inter.noise 2014
MELBOURNE AUSTRALIA
16-19 NOVEMBER

# A simple model of effective elastic properties of materials with inclusions

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## ABSTRACT

The aim of this study is to develop a simple phenomenological model for the elastic moduli of a composite material formed by localized inclusions embedded in an elastic matrix. It is assumed that the material can be characterised by only two aggregated parameters, viz., volume fraction of inclusions and their resonance frequency within the elastic matrix. The values of these two parameters are assumed to be given (i.e. from experimental measurements) or deduced from other models (also presented in the paper). The shear wave velocity in the elastic matrix is assumed to be much smaller than the velocity of longitudinal waves. A simple analytical expression for the effective longitudinal wave velocity that is uniformly valid for the entire frequency domain is derived (including proximity to the resonance frequency of inclusions) and validated with some paradigmatic results of the mean-field theories.

Keywords: Effective modulus, Inclusions, Absorption, Resonance.
I-INCE Classification of Subjects Number: 23.1, 23.9, 35.1, 43.2, 47.1

## 1. INTRODUCTION

Elastic materials with micro-homogeneous structure have been attracting ever-increasing interest in various areas of scientific research and practical applications. Illustrative examples include phononic crystals, acoustic cloaks, sound absorbers, and aircraft and ship structures [1–3]. There is an upsurge of interest in calculating (and predicting) the properties of such materials. This is due to the continuous requirement to engineer new materials targeted at specific applications as well as to reduce the burden associated with the costly experimental programs.

There is a wealth of analytical and numerical methods for calculating the elastic properties of composite materials and there is a vast amount of literature devoted to the subject (see Refs. [1–11] and the references therein). At sufficiently low frequencies (the quasi-static limit), the so-called Effective Medium Approximation (EMA) usually holds. Under this approximation the composite material can be modelled as a homogeneous visco-elastic material with some effective elastic moduli that are determined by only the volume fraction of inclusions in the composite. The analytical description becomes more challenging near the frequencies of internal resonances of the material (natural frequencies of inclusions) where the composites can exhibit the most interesting and nontrivial elastic properties. In fact, in this region the elastic moduli usually have the typical ‘resonant’ singularities, where the simple EMA models become inaccurate.

To reach acceptable agreement with experimental observations more advanced models can be employed that take into account other physical effects that become important near resonance frequencies (such as resonance mode coupling and switching, multi-scattering, frequency-dependent attenuation; see, e.g., [2,8,9,11]). The advanced models can reach an acceptable level of fidelity but unfortunately they usually become analytically intractable. A numerical treatment of advanced models can produce very accurate outcomes, but requires an extensive domain knowledge and significant computational effort to reveal any aggregated trends emerging in the multi-parameter space describing the composite [2]. The inaccuracy of the simplistic EMA approach and the technical challenges associated with application of the advanced numerical models near resonant frequencies necessitates development of interim *conceptual* models which are much easier to use, while still being adequate to capture the complex phenomenology in composite materials near resonances. Being properly validated such models would allow a prompt assessment and interpretation of new trends in

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experimental data as well as rapid evaluation of ‘what-if’ scenarios in prototyping studies.

The aim of this paper is to present a simple, yet scientifically rigorous, model of a composite material with localized inclusions which is characterized by only two aggregated parameters, namely the volume fraction of inclusions $\alpha$ and the resonance frequency $\omega_0$ of an individual inclusion. The values of these parameters are assumed to be given (i.e. from experimental measurement) but can be deduced from other theoretical models, as described in the following sections.

## 2. THEORETICAL FRAMEWORK

### 2.1 Model for elastic wave propagation

The propagation of the longitudinal waves in the medium with resonant scatterers (inclusions) can be written in the form of a single equation [12]

$$\left(\omega_0^2+\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial t^2}u-c_l^2\frac{\partial^2}{\partial x^2}u\right)+\beta\omega_0^2\frac{\partial^2}{\partial t^2}u=0, \tag{1}$$

where $c_l$ is the velocity of longitudinal waves in the elastic matrix of the composite, $\omega_0$ is the resonance frequency of inclusions, and $\beta$ is a concentration parameter proportional to the concentration of inclusions. At this stage $\omega_0$ and $\beta$ are simply two independent parameters of the model, but some analytical models for both parameters, connecting them directly to the volume fraction of inclusions, will be presented below. For $\beta=0$ Eq. (1) reduces to the conventional wave equation.

