# Free vibration analysis of elastically connected multiple-beams with general boundary conditions using improved Fourier series method

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## ABSTRACT

Lots of dynamic systems in science and engineering can be simplified as the elastically connected multiple-beams, such as carbon nano tubes and pipes. However, in the majority of current study, just translational spring is considered on beam coupling interfaces. As we know that, not only the translation but also the rotational deformation will occur during beam bending problem. In this work, general beam vibration model is established, in which both two types of translational and rotational springs are taken into account for the coupling interface and both ends. An improved Fourier series is employed for the vibration displacement description for each beam, in which the additional terms are introduced to meet the continuity requirement when the constructed expression is used as the admissible functions. Modal parameters can be determined through the application of Rayleigh-Ritz procedure to the system energy formulation. The current model is then validated through the comparison with those from other analytical approach in open literature. The results show that the current model can made efficient and accurate prediction on the modal property of such complex beam system. The effect of rotational spring on the modal characteristics is also emphasized.

Keywords: Elastically Connected Multiple-Beams, General Boundary Conditions, Improved Fourier Series Method    I-INCE Classification of Subjects Number(s): 42

## 1. INTRODUCTION

Fundamental one- and two-dimensional simple continuous systems, such as a string, beam, membrane, plate and shell are usually used for modeling real mechanical structure. Some interesting and technically important complex continuous systems can be obtained by connecting these simple systems by constraining springs of different types (1). Elastically restrained multiple-beam system has attracted a lot of research interest from the structural dynamics community.

Seelig and Hoppmann (2) presented the development and solution of the differential equations of motion of a system of *n* elastically connected parallel beams, in which the particular case of a two-beam system was analyzed in detail. Vibration experiments were also performed to validate the theoretical formulation. Kessel (3) obtained the resonance conditions for an elastically connected double-beam system in which one of the members is subjected to a moving point load that oscillates longitudinally along the beam about a fixed point. On the basis of Timoshenko beam theory, Rao (4) solved the differential equation governing the flexural vibration of systems of elastically connected parallel bars with the effects of rotary inertia and shear deformation considered. The natural frequencies and mode shapes of particular three- and two-beam systems are obtained. Hv *etal* (5) presented an exact model for the vibration of a double-beam system subject to harmonic excitation, in which a distributed spring *k* and dashpot *c* in parallel between the two beams are introduced to represent a simplified model of viscoelastic material. Oniszczuk (6) conducted the free vibration analysis of two parallel simply supported beams continuously joined by a Winkler elastic layer, namely the uniform translational coupling spring. Li and Hua (7) used the dynamic stiffness method to analyze the free vibration characteristics of a three-beam system with the coupling springs and dashpots on the common interface. Kelly and Srinivas (8) considered the modal analysis of a set of elastically connected axially loaded Euler-Bernoulli beams. Rosa and Lippiello (9) established prediction model for the free vibrations of parallel double-beams joined by a Winkler-type homogeneous elastic foundation, in which the differential quadrature method (DQM) is used for solving two partial differential equations system.

From the above literature, it can be seen that much work has been done for the vibration analysis of elastically connected multiple-beams. However, in the existing studies, just the translational restraining spring is considered between the coupling interface. From the engineering standpoint, there are two degree freedom at each field point of interface, namely translation and rotation. Motivated by the such gap in the current research, the full coupling with translational and rotational springs will be taken into account.

In this work, free vibration analysis of elastically connected beam system is performed, on the coupling interface, two types of coupling spring are introduced to simulate the mechanical interaction between beam components. On each end of beam structure, both translational and rotational supporting springs are considered, then arbitrary boundary condition of each beam can be easily obtained. Energy principle instead of differential equations is employed for the dynamic description of such coupled structure. In order to ensure the convergence and accuracy of the results from subsequent Rayleigh-Ritz procedure, an improved Fourier series method (12) is chosen for the construction of admissible functions, in which the supplementary trigonometric functions are introduced to the standard Fourier cosine series to remove all the spatial differentiation discontinuities in the whole solving interval. Then, numerical results are given to demonstrate the effectiveness of the current model.

## 2. MATHEMATICAL FORMULATIONS

### 2.1 Model illustration

Consider an elastically connected multiple-beam system, as illustrated in Fig. 1. Several beam member with arbitrary boundary conditions are coupled with other through the elastic restraints on the coupling interfaces. Such mechanical interaction is represented using two types of restraining springs, namely translational and rotational springs. The coordinate systems used in the analysis are also shown, with the equal beam length as $L$. Any boundary condition can be easily obtained by setting the relevant restraining spring stiffnesses. For the beam vibration, there are two freedom degrees at each field point, a full coupling restraints should include the translational and rotational ones. The familiar Winkler type of elastic interface can be readily derived by setting the coefficients of rotational springs into zero.

