Transfer-matrix-based approach for an eigenvalue problem of a coupled rectangular cavity

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ABSTRACT

This study concerns an eigenvalue problem of a vibro-acoustic coupling system. In the conventional method, the dynamics of a vibro-acoustic coupling system is calculated with modal coupling methodology which uses eigenfunctions of a rigid-walled cavity. Therefore, particle velocity on the surface of a panel is always calculated as zero, while it should be coincident with the velocity of the panel. To overcome the problem, this paper presents a new methodology for deriving accurate eigen-pairs of a vibro-acoustic coupling system. First, a transfer matrix is introduced which can describe the characteristics of the sound field. This is followed by the derivation of the sound pressure on the coupled rectangular panel. Then, the vibrational velocity of the panel is derived with the sound pressure being used as the input. Furthermore, the eigenvalue problem is formulated based on the equations of vibration and sound field. Finally, the numerical simulation is carried out, demonstrating the validity of the proposed method.

Keywords: Coupling system, Eigen-pairs, Transfer matrix method, I-INCE Classification of Subjects Numbers: 75.6, 75.9

1. INTRODUCTION

Cabin in cars and airplanes is essentially a vibro-acoustic coupling system in which structural vibration and sound field in the cabin are affected each other. For example, disturbance force from an engine excites the cabin structure, and then the structural vibration radiates the noise into the enclosed field. Furthermore, the noise inside the cabin excites the cabin structure, and thus the coupled system is produced. Since recent mechanical systems tend to be lightweight and flexible due to the realization of high energy efficiency, theoretical understanding of the coupling phenomena is important.

Reviewing the past research on the theoretical analysis of a vibro-acoustic coupling system, the mainstream method is based on the modal coupling method. However, this method utilizes the eigenfunctions of a rigid-walled cavity for describing the sound field in the coupled system. Therefore, particle velocity on the surface of a flexible structure is always calculated as zero, while it should be coincident with the velocity of the structure. It is obvious that the modal coupling method has a limitation in describing the whole dynamics of the coupling system. To overcome the problem described above, the authors presented how to derive the eigen-pairs of a coupled rectangular cavity under the matching condition between the particle velocity of the air and the structural velocity, and clarified its validity from a numerical and experimental point of view. However, in this methodology, vibrational velocity of a flexible panel is treated as the boundary condition of the sound field (that is, there is no coupling term in the right-hand side of the wave equation). On the other hand, it is possible to regard the structural vibration as the input to the sound field. Although the aforementioned modal coupling method is based on this concept, this method cannot realize the precise model of a coupling system as pointed out above.

This study presents the alternative approach for deriving the exact eigen-pairs (that is, eigenfunctions and eigenvalules) of a coupling system which treats vibrational velocity of a flexible

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panel as input to the sound field. First, a transfer matrix is introduced which can describe the characteristics of the sound field. This is followed by the derivation of the sound pressure on the coupled rectangular panel. Then, the vibrational velocity of the panel is derived with the sound pressure being used as the input. Furthermore, the eigenvalue problem is formulated based on the equations of vibration and sound field. Finally, the numerical simulation is carried out, demonstrating the validity of the proposed method.

2. FORMURATION OF AN EIGENVALUE PROBLEM

This paper treats the coupled rectangular cavity consisting of one flexible panel and five rigid walls. The schematic diagram of the coupled cavity is illustrated in Fig. 1. Assuming the harmonic vibration at an angular frequency w, the wave equation with respect to the sound pressure p is written as

$$\nabla_a^2 p(x,y,z) + k^2 p(x,y,z) = \mathrm{j}\omega\rho_a v(x,y)\delta(z-L_z)\,, \tag{1}$$

where ∇_a^2 is the Laplacian, k is wave number, δ is the Dirac's delta function, v is the vibration velocity of the panel. The solution to the inhomogeneous of Eq. (1) is written as

$$p(x,y,z) = \sum_{l,m=0}^{\infty} \cos\frac{l\pi}{L_x}x\cos\frac{m\pi}{L_y}y p_{z,lm}(z)\,, \tag{2}$$

where

$$p_{z,lm}(z) = c_{1,lm}\mathrm{e}^{-\mathrm{j}k_{lm}x} + c_{2,lm}\mathrm{e}^{\mathrm{j}k_{lm}x}, \quad k_{lm} = \sqrt{k^2 - \left(\frac{l\pi}{L_x}\right)^2 - \left(\frac{m\pi}{L_y}\right)^2}\,. \tag{3}$$

