}T_{ji} \tag{28}$$

with

$$T = (1+\nabla \cdot w)I - \nabla^T w + N \quad \text{and} \quad T_{ij} = (1+w_{l,l})\delta_{ij} - w_{j,i} + N_{ij}. \tag{29}$$

The last term in this equation is formulated with:

$$\begin{aligned} N &= \frac{1}{2}\left((\nabla \cdot w)^2 - \nabla w : \nabla w\right) I - (\nabla \cdot w)\nabla^T w + (\nabla w \cdot \nabla w)^T, \\ N_{ij} &= \frac{1}{2}\left(w_{l,l}^2 - w_{m,n}w_{n,m}\right)\delta_{ij} - w_{l,l}w_{j,i} + w_{j,l}w_{l,i}. \end{aligned} \tag{30}$$

2.3.2 Transformation of derivatives

For a material volume V in the perturbed state and a related volume V_0 in the reference state, the following formula expresses the relation between the two volume elements:

$$\mathrm{d}V = J\,\mathrm{d}V_0. \tag{31}$$

The associated outer surface vectors $\mathrm{d}S$ and $\mathrm{d}S_0$ read:

$$\mathrm{d}S = T \cdot \mathrm{d}S_0 \quad \text{and} \quad \mathrm{d}S_i = T_{ij}\mathrm{d}S_{0j}, \tag{32}$$

respectively. To express the time derivative of the Lagrangian perturbation $\tilde{\Phi}(m,t)$ for a given particle m, the chain rule must be applied:

$$\begin{aligned} \frac{\partial \tilde{\Phi}}{\partial t}(m,t) &= \frac{\partial \tilde{\Phi}}{\partial t}(x,t) + \nabla\tilde{\Phi}(x,t) \cdot v_0(x,t) = \frac{\mathrm{D}\tilde{\Phi}}{\mathrm{D}t}(x,t), \\ \frac{\partial \tilde{\Phi}}{\partial t}(m,t) &= \frac{\partial \tilde{\Phi}}{\partial t}(x_i,t) + \tilde{\Phi}(x_i,t)_{,l}\, v_{0l}(x_i,t) = \frac{\mathrm{D}\tilde{\Phi}}{\mathrm{D}t}(x_i,t). \end{aligned} \tag{33}$$

In this sense, the particle velocity v is defined as:

$$\begin{aligned} v &= \frac{\partial y}{\partial t}(m,t) = \frac{\partial x}{\partial t}(m,t) + \frac{\partial w}{\partial t}(m,t) = v_0(x,t) + \tilde{v}(x,t), \\ v_i &= \frac{\partial y_i}{\partial t}(m,t) = \frac{\partial x_i}{\partial t}(m,t) + \frac{\partial w_i}{\partial t}(m,t) = v_{0i}(x_j,t) + \tilde{v}_i(x_j,t). \end{aligned} \tag{34}$$

From equations (33) and (34)

$$\tilde{v}(x,t) = \frac{\mathrm{D}w}{\mathrm{D}t}(x,t) \quad \text{and} \quad \tilde{v}_i(x_j,t) = \frac{\mathrm{D}w_i}{\mathrm{D}t}(x_j,t) \tag{35}$$

follows. Here, the velocity perturbation is equal to the convective time derivative of the particle displacement perturbation. Moreover, the space derivatives of an arbitrary tensor P in the perturbed field in conjunction with the Green-Ostrogradsky theorem read:

$$\begin{aligned} \nabla P(y,t) &= \left(\nabla\left(P_0(x,t) + \tilde{P}(x,t)\right)\right) \cdot G(x,t), \\ P_{,j}(y_l,t) &= \left(P_0(x_l,t) + \tilde{P}(x_l,t)\right)_{,i} G_{ij}(x_l,t), \end{aligned} \tag{36}$$

and

$$\begin{aligned} \nabla \cdot P(y,t) &= \frac{1}{J(x,t)}\nabla \cdot \left[\left(P_0(x,t) + \tilde{P}(x,t)\right) \cdot T(x,t)\right], \\ P_{j,j}(y_l,t) &= \frac{1}{J(x_l,t)}\left[\left(P_{0i}(x_l,t) + \tilde{P}_i(x_l,t)\right) G_{ij}(x_l,t)\right]_{,j}. \end{aligned} \tag{37}$$

All the required information for developing further equations is presented.