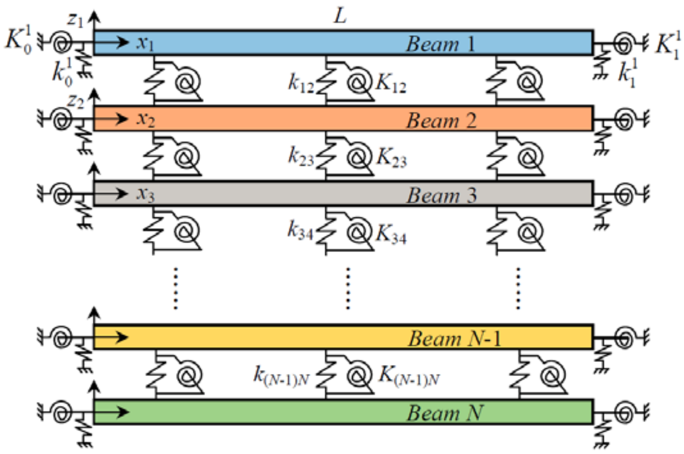



Figure 1 Elastically connected multiple-beams with arbitrary boundary conditions

### 2.2 Dynamic behavior description and solution

For the coupled beam structure as shown in Fig.1, its dynamic behavior is governed by the partial differential equation as well as the boundary and coupling conditions. The solution found in this way is called the *strong* form. On the other hand, the system can be also described from the viewpoint of energy, with its corresponding solution named as the *weak* form. When the admissible functions are constructed smooth enough, these two solutions are equivalent. Here, the energy formulation is used for the description of system dynamic behavior. The Lagrangian for such coupling beam system can be written as

$$L = V - T = \sum_{i=1}^{N} V_i + \sum_{i=1}^{N-1} V_{coupling}^{i,i+1} - \sum_{i=1}^{N} T_i \tag{1}$$

in which, $L$ is the Lagrangian, $V$ and $T$ are the total potential energy and kinetic energy; $V_i$ is the potential energy associated with the $i^{th}$ beam member, $V_{coupling}^{i,i+1}$ is the potential energy stored in interface coupling springs between the $i^{th}$ and $i+1^{th}$ beam member, $T_i$ is the kinetic energy due to the $i^{th}$ vibrating beam.

For the $i^{th}$ beam member with elastically restrained edges, the potential energy $V_i$ is

$$\begin{aligned} V_i = &\frac{1}{2} E_i I_i \int_0^L \left[ \frac{\partial^2 w_i(x)}{\partial x^2} \right]^2 dx \\ &+ \frac{1}{2} k_0^i w_i^2(x)\Big|_{x=0} + \frac{1}{2} K_0^i \left[ \frac{\partial w_i(x)}{\partial x} \right]^2 \Bigg|_{x=0} \\ &+ \frac{1}{2} k_1^i w_i^2(x)\Big|_{x=L} + \frac{1}{2} K_1^i \left[ \frac{\partial w_i(x)}{\partial x} \right]^2 \Bigg|_{x=L} \end{aligned} \tag{2}$$

where $w(x)$ is the transverse vibration displacement field function, $k_0$ and $K_0$ are respectively the stiffness coefficients for the translational and rotational springs at the end $x$=0, and similar meaning can be deduced for the right end of $x$=$L$. The subscript $i$ means that this variable is associated with the $i^{th}$ beam member.

The total kinetic energy of the $i^{th}$ beam structure is

$$T_i = \frac{1}{2} \rho_i S_i \int_0^L \left[ \frac{\partial w_i(x,t)}{\partial t} \right]^2 dx = \frac{1}{2} \omega^2 \rho_i S_i \int_0^L w_i^2(x) dx \tag{3}$$

here, $\omega$ is the radian frequency, $\rho_i$ and $S_i$ are respectively the mass density and cross section area of the $i^{th}$ beam member.

The coupling potential energy between the beam interface can be written as

$$V_{coupling}^{i,i+1} = \frac{1}{2} \int_0^L \left[ k_c^{i,i+1} \left( w_i - w_{i+1} \right)^2 + K_c^{i,i+1} \left( \frac{\partial w_i}{\partial x} - \frac{\partial w_{i+1}}{\partial x} \right)^2 \right] dx \tag{4}$$

in which, $k_c^{i,i+1}$ and $K_c^{i,i+1}$ are respectively the translational and rotational coupling spring stiffness between the $i^{th}$ and $i+1^{th}$ beam member.

