

# On enhanced sound absorption by non-uniform liners

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# ABSTRACT

The use of acoustic liners is a common means of noise reduction in jet engine exhausts. The quest for more effective sound absorption mechanisms in cylindrical ducts has led to the consideration of non-uniform liners, with impedance varying circumferentially, axially, or in both directions. The present paper is based on the theory of mode coupling in a non-uniformly lined cylindrical duct and considers the complementary problem of generation of sound or excitation of coupled modes by a source distribution. The sound field due to an arbitrary source distribution is obtained as a superposition of eigenfunctions corresponding to complex eigenvalues for the radial wavenumbers and natural frequencies taking into account that the radial, axial and azimuthal modes are coupled by the non-uniform wall impedance, and including resonant and non-resonant cases. The waveforms are illustrated for point monopole, dipole and quadrupole sources and a continuous monopole distribution. It is shown that a non-uniform liner provides a greater attenuation than a uniform liner with the same average impedance if it is 'well-matched' to the sound field, that is, if it has higher impedance at the peaks and lower at the nodes of the standing modes.

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## 1. INTRODUCTION

A non-uniform liner can provide greater sound absorption than an uniform liner with the same amount of material if it is well matched to the sound field, that is (Figure 1a) has larger impedance where the sound amplitude is larger; the reverse would happen for poor matching (Figure 1b). A fundamental difference between a non-uniform and a uniform liner is illustrated in the Figure 2 for an axisymmetric incident mode: (a) it is reflected as an axisymmetric mode by a uniform liner; (b) in contrast a non-uniform liner reflects an axisymmetric mode as a non-axisymmetric wave, and thus generates harmonics. In conclusion a non-uniform liner (1, 2, 3, 4) causes coupling of all modes in a duct; the present theory (5, 6, 7, 8) takes into account the mode interaction in a nozzle with a non-uniform liner.



Figure 1 – A non-uniform liner (b) generates harmonics and causes mode coupling, unlike (a) a uniform liner.

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Figure 2 - A non-uniform liner well-matched (a) to the sound field attenuates more than a uniform liner with the same average impedance. The opposite if it is poorly matched (b).

#### 2. CONVECTED WAVE EQUATION WITH UNIFORM AXIAL FLOW

The forced convected wave equation is written in cylindrical coordinates  $(r, \theta, z)$ :

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2\right]p(r,\varphi,z,t) = S(r,\varphi,z,t),$$
(1)

where *p* is the acoustic pressure, *c* the sound speed, *U* the uniform axial velocity and  $S(r, \varphi, z, t)$  an arbitrary source distribution as a function of position and time. The free acoustic fields are the solution of the homogeneous wave equation (1) without sources S = 0. For a doubly-infinite cylindrical duct may be used a Fourier series with azimuthal wave number *m* and a Fourier integral with continuous frequency  $\omega$ , in addition to: (i) a Fourier integral with continuous axial wavenumber:

$$p(r,\varphi,z,t) = \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \int_{-\infty}^{+\infty} e^{i\kappa z} d\kappa \sum_{m=-\infty}^{+\infty} e^{im\varphi} P_m(r;\omega,\kappa),$$
(2)

in the case of uniform wall impedance or wall impedance varying only circumferentially; (ii) a Fourier series with discrete axial wavenumber  $\kappa_l = 2\pi l/L$ :

$$p(r, \varphi, z, t) = \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} P_{lm}(r; \omega)$$
(3)

in the case of axially non-uniform wall impedance with spatial period L.

The radial dependence of the acoustic pressure is determined by substitution in (1), that leads to a Bessel equation:

$$r^{2}P'' + rP' + \left(k^{2}r^{2} - m^{2}\right)P = 0,$$
(4)

with radial wavenumber determined by:

$$(k_l)^2 \equiv \left(\frac{\omega - 2\pi l U/L}{c}\right)^2 - \left(\frac{2\pi l}{L}\right)^2,\tag{5}$$

in the discrete case (3), and by

$$k^{2} \equiv \left(\frac{\omega - \kappa U}{c}\right)^{2} - \kappa^{2}.$$
 (6)

in the continuous case (2).