Substituting $u\sim\exp(i\omega t)$ in Eq. (1) and explicitly evaluating the time derivatives in terms of $\omega$ while leaving the spatial derivative unevaluated leads to

$$-\omega^2\left(\omega_0^2-\omega^2\right)u-\left(\omega_0^2-\omega^2\right)c_l^2\frac{\partial^2}{\partial x^2}u-\beta\omega_0^2\omega^2u=0$$

or

$$\frac{\omega^2}{c_l^2}\left(1+\frac{\beta}{1-(\omega/\omega_0)^2}\right)u+\frac{\partial^2}{\partial x^2}u=0,$$

which is the conventional Helmholtz equation for longitudinal wave propagation

$$\frac{d^2}{dx^2}u+k^2u=0,\quad k=\omega/c_l^e, \tag{2}$$

where $c_l^e$ is the effective, and dispersive, velocity of longitudinal waves in the medium with inclusions,

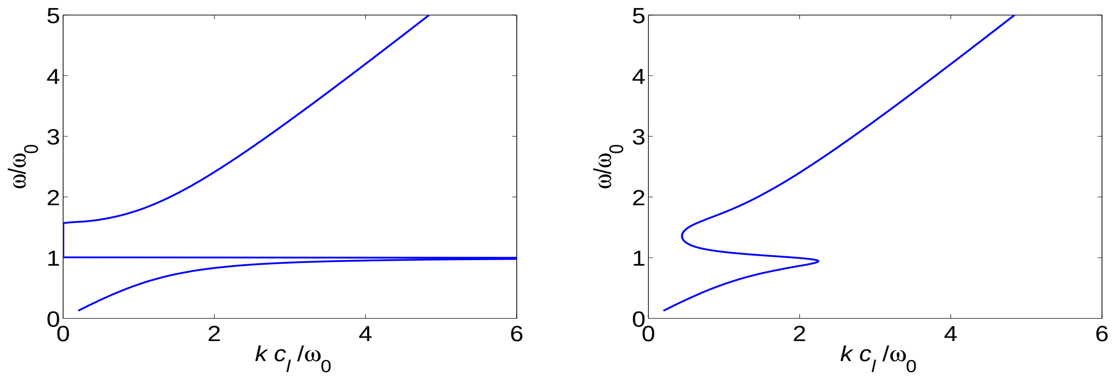
$$c_l^e=c_l\left(1+\frac{\beta}{1-(\omega/\omega_0)^2}\right)^{-1/2}. \tag{3}$$

The effective Lamé elastic moduli $\lambda$ and $\mu$ of the material with inclusions are related to the effective velocities of longitudinal and shear waves, $c_l^e$ and $c_s^e$, through the standard expressions for isotropic materials

$$c_l^e=\sqrt{(\lambda+2\mu)/\rho^e},\quad c_s^e=\sqrt{\mu/\rho^e}, \tag{4}$$

where $\rho^e$ is the effective density of the material. Modulus $\mu$ is the shear modulus. The bulk modulus of the material is $K=\lambda+2\mu/3$ which differs from the compressional (or longitudinal) modulus $\lambda+2\mu$. Since the proposed theory is restricted to rubber-like materials it is assumed that $c_s\ll c_l$ (or $\mu\ll\lambda$). Dissipation processes are equivalent to adding an imaginary part to the wavenumber, and through equations 2 and 4 this correspond to the elastic moduli being complex.

The asymptotic behavior of Eq. (3) is straightforward. As $\omega\to 0$, $c_l^e\to c_l/(1+\beta)^{1/2}\le c_l$, and as $\omega\to\infty$, $c_l^e\to c_l$. More accurate analysis reveals that Eq. (3) corresponds to two dispersion modes (branches) with the different limiting cases. Example dispersion curves, $\omega=\omega(k)$ versus $k=\omega/c_l^e$, directly from Eq. (3), are presented in Fig. 1 for $\beta=1.5$. The $y$ axis is presented in non-dimensional form $\omega/\omega_0$ vs $kc_l/\omega_0$. Fig. 1(a) is the zero-attenuation case. The plot in Fig. 1(b) includes attenuation by making $\omega_0$ complex by multiplying by $1+0.1i$ (in which case $\omega$ is normalised by the real part of $\omega_0$, and $k$ in the figures represents the real part). The introduction of dissipation leads to a reconnection of the two branches of the dispersion curve, resulting in disappearance of a ‘forbidden’ frequency gap. This effect has been previously reported [8]. It will be shown later that the resonance frequency of a void is directly related to the complex shear modulus of the elastic matrix and the radius of the void.