Once the system Lagrangian is obtained, the other thing is to construct the appropriate admissible function. The continuity of the assumed functions has significant effect on the final convergence and accuracy. Here, the improved Fourier series method is employed for this purpose, in which the additional functions are introduced to the standard Fourier series to remove all the discontinuities associated with the spatial differentiation of the displacement field functions. For each beam structure, its flexural vibrating displacement function is expanded as

$$w(x) = \sum_{n=0}^{\infty} a_n \cos(\frac{n\pi}{L} x) + b_1 \zeta_1(x) + b_2 \zeta_2(x) + b_3 \zeta_3(x) + b_4 \zeta_4(x) \tag{5}$$

in which,

$$\zeta_1(x) = \frac{9L}{4\pi} \sin\left(\frac{\pi x}{2L}\right) - \frac{L}{12\pi} \sin\left(\frac{3\pi x}{2L}\right), \tag{6}$$

$$\zeta_2(x) = -\frac{9L}{4\pi} \cos\left(\frac{\pi x}{2L}\right) - \frac{L}{12\pi} \cos\left(\frac{3\pi x}{2L}\right), \tag{7}$$

$$\zeta_3(x) = \frac{L^3}{\pi^3} \sin\left(\frac{\pi x}{2L}\right) - \frac{L^3}{3\pi^3} \sin\left(\frac{3\pi x}{2L}\right), \tag{8}$$

$$\zeta_4(x) = -\frac{L^3}{\pi^3} \cos\left(\frac{\pi x}{2L}\right) - \frac{L^3}{3\pi^3} \cos\left(\frac{3\pi x}{2L}\right). \tag{9}$$

It can be easily proven that the current constructed trigonometric function can satisfy the displacement and its higher order differentiation continuity requirement in the interval [0, $L$]. It should be pointed out that the choice of the supplementary functions is not unique, while the appropriate form will be helpful for simplifying the subsequent mathematical formulations.

Substituting the admissible function Eq.(5) into the elastically connected multiple-beam system Lagrangian Eq. (1-4), minimizing it with respect to all the unknown Fourier series coefficients and truncating the Fourier series into finite number $n$=$N$, one will obtain the system characteristic equation in matrix form as follows:

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\mathbf{A}=\mathbf{0} \tag{10}$$

where **K** and **M** are the stiffness and mass matrix for the elastically connected multiple-beam system, **A** is the unknown Fourier series coefficient vector. Through solving this standard system eigenvalue problem, the associated natural frequency and mode shapes can be easily obtained.

## 3. NUMERICAL EXAMPLES AND DISCUSSIONS

In this section, several numerical examples will be given to demonstrate the effectiveness of the current elastically connected multiple beams with arbitrary boundary conditions. As pointed out in the previous section, all the classical boundary condition, such as simply supported, free and clamped, can be easily obtained by setting the relevant spring stiffness.

The first example is about a double-beam system, which has been analyzed by other approaches in the literature. The model parameter is kept the same as those in the Ref. (11), namely, $E_1I_1$=4×10$^6$$Nm^2$, $E_2I_2$=2× $E_1I_1$, $\rho_1A_1$=100kg/m, $\rho_2A_2$=2×$\rho_1A_1$, $L$=10$m$, $k$=1×10$^5$$N/m^2$. From the comparison tabulated in Table 1, it can be observed that the current results can agree well with those from other approaches.

Table 1 – The first six natural frequencies $\omega_n$ for the double-beams with different boundary conditions ($k$=1×10$^5$N/m$^2$)

| Boundary conditions | | Natural frequencies $\omega_n$ | | | | | |
|---|---|---|---|---|---|---|---|
| Beam 1 | Beam 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| S-S | S-S | 19.7392 | 43.4699 | 78.9564 | 87.9439 | 177.6508 | 181.8239 |
| | | 19.7392[a] | 43.4699 | 78.9568 | 87.9442 | 177.6529 | 181.8256 |
| C-C | C-C | 44.7451 | 59.1790 | 123.3403 | 129.2791 | 241.7925 | 244.8775 |
| | | 44.7466[a] | 59.1799 | 123.3456 | 129.2832 | 241.8068 | 244.8888 |
| C-F | C-F | 7.0320 | 39.3630 | 44.0683 | 58.6688 | 123.3918 | 129.3277 |
| | | 7.0320[a] | 39.3630 | 44.0690 | 58.6692 | 123.3944 | 129.3297 |
| S-S | C-F | 17.42560 | 37.1969 | 51.5913 | 84.9902 | 125.5455 | 180.4213 |
| | | 18.2717[a] | 34.3952 | 48.6778 | 100.4167 | 106.7741 | 208.6526 |
| | | 20.7822[b] | 40.4643 | 56.9722 | 84.8796 | 127.3664 | 179.9064 |
| S-S | C-C | 32.4592 | 53.2683 | 84.7241 | 125.5714 | 180.3970 | 242.8573 |
| | | 30.8364[a] | 49.5064 | 99.9297 | 107.1725 | 208.4953 | 212.0620 |
| | | 30.3365[b] | 58.8628 | 84.3266 | 127.3916 | 179.8562 | 242.7132 |
| C-C | C-F | 21.6177 | 46.0563 | 58.2705 | 123.7272 | 128.9877 | 241.9219 |
| | | 21.6179[a] | 46.0571 | 58.2712 | 123.7303 | 128.9907 | 241.9303 |