The solution that is finite on the cylinder axis is specified by a Bessel function of the first kind both in the case of: (i) discrete axial wavenumber (7a)

$$P_{lm}(r;\boldsymbol{\omega}) = \mathbf{J}_m(k_l r), \tag{7a}$$

$$P_m(r;\kappa,\omega) = \mathbf{J}_m(kr),\tag{7b}$$

(ii) continuous axial wavenumber (7b). The unforced convected wave equation (1) with S = 0 is satisfied by a superposition of: (i) discrete axial modes (3, 5,7a):

$$p(r,\varphi,z,t) = \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} A_{lm}(\omega) J_m(k_l r),$$
(8)

with arbitrary amplitudes  $A_{lm}(\omega)$ ; (ii) continuous axial modes (2, 6,7b):

$$p(r,\varphi,z,t) = \int_{-\infty}^{+\infty} e^{-i\omega t} d\omega \int_{-\infty}^{+\infty} e^{i\kappa z} d\kappa \sum_{m=-\infty}^{+\infty} e^{im\varphi} A_m(\omega,\kappa) J_m(kr),$$
(9)

with arbitrary amplitudes  $A_m(\omega, \kappa)$ . The values of the radial wavenumber in the discrete  $k_l$  (5) and continuous k in (6) axial case are the eigenvalues determined by the boundary condition at the duct wall. The case considered next is the most general locally reacting liner with non-uniform wall impedance varying either circumferentially or axially or both.

#### 3. BOUNDARY CONDITION FOR NON-UNIFORM WALL IMPEDANCE

A locally reacting liner is represented by a linear relation

$$\tilde{p}(a, \boldsymbol{\varphi}, \boldsymbol{z}; \boldsymbol{\omega}) = -Z(\boldsymbol{\varphi}, \boldsymbol{z}; \boldsymbol{\omega}) \,\tilde{v}_r(a, \boldsymbol{\varphi}, \boldsymbol{z}; \boldsymbol{\omega}) \tag{10}$$

between the pressure and minus the radial velocity spectra:

$$p(r, \boldsymbol{\varphi}, z, t) = \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{\omega} \,\mathrm{e}^{-\mathrm{i}\boldsymbol{\omega} t} \,\tilde{p}(r, \boldsymbol{\varphi}, z; \boldsymbol{\omega}). \tag{11a}$$

$$v_r(r, \varphi, z, t) = \int_{-\infty}^{+\infty} \mathrm{d}\omega \,\mathrm{e}^{-\mathrm{i}\omega t} \,\tilde{v}_r(r, \varphi, z; \omega). \tag{11b}$$

The equation (10) applies at the duct walls r = a, where the mean flow vanishes (no-slip condition) for a viscous fluid. When the mean flow can be considered as inviscid, as is often the case in acoustics, its tangential velocity may not vanish at the walls. Then, (10) should be written as

$$\tilde{p}^{w}(a, \boldsymbol{\varphi}, \boldsymbol{z}; \boldsymbol{\omega}) = -Z(\boldsymbol{\varphi}, \boldsymbol{z}; \boldsymbol{\omega}) \,\tilde{v}_{r}^{w}(a, \boldsymbol{\theta}, \boldsymbol{z}; \boldsymbol{\omega}), \tag{12}$$

where  $\tilde{p}^w$  and  $\tilde{v}_r^w$  are the spectra of the acoustic pressure perturbation and radial velocity perturbation at the wall. This condition must be supplemented by the continuity of normal displacement of particles in the fluid and at the wall. The following analysis is similar for continuous (9) or discrete (8) axial wavenumber, and is presented explicitly in the latter case. Assuming that the wall impedquace varies axially with spatial period *L*, both  $\tilde{v}_r$  and  $\tilde{v}_r^w$  can be written as a Fourier series

$$\tilde{v}_r(r, \varphi, z; \omega) = \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} \quad V_{lm}(r; \omega),$$
(13a)

$$\tilde{v}_r^w(r,\varphi,z,t) = \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} \quad V_{lm}^w(r;\omega).$$
(13b)

Continuity of normal displacement leads to (9):

$$V_{lm}^{\omega} = \frac{\omega}{\omega - 2\pi l U/L} V_{lm}.$$
 (14)

because the frequency  $\omega$  has a Doppler shift  $\omega - k_l U$  in the flow.