2.3.3 Perturbations of physical variables

In order to derive the perturbation for any physical variable lets consider the volume integral for the perturbed and reference states for any given quantity Φ:

$$I_\Phi(t) = \int_V \Phi(y,t)\,\mathrm{d}V, \tag{38}$$

$$I_{\Phi_0}(t) = \int_{V_0} \Phi_0(x,t)\,\mathrm{d}V_0. \tag{39}$$

By subtracting the two integrals and using expression (24):

$$\hat{I}_\Phi(t) = \int_{V_0} \left(\Phi(y,t)J(x,t) - \Phi_0(x,t)\right)\,\mathrm{d}V_0. \tag{40}$$

A new perturbation is then formed by the integrand of the integral:

$$\hat{\Phi}(x,t) = \Phi(y,t)J(x,t) - \Phi_0(x,t). \tag{41}$$

Herein $\hat{\Phi}(x,t)$ is the perturbation of Φ per unit volume in the reference flow. In other words, $\hat{\Phi}$ includes the perturbation of Φ for a given particle as well as the change in the amount of particles inside the volume due to the compressibility of the medium. Since mass conservation holds for a given particle, the Eulerian-Lagrange perturbation is related to the "hat"-perturbation in the following form, cf. equations (19) and (41):

$$\hat{\Phi} = \tilde{\Phi}J + (J-1)\Phi_0. \tag{42}$$

Since all quantities have the same argument the (x,t)-dependency is dropped wherever no confusion of the different states is possible.

2.3.4 General equation for $\hat{\Phi}$

It is assumed that the physical variable Φ satisfies the local balance equation

$$\frac{\partial \Phi}{\partial t} + \nabla\cdot(\Phi v) = Q_\Phi + \nabla\cdot\Psi_\Phi. \tag{43}$$

By calculating

$$\frac{\mathrm{D}\hat{I}_\Phi}{\mathrm{D}t}(t) = \frac{\mathrm{D}I_\Phi}{\mathrm{D}t}(t) - \frac{\mathrm{D}I_{\Phi_0}}{\mathrm{D}t}(t) \tag{44}$$

and applying the local balance equation (43),

$$\frac{\mathrm{D}\hat{\Phi}}{\mathrm{D}t}(x,t) + \hat{\Phi}(x,t)\nabla\cdot v_0(x,t) = \hat{Q}_\Phi(x,t) + \widehat{\nabla\cdot\Psi_\Phi}(x,t) \tag{45}$$

follows, where

$$\widehat{\nabla\cdot\Psi_\Phi}(x,t) = \nabla\cdot\left[\left(\Psi_{\Phi_0}(x,t) + \tilde{\Psi}_\Phi(x,t)\right)\cdot T - \Psi_{\Phi_0}(x,t)\right]. \tag{46}$$

The divergence of the reference velocity can be transformed using the mass balance, cf. equation (13), which leads to:

$$\frac{\mathrm{D}\hat{\Phi}}{\mathrm{D}t} - \frac{\hat{\Phi}}{\rho_0}\frac{\mathrm{D}\rho_0}{\mathrm{D}t} = \hat{Q}_\Phi + \widehat{\nabla\cdot\Psi_\Phi}, \tag{47}$$

hence

$$\rho_0\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\hat{\Phi}}{\rho_0}\right) = \hat{Q}_\Phi + \widehat{\nabla\cdot\Psi_\Phi}. \tag{48}$$

2.3.5 Application to density

Equation (48) is applied to the density perturbation. Since the mass balance is valid for any material domain, no source terms are present and therefore $Q_\rho = 0$ and $\Psi_\rho = 0$. Hence:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\hat{\rho}}{\rho_0}\right) = 0 \Rightarrow \frac{\hat{\rho}}{\rho_0} = \text{constant}. \tag{49}$$

For times $t \leq 0$ no perturbations are present. In conjunction with equation (42) it stems:

$$\tilde{\rho} = \frac{1-J}{J}\rho_0. \tag{50}$$

This implies that the density perturbation is only a function of ∇w, cf. equation (23) and (24), and the reference field.