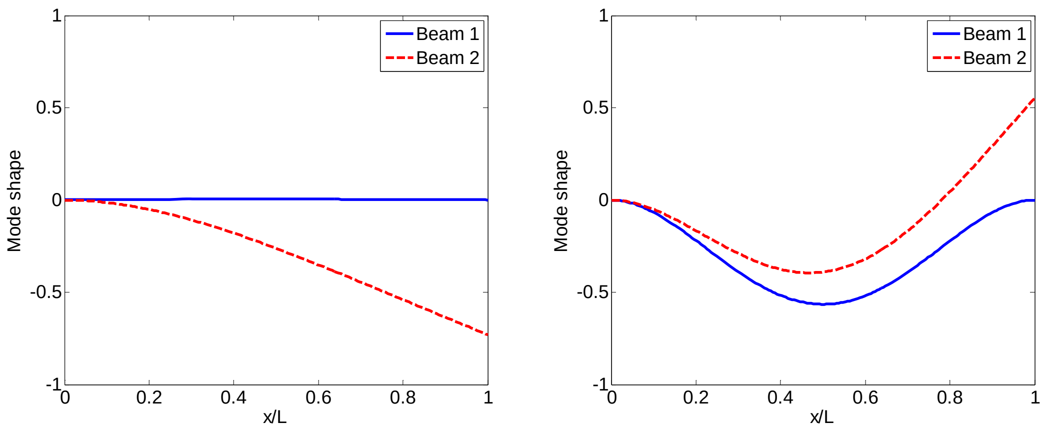
[a]Results from Ref. (11).
[b]Results from FEM with 500 elements.

Table 2 – The first six natural frequencies ωn for the double-beams with different boundary conditions ($K$=1×10$^5$N/m$^2$)

| Boundary conditions | | Natural frequencies $\omega_n$ | | | | | |
|---|---|---|---|---|---|---|---|
| Beam 1 | Beam 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| S-S | S-S | 19.7392 | 23.1879 | 78.9564 | 82.6214 | 177.6508 | 181.3623 |
| | | 19.7298[a] | 24.2336 | 78.8479 | 83.7046 | 177.1544 | 182.0741 |
| C-C | C-C | 44.7451 | 46.7575 | 123.3403 | 126.1081 | 241.7925 | 244.8432 |
| | | 44.6841[a] | 47.3381 | 122.9808 | 126.6313 | 240.5769 | 244.5856 |
| C-F | C-F | 7.0319 | 10.6573 | 44.0683 | 49.2300 | 123.3918 | 128.0019 |
| | | 7.0296[a] | 11.5366 | 44.0181 | 50.7681 | 123.0876 | 129.1886 |
| S-S | C-F | 8.4523 | 21.9219 | 45.9457 | 81.3350 | 124.9995 | 180.0818 |
| | | 9.6120[a] | 21.7731 | 47.6781 | 81.1536 | 126.2732 | 179.5295 |
| S-S | C-C | 22.0301 | 45.4556 | 81.3663 | 124.3000 | 180.0971 | 242.8361 |
| | | 21.9660[a] | 46.0872 | 81.2165 | 124.8783 | 179.5546 | 242.6378 |
| C-C | C-F | 8.4784 | 44.6111 | 47.3107 | 123.5349 | 126.5879 | 241.9258 |
| | | 9.5610[a] | 44.7382 | 48.7537 | 123.2447 | 127.7120 | 240.8219 |

[a]Results from FEM with 500 elements.

Listed in Table 2 are the natural frequencies of two-beams with different boundary condition, in which the translational coupling spring is removed, and the rotational coupling spring is applied on the interface with the same stiffness value as $K$=1×10$^5$N/m$^2$. Since there is little data that can be found in literature for comparison, the finite element analysis with 500 elements is also used to obtain relevant data for validation. Again, it can be found that the comparison is satisfactory. From these two Tables, it can be also observed that for different boundary conditions, the trend is different. For the symmetric end condition, both two coupling springs lead to the similar results, such S-S and C-C, while for other cases, the two coupling stiffness yields different natural frequencies. This also implies that for the elastically connected beam system, the coupling type on the interface should be specified. The corresponding mode shapes for the double-beam system with C-C and C-F boundary conditions under such rotational interface restraints are also plotted in Fig. 2.