With the same manner, it is possible to derive the particle velocity for the z direction, and then the state vector of the enclosed sound field is defined as

$$\mathbf{z}_z(x,y,z) = \begin{pmatrix} p(x,y,z) & v_z(x,y,z) \end{pmatrix}^{\mathrm{T}} = \sum_{l,m=0}^{\infty} \cos\frac{l\pi}{L_x}x\cos\frac{m\pi}{L_y}y\mathbf{z}_{z,lm}(z)\,. \tag{4}$$

Here $\mathbf{z}_{z,lm}(z)$ is the state vector of the (l, m) mode group which is given by

$$\mathbf{z}_{z,lm}(z) = \begin{pmatrix} p_{z,lm}(z) \\ v_{z,lm}(z) \end{pmatrix} = \mathbf{K}_{lm}\mathbf{D}_{lm}(z)\mathbf{c}_{lm}\,, \tag{5}$$

where

$$\mathbf{K}_{lm} = \begin{pmatrix} 1 & 1 \\ \dfrac{k_{lm}}{\omega\rho_a} & \dfrac{-k_{lm}}{\omega\rho_a} \end{pmatrix}, \quad \mathbf{D}_{lm}(z) = \begin{pmatrix} \mathrm{e}^{-\mathrm{j}k_{lm}z} & 0 \\ 0 & \mathrm{e}^{\mathrm{j}k_{lm}z} \end{pmatrix}, \quad \mathbf{c}_{lm} = \begin{pmatrix} c_{1,lm} \\ c_{2,lm} \end{pmatrix}. \tag{6}$$

Using the matrices listed above, the transfer matrix of the (l, m) mode group is defined, and the

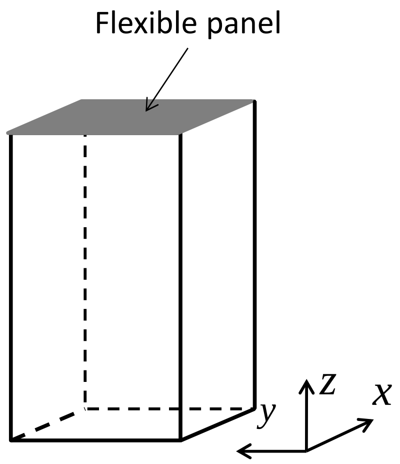

Figure 1 – Schematic diagram of a coupled rectangular cavity

relationship between the state vectors at node i and i-1 is given by

$$ {}_{i}\mathbf{z}_{z,lm} = \mathbf{K}_{lm}\mathbf{D}_{lm}(l_z)\mathbf{K}_{lm}^{-1}\,{}_{i-1}\mathbf{z}_{z,lm} = {}_{i,i-1}\mathbf{T}_{lm}\,{}_{i-1}\mathbf{z}_{z,lm}\,, \tag{7} $$

where ${}_{i}\mathbf{z}_{z,lm}$ and ${}_{i-1}\mathbf{z}_{z,lm}$ are the state vectors at node i and i-1, respectively, and ${}_{i,i-1}\mathbf{T}_{lm}$ is the transfer matrix between the two nodes.

Next, defining the left and right boundaries as nodes 1 and 2, respectively, the state vector of the rigid-walled cavity is written as

$$ \begin{gathered} {}_{1}\mathbf{z}_{z,lm} = {}_{10}\mathbf{T}_{lm}\,{}_{0}\mathbf{z}_{z,lm} + \mathbf{v}_{lm} \\ \begin{pmatrix} {}_{1}p_{z,lm} \\ 0 \end{pmatrix} = \begin{pmatrix} {}_{10}t_{11,lm} & {}_{10}t_{12,lm} \\ {}_{10}t_{21,lm} & {}_{10}t_{22,lm} \end{pmatrix} \begin{pmatrix} {}_{0}p_{z,lm} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_{lm} \end{pmatrix}, \end{gathered} \tag{8} $$