2.3.6 Application to other variables

It is assumed that a value per unit volume of any quantity can be written as $\Phi = \rho\Phi_m$, where Φ_m is the value per unit mass of the quantity. Using equation (41) and $\rho_0 = J\rho$ yields

$$\rho(\Phi_0 + \hat{\Phi}) = \rho_0\Phi \tag{51}$$

and

$$\frac{\hat{\Phi}(x,t)}{\rho_0(x,t)} = \frac{\Phi(y,t)}{\rho(y,t)} - \frac{\Phi_0(x,t)}{\rho_0(x,t)} = \widetilde{\left(\frac{\Phi}{\rho}\right)} \tag{52}$$

or in a more convenient way,

$$\hat{\Phi} = \rho_0\tilde{\Phi}_m. \tag{53}$$

Taking this result and replacing $\hat{\Phi}$ in equation (48) yields:

$$\rho_0 \frac{\mathrm{D}\tilde{\Phi}_m}{\mathrm{D}t} = \hat{Q}_\Phi + \nabla\cdot\left[\left(\Psi_{\Phi_0}(x,t) + \tilde{\Psi}_\Phi(x,t)\right)\cdot T - \Psi_{\Phi_0}(x,t)\right] \tag{54}$$

which can be written in conservative form:

$$\frac{\partial\rho_0\tilde{\Phi}_m}{\partial t} + \nabla\cdot\left(\rho_0\tilde{\Phi}_m v_0\right) = \hat{Q}_\Phi + \nabla\cdot\left[\left(\Psi_{\Phi_0}(x,t) + \tilde{\Psi}_\Phi(x,t)\right)\cdot T - \Psi_{\Phi_0}(x,t)\right]. \tag{55}$$

2.3.7 Application to the momentum equation

Now the momentum equation in its conservative form reads:

$$\frac{\partial\rho v}{\partial t} + \nabla\cdot(\rho vv) = \nabla\cdot(\tau - p) + f \quad \text{and} \quad \frac{\partial\rho v_i}{\partial t} + (\rho v_i v_j)_{,j} = (\tau_{ij} - p_{ij})_{,j} + f_i. \tag{56}$$

Identifying the source terms according to equation (55), yields:

$$\begin{aligned} \tilde{Q}_{\rho v} = \hat{f} \quad &\text{and} \quad \tilde{Q}_{\rho v} = \hat{f}_i, \\ \Psi_{(\rho v)_0} = \tau_0 - p_0 \quad &\text{and} \quad \Psi_{(\rho v)_0} = \tau_{0ij} - p_0\,\delta_{ij}, \\ \tilde{\Psi}_{\rho v} = \tilde{\tau} - \tilde{p} \quad &\text{and} \quad \tilde{\Psi}_{\rho v} = \tilde{\tau}_{ij} - \tilde{p}\,\delta_{ij}. \end{aligned}$$

yields. Inserting the source terms into equation (55) and applying equation (35) results in:

$$\begin{aligned} \rho_0 \frac{\mathrm{D}^2 w}{\mathrm{D}t^2} &= \hat{f} + \nabla\cdot[(\tau_0 - p_0 + \tilde{\tau} - \tilde{p})\cdot T - \tau_0 + p_0], \\ \rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} &= \hat{f}_i + \left[(\tau_{0il} - p_0\,\delta_{il} + \tilde{\tau}_{il} - \tilde{p}\,\delta_{il})\,T_{lj} - \tau_{0ij} + p_0\,\delta_{ij}\right]_{,j}. \end{aligned} \tag{57}$$

For the sake of clearness further formulations are restricted to the component formulation in order to give a clear insight to the summations and simplifications. Considering the characteristics of the δ_{ij}-function and showing that $T_{ij,j} = 0$ for any i, equation (57) can be simplified to:

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \hat{f}_i + \left[(\tau_{0il} + \tilde{\tau}_{il})\,T_{lj} - (p_0 + \tilde{p})\,T_{ij} - \tau_{0ij} + p_0\,\delta_{ij}\right]_{,j} \tag{58}$$

and further

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \hat{f}_i + T_{lj}\,(\tau_{0il} + \tilde{\tau}_{il})_{,j} - T_{ij}\,(p_0 + \tilde{p})_{,j} - \tau_{0ij,j} + p_{0,i}. \tag{59}$$

If one is expanding T_{mn} in its first order terms it is clear to see that all gradients of the reference pressure $p_{0,i}$ and the divergence of the viscous stress tensor $\tau_{0ij,j}$ cancel exactly, i.e. these terms are not involved in the process of generating particle movement or sound waves. This seems logical from a physical point of view.

2.3.8 Application to inviscid ideal gas

For an ideal gas without frictional stress components and neglecting heat flux and other sources, equation (16) takes the simple form of:

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0. \tag{60}$$

Utilizing equation (55) the entropy perturbation reads:

$$\frac{\mathrm{D}\tilde{s}}{\mathrm{D}t} = 0. \tag{61}$$

Moreover the thermodynamic relations, cf. equation (12) can be applied leading to

$$\tilde{s} = s - s_0 = C_v \ln\left[\frac{p}{p_0}\left(\frac{\rho_0}{\rho}\right)^{\gamma}\right] = 0. \tag{62}$$

Hence,

$$1 + \frac{\tilde{p}}{p_0} = \left(1 + \frac{\tilde{\rho}}{\rho_0}\right)^{\gamma}, \tag{63}$$

and together with equation (50) $\tilde{p}$ can be expressed as:

$$\tilde{p} = \left(J^{-\gamma} - 1\right) p_0. \tag{64}$$

Replacing $\tilde{p}$ in equation (59) and neglecting all viscous stress terms leads to:

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \hat{f}_i - T_{ij}\left(p_0 + \left(J^{-\gamma} - 1\right) p_0\right)_{,j} + p_{0,i}. \tag{65}$$

By multiplying eq. (65) with T^{-1} resulting from equations (23) and (28),

$$\rho_0 T_{li}^{-1} \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = T_{li}^{-1} \hat{f}_i - \left(p_0 + \left(J^{-\gamma} - 1\right) p_0\right)_{,l} + T_{li}^{-1} p_{0,i} \tag{66}$$

follows, where:

$$T_{li}^{-1} = \frac{1}{J}\left(\delta_{li} + w_{i,l}\right). \tag{67}$$

Finally:

$$\frac{\rho_0}{J}\left(\delta_{li} + w_{i,l}\right) \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \frac{1}{J}\left(\delta_{li} + w_{i,l}\right) \hat{f}_i - \left(p_0 + \left(J^{-\gamma} - 1\right) p_0\right)_{,l} + \frac{1}{J}\left(\delta_{li} + w_{i,l}\right) p_{0,i} \tag{68}$$

and after further simplification due to multiplying eq. (68) with J

$$\rho_0\left(\delta_{li} + w_{i,l}\right) \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \left(\delta_{li} + w_{i,l}\right) \hat{f}_i - J\left(J^{-\gamma} p_0\right)_{,l} + \left(\delta_{li} + w_{i,l}\right) p_{0,i}. \tag{69}$$

Applying the chain derivation rule leads to:

$$\rho_0\left(\delta_{li} + w_{i,l}\right) \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \left(\delta_{li} + w_{i,l}\right) \hat{f}_i + \gamma J^{-\gamma} J_{,l}\, p_0 - J^{1-\gamma} p_{0,l} + \left(\delta_{li} + w_{i,l}\right) p_{0,i}. \tag{70}$$

Equation (69) represents an exact nonlinear formulation of the perturbation of the particle displacement excited by external force fluctuations. With a closer look one can see that the gradients of the reference pressure $p_{0,l}$ cancel when expanding J in its first order. This fact appears to be correct from a physical point of view, since the gradient of the reference pressure produces no acoustic waves but the flow itself. In other words, if one neglects all external force fluctuations this formulation is free of any sources. Therefore, the boundary conditions must be carefully chosen in order to compute aeroacoustic problems correctly.