(a) (b)

Figure 1 – Dispersion relation $\omega(k)$ for Eq. (3): (a) case of two independent modes; (b) merging two modes caused by dissipation in the material.

### 2.2 Model for concentration parameter $\beta$

Following Ostrovsky [13] one can consider a 1D system with the axis $z$ pointing in the direction of wave propagation. Assume $u$ is particle displacement in the $z$ direction and $v$ is the deviation of a single inclusion volume from its equilibrium value. The set of coupled equations consists of the equation for the longitudinal waves

$$\frac{\partial^2}{\partial t^2}u = \frac{\partial}{\partial z}\sigma_{zz}, \tag{5}$$

where $\sigma_{zz}$ is the component of the stress tensor, and the equation for the oscillations of the inclusions is

$$\frac{\partial^2}{\partial t^2}v + \omega_0^2 v = 4\pi a c_l^2 \sigma_{zz}, \tag{6}$$

where $\omega_0$ is the resonance frequency of the inclusion.

For the sake of simplicity only the case of void inclusions is considered, where the density of the inclusions is much less than the density $\rho$ of the host matrix. In such a case $\sigma_{zz} = (\lambda+2\mu)\frac{\partial u}{\partial z} - (\lambda+2\mu)nv$, and

$$\sigma_{zz} = \rho c_l^2 \left(\frac{\partial u}{\partial z} - nv\right), \tag{7}$$

where $n$ is the number of voids per unit volume, so that the volume fraction of voids relative to the whole is

$$\alpha = (4/3)\pi r_0^3 n, \tag{8}$$

which can be connected with the parameter $\beta$ as follows.

A dispersion relation for this model follows from an assumed solution of the form $u, v \sim \exp(i\omega t - ikx)$ in Eq. (7), giving

$$\omega^4 - (\Omega_0^2 + c_l^2 k^2)\omega^2 + \omega_0^2 c_l^2 k^2 = 0, \tag{9}$$

where $\Omega_0^2 = \omega_0^2 + 4\pi a c_l^2 n = \omega_0^2\left(1 + \frac{3}{4}\alpha\left(\frac{c_l}{c_s}\right)^2\right)$. One can write $k(\omega)$ from Eq. (9) as

$$k^2(\omega) = \frac{\omega^2}{c_l^2}\left(1 + \frac{(\Omega_0/\omega_0)^2 - 1}{1-(\omega/\omega_0)^2}\right). \tag{10}$$

By comparing the expression for $c_l^e = \omega/k(\omega)$ with Eq. (3), the following formula for the aggregated parameter $\beta$ can be derived:

$$\beta = \frac{3}{4}\alpha\frac{\lambda+2\mu}{\mu} \approx \frac{3}{4}\alpha\left(\frac{\lambda}{\mu}\right), \tag{11}$$

assuming $\mu \ll \lambda$.

The aggregated parameter $\beta$ describes the relative effects of the elastic properties of the host material ($\lambda$ and $\mu$) and the volume fraction of inclusions ($\alpha$). This enables informative comments and consistent design of new elastic materials that provide a desirable value for the aggregated parameter $\beta$ (perhaps derived from an optimisation study).

### 2.3 Models for resonant frequency $\omega_0$

For a number of paradigmatic shapes the resonance frequency $\omega_0$ can be calculated analytically and employed in Eqs. (3). For instance, for a spherical inclusion of radius $r_0$ in a rubber-like material the lowest resonance frequency can be estimated by employing analytical results for the response scattering from a single inclusion [6, 14]

$$\omega_0 = \xi_1 \frac{c_s}{r_0} = \xi_1 \left(\frac{4\pi}{3v_0}\right)^{1/3} c_s, \tag{12}$$

where $v_0 = (4\pi/3)r_0^3$ is the volume of the cavity. Parameter $\xi_1$ is approximately 2 for a void inclusion [15]. For the case of a hard spherical inclusion one can use the formula [6, 10, 15]

$$\xi_1 = \frac{3}{\sqrt{2(\rho_i/\rho)+1}}, \tag{13}$$

where $\rho_i$ is the mass density of the inclusions and $\rho$ is the density of the matrix. For the most realistic composites $\rho_i/\rho$ is generally less than 10 and this leads to a reasonably narrow range for the parameter $\xi_1$ to capture a variety of the elastic properties of the materials that are of interest for practical applications. Typically, $0.5 \leq \xi_1 \leq 0.7$.