(a) (b)

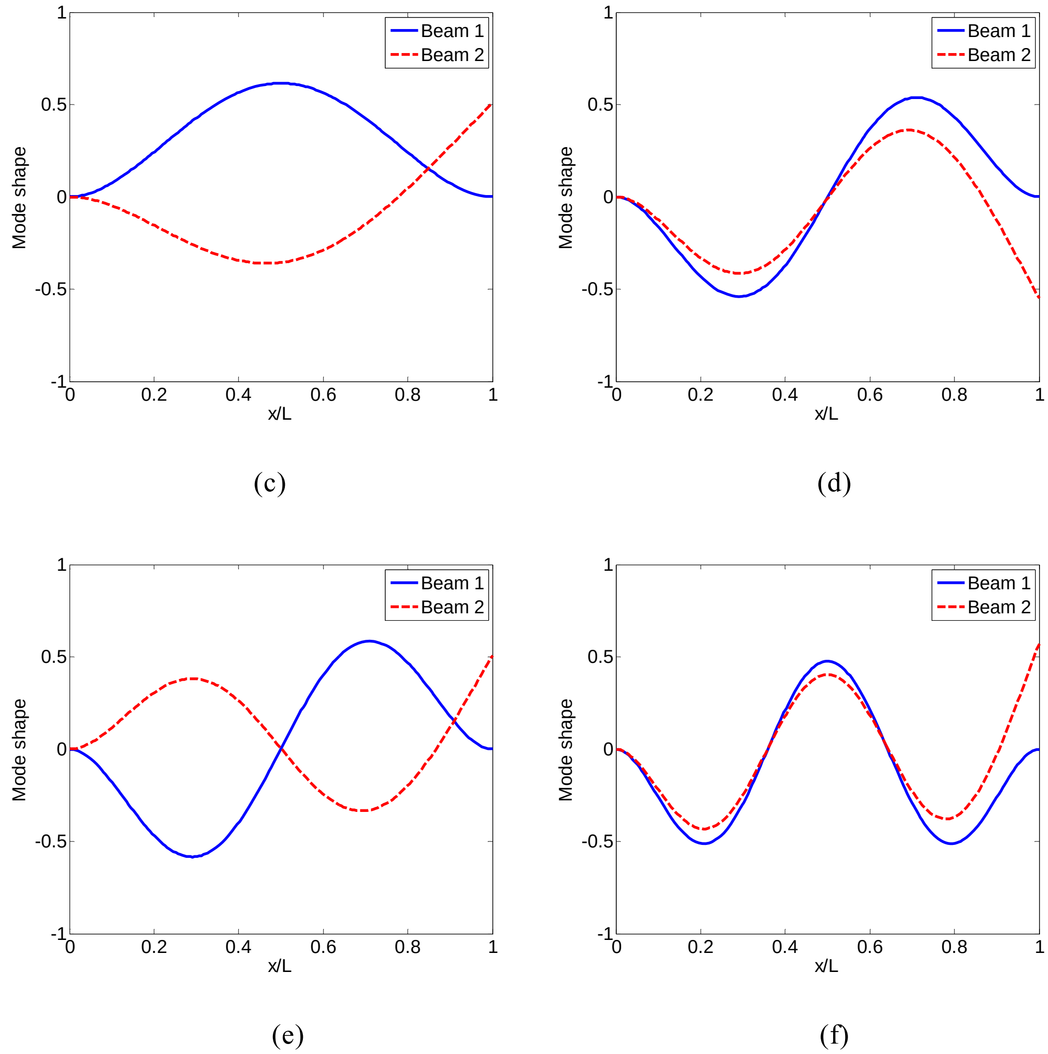



Figure. 2 Mode shapes of C-C and C-F double beam system, with rotational elastic restraint as $K=1\times10^5 N/m^2$

The second example is about a three-beams coupling system, with the parameters as: $E_1I_1=E_2I_2=E_3I_3=4\times10^6$ Nm$^2$, $\rho_1A_1=\rho_2A_2=\rho_3A_3=100$ kg/m, $k_1=k_2=1\times10^5$N/m$^2$, $L=10$. In order to compare the results with those given in Ref. (11), the above parameter is used, and the rotational restraining stiffness is set into zero. From the Table 3, it can be seen that the agreement is very good for this three-beams with the Winkler elastic interface.

Table 3 – The first six natural frequencies $\omega_n$ for the three-beams with various classical boundary conditions ($k_{12}= k_{23}=1\times10^5 N/m^2$, $K_{12}=K_{23}=0$)

| Boundary conditions | | | Natural frequencies $\omega_n$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| Beam 1 | Beam 2 | Beam 3 | 1 | 2 | 3 | 4 | 5 | 6 |
| S-S | S-S | S-S | 19.7392 | 37.2778 | 58.2206 | 78.9566 | 85.0538 | 96.0944 |
| | | | 19.7392[a] | 37.2778 | 58.2206 | 78.9568 | 85.0540 | 96.0947 |
| S-S | C-C | S-S | 28.0017 | 37.2778 | 67.8610 | 83.9412 | 85.0538 | 131.9086 |
| C-C | C-C | C-C | 44.7457 | 54.7921 | 70.7261 | 123.3425 | 127.3317 | 134.9569 |
| | | | 44.7466[a] | 54.7928 | 70.7266 | 123.3456 | 127.3348 | 134.9598 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| C-C | C-C | C-F | 29.7313 | 46.9018 | 54.4353 | 70.2139 | 123.7098 | 127.3594 |
| C-C | S-S | C-F | 25.0730 | 41.6824 | 54.3269 | 63.2773 | 89.9353 | 127.3573 |
| | | | 25.0731[a] | 41.6827 | 54.3274 | 63.2776 | 89.9355 | 127.3595 |

[a]Results from Ref. (11).