Bearing in mind that the pressure is continuous, substitution of (13b) and (14) in (12) leads to

$$\tilde{p}(a,\varphi,z;\omega) = -Z(\varphi,z;\omega) \sum_{l=-\infty}^{+\infty} e^{i2\pi l/L} \frac{\omega}{\omega - 2\pi lU/L} V_{lm}(r;\omega).$$
(15)

Comparing (11a) with (8) it follows that the spectrum of the acoustic pressure is given by

$$\tilde{p}(r,\varphi,z;\omega) = \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} A_{lm}(\omega) \mathbf{J}_m(k_l r).$$
(16)

The radial acoustic velocity is related to the pressure by the radial component of the linearized momentum equation:

$$\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)v_r(r,\varphi,z,t) + \frac{\partial p(r,\varphi,z,t)}{\partial r} = 0.$$
(17)

Substituting (13a) and (16) in (17) leads to:

$$i\rho(\omega - 2\pi lU/L)V_{lm}(r;\omega) - k_l A_{lm}(\omega) J'_m(k_l r) = 0.$$
(18)

It remains to relate the amplitudes of the pressure spectrum and the radial velocity spectra in (18).

Substitution of (16) and (18) in (15), with r = a the radius of the duct, results in the impedance boundary condition:

$$\sum_{l=-\infty}^{+\infty} e^{i2\pi lz/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} A_{lm}(\omega) J_m(k_l a) =$$
$$= -Z(\varphi, z; \omega) \sum_{l=-\infty}^{+\infty} e^{i2\pi lz/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} \frac{\omega}{i\rho \left(\omega - 2\pi lU/L\right)^2} k_l A_{lm}(\omega) J'_m(k_l a), \quad (19)$$

in the case of the discrete axial wavenumber (8) applying to an axially non-uniform impedance with spatial period *L*, that may or not also vary circumferentially. If the impedance is axially uniform, regardless of whether it varies circumferentially or not, a similar calculation using (9) leads to:

$$\int_{-\infty}^{+\infty} e^{i\kappa z} d\kappa \sum_{m=-\infty}^{+\infty} e^{im\varphi} A_m(\omega,\kappa) J_m(ka) =$$
  
=  $-Z(\omega;\varphi) \int_{l=-\infty}^{+\infty} e^{i\kappa z} d\kappa \sum_{m=-\infty}^{+\infty} e^{im\varphi} \frac{\omega}{i\rho (\omega - \kappa U)^2} k A_m(\omega,\kappa) J'_m(ka), \quad (20)$ 

as the wall boundary condition. When the wall liner impedance is constant the boundary condition reduces in the case of discrete axial spectrum (19) to

$$0 = k_l Z(\boldsymbol{\omega}) \mathbf{J}'_m(k_l a) + \mathbf{i} \frac{(\boldsymbol{\omega}/c - 2\pi l M/L)^2}{\boldsymbol{\omega}/c} \mathbf{J}_m(k_l a) = B \prod_{n=1}^{\infty} (k_l - k_{lmn}).$$
(21)

(ii) continuous axial spectrum (20) to:

$$0 = kZ(\boldsymbol{\omega})\mathbf{J}'_{m}(ka) + \mathbf{i}\frac{\boldsymbol{\omega}/c - \kappa M}{\boldsymbol{\omega} - c}\mathbf{J}_{m}(ka) = B\prod_{n=1}^{\infty}(k - k_{mn}).$$
(22)

Both in (21) and (22) the non-zero constant  $B \neq 0$  does not affect the roots that depend on the Mach number (23a):

$$M \equiv U/c \tag{23a}$$

$$\bar{Z}(\varphi, z; \omega) \equiv Z(\varphi, z; \omega) / \rho c$$
 (23b)

Z is the specific impedance (23b), defined as the impedance divided by that of a plane wave. In (23b) and subsequently an overbar is used to indicate a dimensionless quantity. The radial wavenumbers are the roots  $k_n$  of (22) for continuous axial spectrum and the roots  $k_{ln}$  of (21) for discrete axial spectrum; in both cases

woth uniform impedance there is a separate boundary condition for each m. Thus a uniform wall impedance does not couple the azimuthal modes. That is not the case for a non-uniform wall impedance as shown by the following 'thought experiment': (i) consider an axysymmetric mode m = 0; (ii) a uniform liner reflects it as another axisymmetric mode m = 0; (iii) a circumferentially non-uniform liner will cause a reflection with amplitude depending on the angle  $\varphi$ ; (iv) the case (iii) is equivalent to the generation of other azimuthal modes  $m \neq 0$ . Thus a non-uniform liner couples the modes in a duct as shown next for impedance varying either circumferentially or axially.