To evaluate other inclusion shapes a phenomenological approach initially proposed in Refs. [4, 16] can be used. According to this approach the resonance frequency of a spheroidal cavity can be estimated by introducing a shape factor $\xi_2$:

$$\omega_0^e = \xi_2 \omega_0, \tag{14}$$

where $\omega_0$ is the resonance frequency of a spherical inclusion of the same volume $v_0 = (4\pi/3)r_0^3 \approx (4\pi/3)a^2c$, and $a$ and $c$ are the main axes of the spheroidal inclusion. The corresponding effective radius is $r_0 = (ca^2)^{1/3} = ae^{1/3}$, where $e = a/c$ is the eccentricity of the spheroid. The simplest formula for the ‘shape’ factor can be conjectured from some analytical arguments [16] and is

$$\xi_2 = \frac{\kappa}{\ln Z}, \quad e \geq 1, \tag{15}$$

and

$$\xi_2 = (1/\kappa)\ln Z, \quad e \leq 1, \tag{16}$$

where $Z = 2e + 2/e$ and $\kappa = \ln(4)$. For a spherical inclusion $e = 1$ and $\xi_2 = 1$. The limit $e \to 0$ or $e \to \infty$ corresponds to the extreme cases of an oblate or prolate spheroid, respectively.

A very slender inclusion can be modeled as a cylinder. At this limit the analytical results for the resonance scattering in the cylindrical geometry [13] can be employed. For a cylindrical void of radius $r_0$ Eq. (12) can still be used but with an additional factor $\xi_2 = 1/\ln(c_l/c_s) \ll 1$. For the case of a conventional polymer matrix this implies that the shape parameter $\xi_1$ can take rather lower values (down to $\xi_1 \simeq 0.3$).

If the volume fraction of inclusions is not very low, then the effect of multiple scattering (or coherence of sound waves scattered by different inclusions) may become important. In this case the scatterers can no longer be considered independent and the proposed framework must be refined. Formally, the resonance frequency $\omega_0$ then becomes a function of the volume fraction $\alpha$. To describe this effect the analytical models of Refs. [7, 9] are employed, where the resonance frequency of a system of identical inclusions can be written in the scaling form

$$\omega_0^e = \xi_3 \omega_0, \tag{17}$$

where $\omega_0$ is the resonance frequency of a single inclusion in an infinite medium (i.e., given by Eqs. (12), (14)), and factor $\xi_3$ can be estimated from the explicit relationship:

$$\xi_3 = \mathscr{Q}\mathscr{P}, \tag{18}$$

$$\mathscr{Q}(\alpha) = \sqrt{\frac{1+(2/3)\alpha^{5/3}}{1-(3/2)\alpha^{1/3}+(3/2)\alpha^{5/3}-\alpha^2}}, \tag{19}$$

$$\mathscr{P}(\alpha,\rho_i,\rho) = \sqrt{\frac{2(\rho_i/\rho)+1}{2(\rho_i/\rho)+\zeta}}, \tag{20}$$

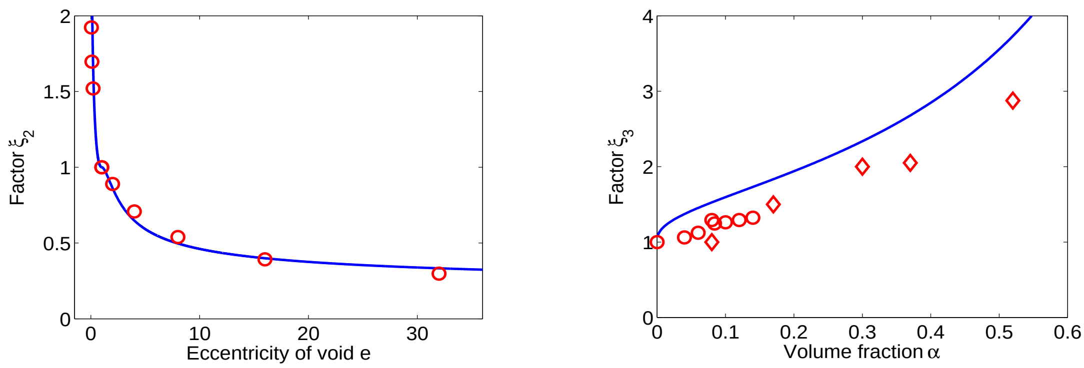
with $\zeta = (1+2\alpha)/(1-\alpha)$. One can see that for low concentrations of inclusions ($\alpha \to 0$), $\mathscr{P} \to 1$ and $\zeta \to 1$, and $\xi \to 1$, so this expression is consistent with Eq.(12) for sparse composites.