Table 4 –Three cases of elastically restrained boundary conditions of three-beam system

| Three cases | Boundary restraining conditions for three-beam system | | | | | |
|---|---|---|---|---|---|---|
| | Beam 1 | | Beam 2 | | Beam 3 | |
| | $x$=0 | $x$=$L$ | $x$=0 | $x$=$L$ | $x$=0 | $x$=$L$ |
| Case 1 | Simply supported | Simply supported | Clamped | $\hat{h}_1=\hat{H}_1=100$ | $\hat{h}_0=\hat{H}_0=50$ | Clamped |
| Case 2 | $\hat{h}_0=\hat{H}_0=100$ | $\hat{h}_1=\hat{H}_1=100$ | $\hat{h}_0=\hat{H}_0=300$ | $\hat{h}_1=\hat{H}_1=100$ | $\hat{h}_0=\hat{H}_0=50$ | Clamped |
| Case 3 | $\hat{h}_0=\hat{H}_0=75$ | $\hat{h}_1=\hat{H}_1=75$ | $\hat{h}_0=\hat{H}_0=150$ | $\hat{h}_1=\hat{H}_1=150$ | $\hat{h}_0=\hat{H}_0=300$ | $\hat{h}_1=\hat{H}_1=300$ |

Now, let us consider more complicated boundary restraining conditions for this three-beams, tabulated in Table 4 are the three cases considered in the subsequent calculation, in which the non-dimensional boundary restraining stiffness is used, with its definition as $\hat{h} = kL^3 / EI$ and $\hat{H} = KL / EI$.

Table 5 – The first six natural frequencies $\omega_n$ for the three-beams with various boundary conditions ($k_{12}$=$k_{23}$=$1\times10^5 N/m^2$, $K_{12}$=$K_{23}$=0)

| Boundary conditions | Natural frequencies $\omega_n$ | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Case1 | 27.9139 | 38.6784 | 56.4838 | 71.8640 | 79.1605 | 87.4107 |
| | 27.9141[a] | 38.6784 | 56.4838 | 71.8644 | 79.1610 | 87.4110 |
| Case 2 | 27.5359 | 38.9941 | 49.2193 | 58.7721 | 70.2931 | 74.7400 |
| | 27.5360[a] | 38.9941 | 49.2193 | 58.7721 | 70.2935 | 74.7401 |
| Case 3 | 27.3149 | 43.3336 | 45.5920 | 61.7903 | 65.2249 | 78.3226 |

[a]Results from Ref. (11).

Table 5 shows the calculated natural frequencies of the three-beams with various boundary conditions, for these complex boundary restraints, the results obtained in Ref. (11) are also presented. The corresponding mode shapes for this three-beams with the boundary conditions of case 3 are plotted in Fig. 3.