2.3.9 Simplification due to small perturbations: The Galbrun Equation

Considering only small perturbations, i.e. only terms with $O(\nabla w)$, the following expressions can be stated:

$$J = 1 + w_{l,l}, \tag{71}$$
$$T_{ij} = \delta_{ij} - w_{j,i} + w_{l,l}\delta_{ij}, \tag{72}$$
$$\tilde{\rho} = -\rho_0 w_{l,l}, \tag{73}$$
$$\tilde{p} = -\gamma p_0 w_{l,l}. \tag{74}$$

and using $c_0^2 = \gamma\frac{p_0}{\rho_0}$, it stems:

$$\tilde{p} = c_0^2\tilde{\rho}. \tag{75}$$

Equation (59), which is stated here again while neglecting viscous stress terms reads:

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \hat{f}_i - T_{ij}\left(p_0 + \tilde{p}\right)_{,j} + p_{0,i}. \tag{76}$$

Inserting equations (71) – (75) into equation (76) results in:

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \hat{f}_i - \left(\delta_{ij} - w_{j,i} + w_{l,l}\delta_{ij}\right)\left(p_0 - c_0^2\rho_0 w_{k,k}\right)_{,j} + p_{0,i} \tag{77}$$

and further in:

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} = \hat{f}_i + \left(c_0^2\rho_0 w_{k,k}\right)_{,i} + p_{0,j}w_{j,i} - p_{0,i}w_{l,l} + \left(c_0^2\rho_0 w_{k,k}\right)_{,j}\left(-w_{j,i} + w_{l,l}\delta_{ij}\right). \tag{78}$$

Neglecting the terms of $O(||\nabla w||^2)$ the usual Galbrun's equation reads

$$\rho_0 \frac{\mathrm{D}^2 w_i}{\mathrm{D}t^2} - p_{0,j}w_{j,i} + p_{0,i}w_{l,l} - \left(c_0^2\rho_0 w_{k,k}\right)_{,i} = \hat{f}_i \tag{79}$$

where all external forces are set to zero. In this way, equation (79) can be seen as a inhomogeneous form of the usual Galbrun equation, having source terms in form of an external force fluctuations, which is in an exact agreement with the outlines of Bonnet-Ben Dhia et al. (15).

3. NUMERICAL METHODS

In order to solve equation (69) or (79) numerous numerical schemes such as the Finite Volume, Finite Difference or Finite Element Method (FEM) can be utilized. Since finding closed solutions is usually not possible for realistic problems the key idea is to discretize the mathematical model, i.e. the partial differential equations into a system of local equations.
Several attempts have been done to solve equation (79) with the help of the usual Galerkin discretization method as a weak formulation scheme and the use of the FEM. Retka et al. (11) and Dietzsch et al. (16) show that difficult problems such as spurious modes and non physical results can arise when using FEM for solving the Galbrun Equation. An attractive approach seems to be the Discontinuous Galerkin Finite Element Method (DGFEM) for discretizing the global domain since this scheme combines flexible approximation order and complex mesh structures, local conservation of physical quantities such as mass, momentum and energy, the increase of robustness and accuracy as well as the facilitation of parallelization.

4. CONCLUSION

The authors presented a detailed outline of how to derive an inhomogeneous nonlinear form of the Galbrun equation based on the well known Naver-Stokes equations. Utilizing a mixed Eulerian/Lagrange formalism, a displacement based equation was derived. Having a closer look at this novel expression, it was clear to see that neither the gradient of the reference pressure nor the divergence of the friction related stress tensor can be sources of sound. Further, if one considers an inviscid ideal fluid and neglects any volume force fluctuations the nonlinear form becomes homogeneous and is therefore free of sources, implying that the boundary conditions must be well defined in order to compute aeroacoustic phenomenons correctly.
In the future work the formulation will be extended to include the friction related stress tensor to account for real fluid behavior and in addition, numerical schemes such as the DGFEM will be tested.

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