To summarise, the main outcome of the proposed models is that the natural frequencies of an inclusion in a rubber-like material can be approximated in the simple scaling form, $\omega_0^e = \xi(c_s/r_0)$, where $r_0$ is the effective radius, and $\xi = \xi_1\xi_2\xi_3$, with factors $\xi_1, \xi_2, \xi_3$ determined by the density, shape and relative proximity of inclusions as defined above. This form provides a simple, yet rigorous, framework for prediction and control of the resonance properties of composite materials with inclusions.

## 3. NUMERICAL RESULTS

Eqs. (3), (11), (12) and (14)–(20) completely characterise the elastic properties (effective elastic moduli) of the composite material with resonant inclusions. At low concentration of inclusions the volume fraction $\alpha$ does not enter the expression for the parameter $\omega_0$, so the parameters $\alpha$ (or $\beta$) and $\omega_0$ can be varied independently. At higher concentrations the correction for $\omega_0$ given by Eq. (17) needs to be applied.

As the first step, the scaling factors $\xi_1$, $\xi_2$ and $\xi_3$ in Eqs.(12), (14), and (17) are validated. The formula (12) for a spherical void is well-known and is the material of textbooks [6, 14]. The expression (12) has been previously used in numerous studies [10]. The expression for the shape parameter $\xi_2$ (Eqs. (15) and (16) can be validated by comparison with numerical calculations for spheroidal voids in rubber-like materials [4, 16]. Fig. 2(a) compares the calculated resonance frequencies with the analytical approximation of Eqs. (15) and (16)) as a function of eccentricity $e$. It can be seen that the simple analytical approximation provides a good fit.




(a) (b)

Figure 2 – Factors for estimation of resonance frequency: (a) Shape factor, Eq.(14), for a spheroidal inclusion; (b) Scaling factor, Eq.(18), for a material with spherical voids embedded in the host rubber-like matrix as a function of the inclusion volume fraction of $\alpha$. In both figures, markers correspond to numerical results [4,16] or experimental data [7, 9]

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Next, Eq. (18) for the concentration correction for the resonant scattering by a composite with inclusions is validated. In Fig. 2(b) measurements of the resonance frequencies of a composite material are compared with the results of the equation, presented as a function of concentration. The markers show data for steel inclusions embedded in a host rubber-like material with $c_l = 1500$ m/s and $c_s = 200$ m/s [7]. Although the agreement for the scaling factor in the figure is not as good as for the shape factor, these results, and others [9], clearly show the general trend of an increase in resonance frequency with an increase in the volume fraction of inclusions (for a given inclusion size). This trend is supported by the analytical model of Eq. (18) for a broad range of parameters. For low volume concentration the asymptote of Eq. (18) is $\xi_2 \approx 1 + (3/4)\alpha^{1/3}$.

As a demonstration of the proposed approach the following optimisation problem is considered. More specifically, the optimal performance of a vibro-acoustic panel formed by a visco-elastic matrix with resonant inclusions (voids) is determined. The performance of the panel can easily be defined in terms of parameters of the proposed framework (such as $\beta$ and $\omega_0$, or $\alpha$ and $r_0$) that can be tuned.