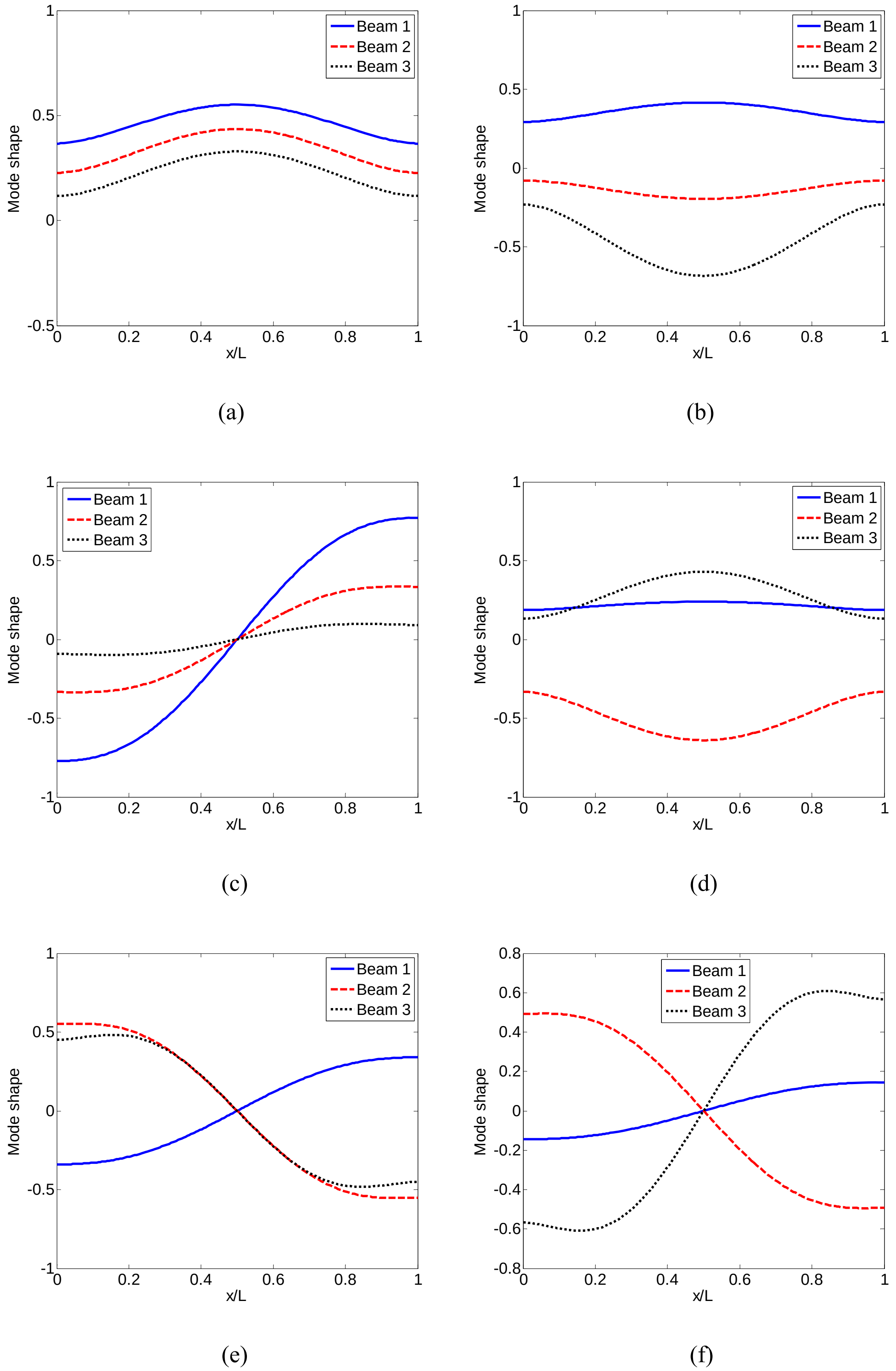
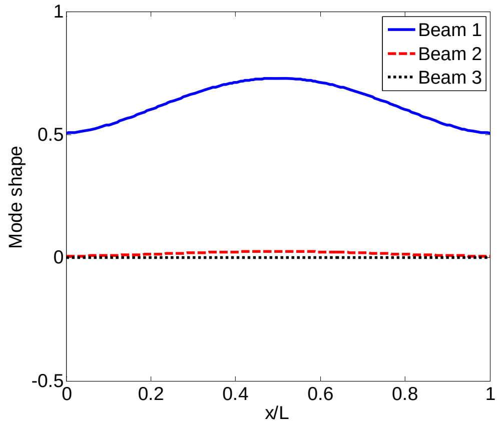


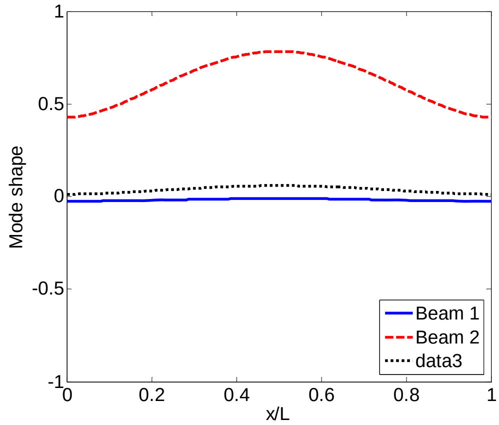

Figure 3 – The first six mode shapes for the three-beams with case 3 boundary conditions ($k_{12}$= $k_{23}$=1×10$^5$$N/m^2$, $K_{12}$=$K_{23}$=0)

Table 6 – The first six natural frequencies $\omega_n$ for the three-beams with various classical boundary conditions ($K_{12}$=$K_{23}$=1×10$^5$$N/m^2$, $k_{12}$= $k_{23}$=0)