## 4. RADIAL WAVENUMBERS FOR CIRCUMFERENTIALLY OR AXIALLY NON-UNI-FORM LINERS

In the case of an acoustic liner with circumferentially non-uniform impedance  $Z(\varphi; \omega)$  the impedance is represented by a Fourier series:

$$\bar{Z}(\theta,\omega) = \sum_{m'=-\infty}^{+\infty} \bar{Z}_{m'}(\omega) e^{im'\theta}$$
(24)

where the coefficients

$$\bar{Z}_{m'}(\omega) = \frac{1}{2\pi} \int_0^{2\pi} \bar{Z}(\theta; \omega) \mathrm{e}^{-\mathrm{i}m'\theta} \mathrm{d}\theta, \qquad (25)$$

specify the amplitudes of the 'harmonics' of the impedance. Note that the Fourier series representation applies to an impedance distribution that is an arbitrary function of bounded variation (10) of position, e.g., it may have a finite number of finite discontinuities; thus it applies to liner patches and splices.

Since the liner is circumferentially non-uniform (24) but axially uniform the axial wavenumber is continuous (6) and the impedance boundary condition (20) is used:

$$\sum_{m'=-\infty}^{+\infty} \bar{Z}_{m'} e^{im'\varphi} \sum_{m=-\infty}^{+\infty} k J'_m(ka) e^{im\varphi} A_m(\omega,\kappa) + \sum_{m=-\infty}^{+\infty} i \frac{(\omega/c - \kappa M)^2}{\omega/c} J_m(ka) e^{im\varphi} A_m(\omega,\kappa) = 0, \quad (26)$$

that can be rearranged:

$$\sum_{n',m=-\infty}^{+\infty} A_m(\omega,\kappa) \mathrm{e}^{\mathrm{i}m\varphi} \left\{ \mathrm{i}\frac{(\omega/c - \kappa M)^2}{\omega/c} \delta_{m'm} \mathrm{J}_m(ka) - \bar{Z}_{m'-m} k \, \mathrm{J}'_m(ka) \right\} = 0, \tag{27}$$

where  $\delta_{m'm}$  is the identity matrix. Since the amplitudes  $A_m$  cannot be all zero, the determinant of the coefficients must vanish.

$$0 = \det\left\{i\frac{(\omega/c - \kappa M)^2}{\omega/c}\delta_{m'm}J_m(ka) - \bar{Z}_{m'-m}kJ'_m(ka)\right\} = B\prod_{n=1}^{\infty}(k - k_{mn}).$$
(28)

The roots of (28) are radial wavenumbers  $k_{mn}$ , and they are generally coupled between different *m*. In the case of uniform impedance  $\overline{Z} = \overline{Z}_0 = \text{const}$  and  $\overline{Z}_m = 0$  for all  $m \neq 0$ . the determinant (28) is diagonal, and vanishes when a single diagonal term is zero, leading to (22). If the wall impedance varies axially with spatial period *L*, the specific impedance may again be represented by a Fourier series:

$$\bar{Z}(z;\omega) = \sum_{l'=-\infty}^{+\infty} \bar{Z}^{l'}(\omega) \,\mathrm{e}^{\mathrm{i}2\pi l' z/L},\tag{29}$$

with coefficients

$$\bar{Z}^{l'}(\boldsymbol{\omega}) = \frac{1}{L} \int_0^L \bar{Z}(z; \boldsymbol{\omega}) e^{-i2\pi l' z/L} dz.$$
(30)

In this case of discrete axial wavenumbers (5) the specific impedance (29) is substituted in the boundary condition (19) leading to

$$\sum_{m=-\infty}^{+\infty} e^{im\varphi} \left[ \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} A_{lm}(\omega) J_m(k_l a) - \sum_{l'=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} e^{i2\pi (l+l')z/L} \bar{Z}^{l'} \frac{\omega/c}{i\left(\frac{\omega}{c} - \frac{2\pi l}{L}M\right)^2} k_l A_{lm}(\omega) J_m'(k_l a) \right] = 0.$$
(31)