where ${}_{i,i-1}t_{op,lm}$ denotes the oth row and pth column variable in the transfer matrix ${}_{i,i-1}\mathbf{T}_{lm}$. Furthermore, v_{lm} is the effect of the panel vibration to the (l, m) mode group which is written as

$$ v_{lm} = e_{lm}\int_S \cos\frac{l\pi}{L_x}x\cos\frac{m\pi}{L_y}y\,v(x,y)dS\,, \tag{9} $$

where e_{lm} is the constant determined by the modal indices. It should be noted that the second row in the left-hand side of Eq. (8) is zero. This indicates that in the uncoupled state, the boundary condition is a rigid wall. Developing Eq. (8), the sound pressure of the (l, m) mode group at the node 1 is obtained as

$$ {}_{1}p_{z,lm} = \lambda_{lm}v_{lm}\,, \tag{10} $$

where

$$ \lambda_{lm} = -\frac{{}_{10}t_{11,lm}}{{}_{10}t_{21,lm}}\,. \tag{11} $$

Next, a general solution of vibration velocity of a flexible panel is written as

$$ v(x,y) = \sum_{i,j=1}^{\infty} b_{ij}\sin\frac{i\pi}{L_x}x\sin\frac{j\pi}{L_y}x\,, \tag{12} $$

where b_{ij} denotes the modal coefficient of the (i, j) mode of the panel. Substituting the above equation into Eq. (9), we have

$$ \begin{aligned} v_{lm} &= \sum_{i,j=1}^{\infty} b_{ij}e_{lm}\int_S \cos\frac{l\pi}{L_x}x\cos\frac{m\pi}{L_y}y\sin\frac{i\pi}{L_x}x\sin\frac{j\pi}{L_y}y\,dS \\ &= \sum_{i,j=1}^{\infty} b_{ij}e_{lm}\beta_{lm,ij} \qquad\qquad , \\ &= e_{lm}\boldsymbol{\gamma}_{lm}^{\mathrm{T}}\mathbf{b} \end{aligned} \tag{13} $$

where

$$ \boldsymbol{\gamma}_{lm}^{\mathrm{T}} = \begin{pmatrix} \beta_{lm,11} & \beta_{lm,12} & \cdots \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_{11} & b_{12} & \cdots \end{pmatrix}^{\mathrm{T}}\,. \tag{14} $$

Considering all modes, Eq. (10) is rewritten in the vector form as

$$ \begin{aligned} \begin{pmatrix} {}_{1}p_{z,00} \\ {}_{1}p_{z,10} \\ {}_{1}p_{z,01} \\ \vdots \end{pmatrix} &= \begin{pmatrix} \lambda_{00} & & & \mathbf{0} \\ & \lambda_{10} & & \\ & & \lambda_{01} & \\ \mathbf{0} & & & \ddots \end{pmatrix} \begin{pmatrix} v_{00} \\ v_{10} \\ v_{01} \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} \lambda_{00}e_{00} & & & \mathbf{0} \\ & \lambda_{10}e_{10} & & \\ & & \lambda_{01}e_{01} & \\ \mathbf{0} & & & \ddots \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_{00}^{\mathrm{T}} \\ \boldsymbol{\gamma}_{10}^{\mathrm{T}} \\ \boldsymbol{\gamma}_{01}^{\mathrm{T}} \\ \vdots \end{pmatrix} \mathbf{b}\,. \\ \mathbf{p}_z(L_z) &= \boldsymbol{\Lambda}\mathbf{C}\mathbf{b} \end{aligned} \tag{15} $$

Next, consider the modal coefficient of the vibration velocity written in Eq. (12). This coefficient is generally described as

$$b_{ij} = -\frac{\mathrm{j}\omega}{M_s\left(\omega_{ij}^2 - \omega^2\right)} g_{ij}, \tag{16}$$

where M_s is quarter mass of the panel, ω_{ij} is the (i, j) modal angular frequency. Furthermore, f_{ij} and g_{ij} are defined as

$$g_{ij} = \int_S p(x, y, L_z) \sin\frac{i\pi}{L_x} x \sin\frac{j\pi}{L_y} y dS . \tag{17}$$