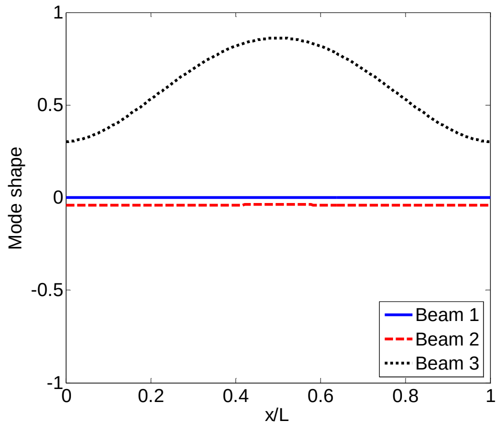
| Boundary conditions | Natural frequencies $\omega_n$ | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Case1 | 21.9587 | 24.2448 | 30.6282 | 64.9618 | 71.2026 | 81.4105 |
| Case 2 | 24.2420 | 24.6446 | 28.5599 | 44.5832 | 56.9182 | 65.0601 |
| Case 3 | 22.0001 | 28.7643 | 34.7959 | 40.3089 | 52.7630 | 66.9391 |



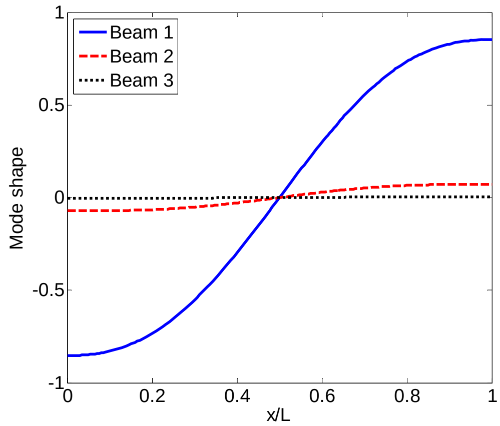

(a)



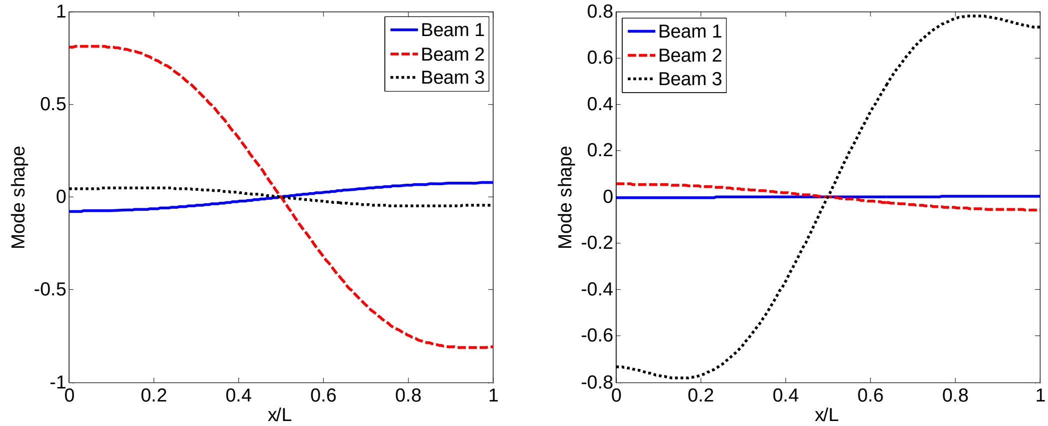

(b)




(c)




(d)




(e) (f)

Figure 4 – The first six mode shapes for the three-beams with case 3 boundary conditions ($K_{12}$=$K_{23}$=$1\times10^5 N/m^2$, $k_{12}$= $k_{23}$=0)

Finally, the translational restraining spring between the interface of three-beams is removed, while the rotational restraining spring of the same value is applied. Tabulated in Table 6 are the first six natural frequencies of such three-beams, the effect of the rotational restraint on the modal characteristics of this coupling system can be observed from the comparison between Table 5 and Table 6. The corresponding mode shapes for the system with the case 3 boundary condition and rotational restraint are also plotted in Figure 4.

## 4. CONCLUSIONS

In this paper, free vibration analysis of the elastically connected beams is performed, two types of restraining springs are introduced at the both ends and the coupling interfaces of the multiple beam systems. The energy principle is employed to describe the system dynamics of the coupling system. In order to ensure the convergence and accuracy of the Fourier series solution through Rayleigh-Ritz procedure, supplementary functions are added to remove the relevant spatial derivative discontinuities in the whole solving interval. All the modal characteristics can be obtained from a standard eigenvalue problem of the final system matrix.

Theoretical formulation is then implemented by writing the simulation codes, the model parameters used in other literature is chosen with the aim of validating the current methodology. When setting the rotational spring stiffness into zero on the common interface, the model will be degenerated into that in the literature. The comparison shows that the these two models agree very well. The effect of inclusion of rotational spring stiffness is also studied, the results show that such type of coupling stiffness has some influence on the modal characteristics of the multiple elastically connected beams. Although two and three coupled beams are solved in this work, this model can be easily extended to analyze the dynamics of elastically connected beams of any number.

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