An elastic planar panel of fixed thickness in water is considered and the reflection and transmission of sound at normal incidence is computed using plane-wave reflection/transmission theory (see, for example [17–19] and the physical properties shown in Table 1. The theory is based on plane waves incident on finite thickness layers of infinite lateral extent. The amplitude of reflected and transmitted plane waves is obtained mathematically by using the equations of continuity of pressure, shear stress and particle velocity across the boundaries of the layers at arbitrary angles of incidence. In the example here at normal incidence the shear modulus of the voided panel does not affect the energy reflection or transmission coefficients $R$ and $T$ (although the shear modulus of the matrix *does* affect the compressional modulus $\lambda + 2\mu$ through Eqs. (3)

Table 1 – Material Properties for Figs. 3 and 4

| Material | Parameter | Value | Units |
|---|---|---|---|
| Water | Density | 1000 | $\mathrm{kg/m^3}$ |
| | Sound speed | 1500 | m/s |
| Elastic Material | Density | 1300 | $\mathrm{kg/m^3}$ |
| | Compressional modulus $\lambda + 2\mu$ | $5 \times 10^9$ | Pa |
| | Loss factor $\eta$ | 0.001 | — |
| | Shear modulus $\mu$ | $3 \times 10^6$ | Pa |
| | Loss factor $\eta$ | 0.1 | — |
| Voided panel | Thickness $h$ | 0.05 | m |
| | Fig. 3 void radius $r_0$ | 0.005 | m |
| | Fig. 3 void volume fraction $\alpha$ | 0.01 | — |

and (11)). Furthermore, with only one layer in the panel and the same medium on either side of the panel the equations can be written analytically. For example, for a panel of thickness $h$ [20],

$$T = \frac{4}{4\cos^2 kh + \left(\frac{\rho_w c_w}{\rho^e c_l^e} + \frac{\rho^e c_l^e}{\rho_w c_w}\right)^2 \sin^2 kh}, \tag{21}$$

where $\rho_w$ and $c_w$ and the density and sound speed in water, respectively, and the wavenumber $k$ is as in Eq. (2). With damping included, as $\mu^* = \mu(1+i\eta)$, etc. (see Table 1), both $k$ and $c_l^e$ are complex.

Fig. 3 shows the results using the simplified effective elastic properties given by Eqs. (3), (11) and (12) compared with results of the well established theories of Gaunaurd [21] and Chaban [15]. Fig. 3(a) shows the energy reflection $R$ and transmission $T$ coefficients for the panel, and Fig. 3(b) shows the fraction of energy absorbed by the panel $(1-R-T)$ according to these three theories. The results of the simplified model and [15] are almost indistinguishable, and good agreement is obtained for all models. The effect of void shape and high volume concentration is not included in the comparison as models [21] and [15] do not include this effect. The resonance $\omega_0$ around 3 kHz can be clearly seen.

The high frequency absorption tail of Gaunaurd's theory in Fig. 3 is due to the inclusion of the direct scattering from the voids themselves, whereas the Chaban and simplified theories only include the loss from the material resonance due to those voids. If the loss factors are set to zero in the modelling then the sound absorption becomes zero for these two theories, unlike the Gaunaurd theory. The effect of the absorption tail is not insignificant above the resonance frequency, as can be seen in the blue curve of Fig. 3(a).

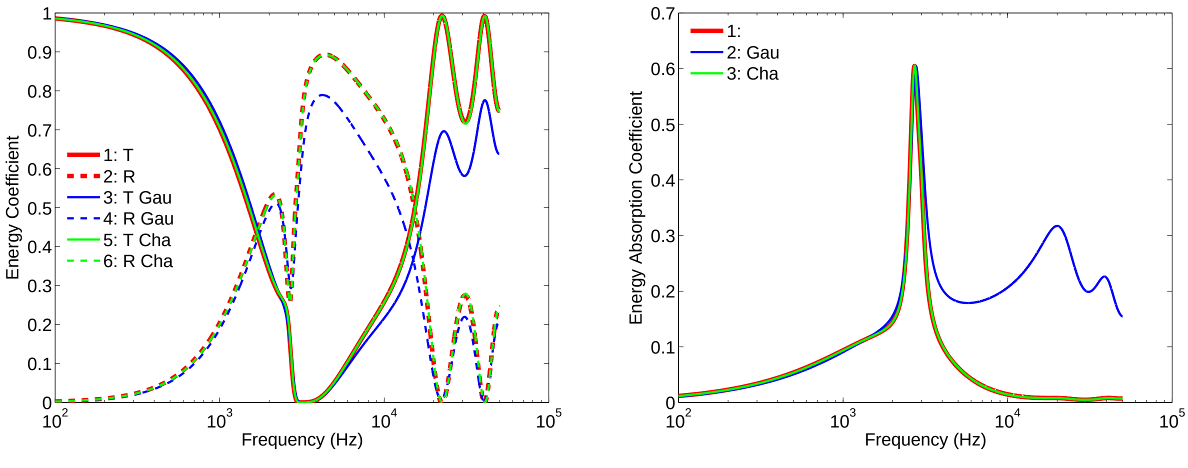


Figure 3 – Comparison of the simplified model, Eqs. (3), (11) and (12) with established theories [21], labelled 'Gau', and [15], 'Cha': (a) sound reflection $R$ and transmission $T$ ; (b) sound absorption $(1-R-T)$.