Then, substituting the above equation into Eq. (2) yields

$$g_{ij} = \boldsymbol{\beta}_{ij}^{\mathrm{T}} \mathbf{p}_z(L_z), \tag{18}$$

where

$$\boldsymbol{\beta}_{ij}^{\mathrm{T}} = \begin{pmatrix} \beta_{00,ij} & \beta_{10,ij} & \beta_{01,ij} & \cdots \end{pmatrix}. \tag{19}$$

Therefore, the modal coefficient of the panel is described in the vector form as

$$\begin{pmatrix} b_{11} \\ b_{12} \\ \vdots \end{pmatrix} = -\begin{pmatrix} \frac{\mathrm{j}\omega}{M_s\left(\omega_{11}^2 - \omega^2\right)} & & \mathbf{0} \\ & \frac{\mathrm{j}\omega}{M_s\left(\omega_{12}^2 - \omega^2\right)} & \\ \mathbf{0} & & \ddots \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{12} \\ \vdots \end{pmatrix}. \tag{20}$$

$$\mathbf{b} = -\mathbf{B}\mathbf{C}^{\mathrm{T}} \mathbf{p}_z(L_z)$$

From Eqs. (15) and (19), an eigenvalue problem of the coupled cavity is formulated as

$$\bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{0}, \tag{21}$$

where

$$\bar{\mathbf{A}} = \begin{pmatrix} \mathbf{I} & -\boldsymbol{\Lambda}\mathbf{C} \\ \mathbf{B}\mathbf{C}^{\mathrm{T}} & \mathbf{I} \end{pmatrix}, \quad \bar{\mathbf{x}} = \begin{pmatrix} \mathbf{p}_z(L_z) \\ \mathbf{b} \end{pmatrix}. \tag{22}$$

Therefore, the eigenfrequencies of the coupled cavity are calculated as non-trivial solutions of Eq.(21), and the corresponding eigenfunctions are obtained from the eigenvectors that are also calculated from Eq.(21).

3. NUMERICAL SIMULATION

Table 1 shows the physical parameters of the coupled rectangular cavity used in this paper. Based on those parameters, the modal frequencies before and after coupling are calculated, being listed in table 2. It should be noted that there are no notations of modal indices for the modal frequencies of the coupled cavity. This is because it is impossible to express the coupled modes as (l, m, n) or (i, j) modes. Before coupling, there are two acoustical modes in the first nine modes. Those modes have the same coupling characteristics, that is, those are coupled only with odd-odd vibration modes. Thus, the (1, 1) mode of the panel is coupled with the (0, 0, 0) mode of the cavity, and the modal frequency is shifted from 73.2 Hz to 76.2 Hz. The reason this shift is for increment is because the (0, 0, 0) mode acts as an air spring. In contrast, the (1, 2) modal frequency of the panel is slightly

Table 1 – Physical parameters of the coupled rectangular cavity

Dimensions	Air density	Speed of sound	Youg's modulus
0.18 m x 0.38 m x 0.58 m	1.21 kg/m^3	340 m/s	2.06x10^9 Pa
Panel density	Poisson's ratio	Thicness of the panel	
7900 kg/m^3	0.29	0.008 m	

Table 2 – Eigenpairs of the coupled rectangular cavity before and after coupling

	in vacuo		coupled
Mode number	Modal indices (l, m, n) or (i, j)	Frequency Hz	Frequency Hz
1	(0, 0, 0)	0	76.2
2	(1, 1)	73.2	112.2
3	(1, 2)	113.4	179.6
4	(1, 3)	180.4	250.9
5	(2, 1)	252.4	273.4
6	(1, 4)	274.2	291
7	(2, 2)	292.6	295.8
8	(0, 0, 1)	293.1	358.3
9	(2, 3)	359.6	393.6

reduced from 113.4 Hz to 112.2 Hz. This vibration mode is mainly coupled with the (0, 1, 0) mode of the cavity at 447.4 Hz which is not listed in table 2. In this case, the modal frequency before coupling, 112.2 Hz, is lower than the cut-on frequency of the (0, 1) mode groups of the cavity, and hence the coupled mode becomes a evanescent mode.

Next, the vibrational velocity distribution is compared with the particle velocity distribution. Figure 2 shows the normalized mode shape of vibrational velocity and particle velocity on the panel at the first modal frequency. As shown in the figure, the both distributions are fairly coincident. Therefore, it can be concluded that the proposed method overcome the problem of the conventional modal coupling method. Figure 3 shows the normalized mode shape of vibrational velocity and particle velocity on the panel at the second modal frequency. In this case, the particle velocity along the perimeters of the panel is not close to zero, as compared to the case of the first mode. This result

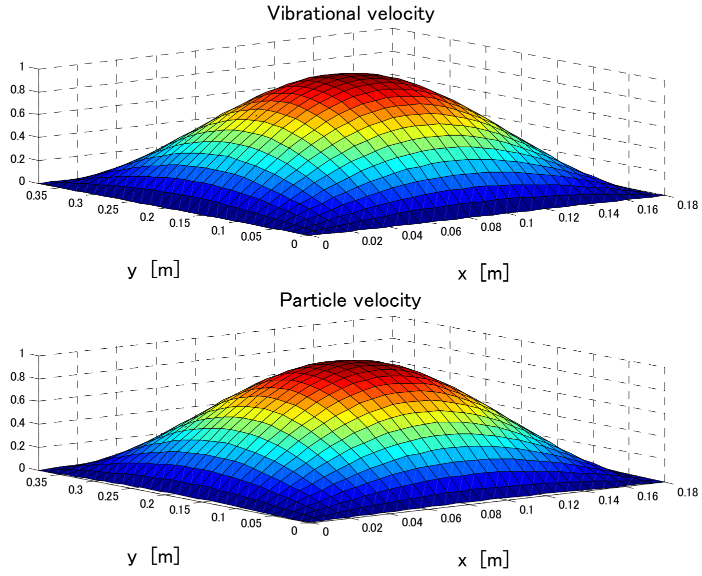

Figure 2 – Normalized mode shape of vibrational velocity and particle velocity on the panel at the first modal frequency

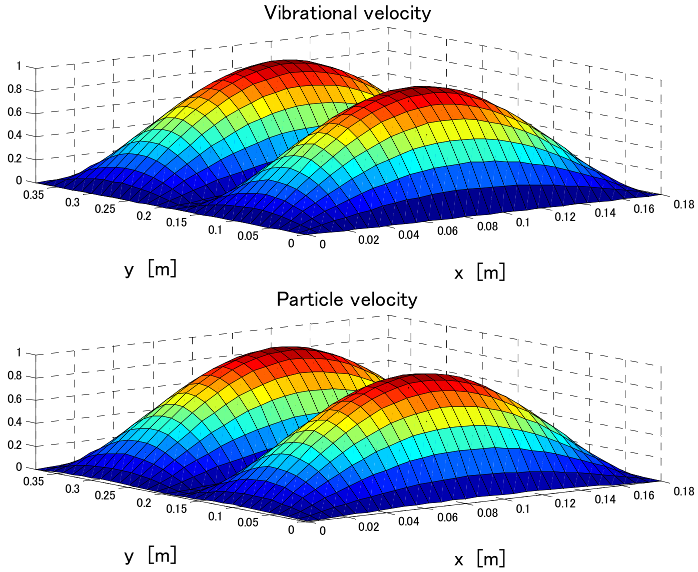

Figure 3 – Normalized mode shape of vibrational velocity and particle velocity on the panel at the second modal frequency

indicates that the number of the cavity mode groups is not sufficient for describing the sound field around the flexible panel, that is, a number of the cavity mode groups are required in order that vibration distribution described by sine functions is coincident with particle velocity distribution described by cosine functions.

4. CONCLUSION

This paper has discussed the alternative approach for deriving the eigenparis of a coupled rectangular cavity. First, a transfer matrix method for an enclosed sound field is introduced, which can treats vibration velocity of a flexible panel as input to a cavity. This is followed by the derivation of sound pressure on a flexible panel. Next, vibration velocity distribution of a flexible panel is derived by regarding sound pressure as input to the panel. Furthermore, using the two equations of the sound pressure and vibration velocity, an eigenvalue problem of a couple cavity is formulated. Finally, numerical simulations are carried out, demonstrating that the proposed method overcomes the problem of the conventional modal coupling method.

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