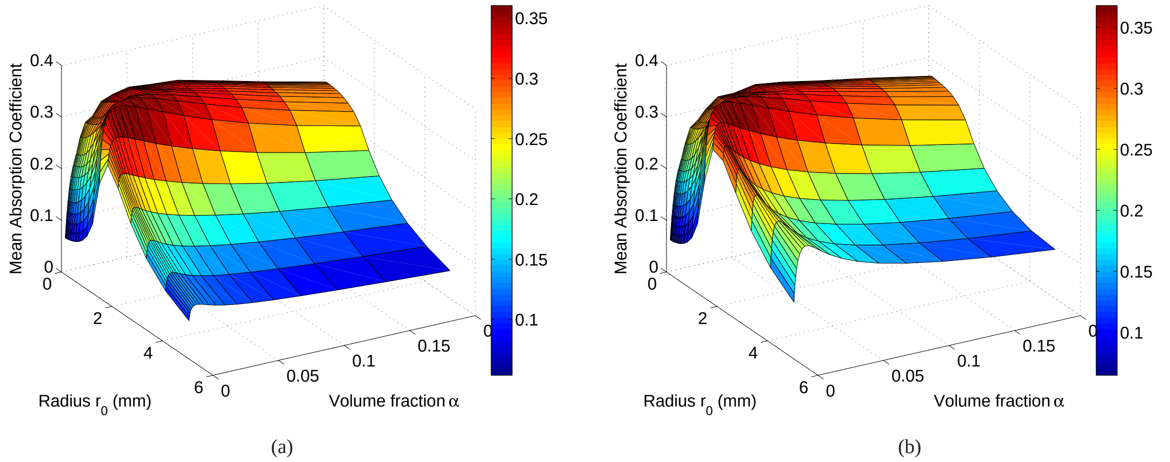


Figure 4 – Mean sound absorption over 1–10 kHz as a function of volume fraction $\alpha$ and radius of inclusion $r_0$: (a) simplified model, Eqs. (3), (11) and (12); (b) established theory [21].

Fig. 4 shows a measure of the effect of varying void radius and volume fraction on the energy absorbed. In this case the measure chosen is the mean energy absorption coefficient over the frequency range from 1–10 kHz. Fig. 4(a) uses the simplified model and Fig. 4(b) uses the theory of Gaunard [21]. Calculations have not been shown for the theory of Chaban [15] as it agrees closely with the simplified model. The advanced models and the simplified model are in good quantitative agreement. A global maximum in sound absorption of 0.36 is observed at $\alpha = 0.01$–$0.02$, $r_0 = 1.5$ mm for both models, although the peak is broad. Different results would be obtained depending on the measure chosen for optimisation.

## 4. CONCLUDING REMARKS

The two-parameter model presented above can enable a significant simplification in modelling of composite materials with resonant inclusions. It formulates the effective elastic properties of the materials in terms of only two aggregated parameters (volume fraction of inclusions and the resonance frequency of an individual inclusion) that realistically describe the elastic properties of the composite if the base material sound speed is known. The specific values of these parameters can be translated to the particular structure of the material in many different ways, such as by changing volume fraction, size, shape and density of inclusions. Importantly, the acoustic properties of the composite should be similar as long as the value of two aggregated parameters remain the same. Furthermore, the resonance frequency of the inclusions can be related to the shear modulus of the base material if it is known. This approach may help to establish a feasible strategy for consistent assessment of candidate materials and insightful evaluation studies, especially when the process of sound–material interaction exhibits complex wave phenomenology (see Fig. 3).

The proposed framework can easily be integrated in any 1D model for the sound wave propagation (see Eq.(2)). This enables straightforward numerical calculations of the effective elastic moduli of elastic structures (laminates) consisting of a number of layers with resonant inclusions. These calculations may provide revealing guidance at the design stage of vibro-elastic panels and reduce the experimental burden associated with exploring their optimal configurations.

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