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The terms in square brackets, which are the coefficients of the sum in *m*, must vanish. They may be rearranged:

$$\sum_{l=-\infty}^{+\infty} e^{i2\pi lz/L} \left\{ \sum_{l'=-\infty}^{+\infty} \left[ A_{lm}(\omega) J_m(k_l a) - \frac{Z^{l-l'} \omega/c}{i\left(\frac{\omega}{c} - \frac{2\pi l'}{L}M\right)^2} k_l A_{l'm}(\omega) J_m'(k_l a) \right] \right\} = 0.$$
(32)

The terms in curly brackets must vanish, which leads to

$$\sum_{l'=-\infty}^{+\infty} A_{l'm}(\boldsymbol{\omega}) \left[ \delta_{l'l} \mathbf{J}_m(k_l a) - \bar{Z}^{l-l'} \frac{\boldsymbol{\omega}/c}{\mathbf{i} \left(\frac{\boldsymbol{\omega}}{c} - \frac{2\pi l'}{L} M\right)^2} k_l \mathbf{J}'_m(k_l a) \right] = 0.$$
(33)

This represents a set of homogeneous equations (one for each value of l and m) for the coefficients  $A_{lm}$ . For fixed m, and since not all the  $A_{lm}$  can vanish,

$$0 = \det\left\{\delta_{ll'} \mathbf{J}_m(k_l a) - Z^{l-l'} \frac{\omega/c}{i\left(\frac{\omega}{c} - \frac{2\pi l'}{L}M\right)^2} k_l \mathbf{J}_m'(k_l a)\right\} = B \prod_{n=1}^{\infty} (k_l - k_{lmn}).$$
(34)

The radial wavenumbers are the roots of the determinant (34), and thus generally coupled for different *l*. In the case of a uniform liner  $\overline{Z} = \overline{Z}^0 = \text{const}$  and  $\overline{Z}^l = 0$  for  $l \neq 0$ , the determinant (34) is diagonal, and vanishes if one term is zero, leading to the boundary condition (21) with decoupled *l*.

The case of a liner non-uniform both axially and radially can be treated as a combination of the preceding; it leads to a doubly infinite determinant, *i.e.* an infinite determinant whose elements are infinite determinants (6). In all cases of axially non-uniform liners, since the radial wavenumbers that are the roots of (34) are generally complex the corresponding frequencies (5):

$$\omega_{lmn} = \frac{2\pi l U}{L} \pm c \sqrt{\left(\frac{2\pi l}{L}\right)^2 + (k_{lmn})^2}$$
(35)

are also complex; the complex natural frequencies imply that

$$\exp(-i\omega_{lmn}t) = \exp[-it\operatorname{Re}(\omega_{lmn})]\exp[-t\operatorname{Im}(\omega_{lmn})], \qquad (36)$$

their imaginary part specifies the decay of the acoustic field with time. In the case of a circumferentially non-uniform liner the radial wavenumbers, that are the roots of (28), are generally complex and lead (6) to complex natural frequencies:

$$\omega_{mn} = \kappa U \pm c \sqrt{\kappa^2 + (k_{mn})^2}$$
(37)

whose imaginary part again specifies (36) temporal decay or growth respectively for  $Im(\omega) > 0$  and  $Im(\omega) < 0$ .

#### 5. WAVE FIELD DUE TO INITIAL CONDITIONS OR SOURCES

The acoustics of non-uniformly lined ducts is analogous for discrete (8, 28) and continuous (9, 34) axial wavenumbers; henceforth only the discrete case will be considered as appropriate to axially non-uniform liners. The amplitudes of each mode (28) in the acoustic pressure (8) can be specified either by (i) initial conditions or (ii) by sound sources. Considering the first case the initial condition applies to the pressure perturbation (8) for all radial modes (28):

$$p(r,\boldsymbol{\varphi},z,t) = \mathrm{e}^{-\mathrm{i}\omega t} \sum_{l=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}2\pi l z/L} \sum_{m=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}m\boldsymbol{\varphi}} \sum_{n=1}^{\infty} A_{lmn}(\boldsymbol{\omega}) \mathbf{J}_m(k_{lmn}r), \tag{38}$$

assumed to be known in the whole duct at the initial time t = 0:

$$p(r,\boldsymbol{\varphi},z,0) = \sum_{l=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}2\pi l z/L} \sum_{m=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}m\boldsymbol{\varphi}} \sum_{n=1}^{+\infty} A_{lmn}(\boldsymbol{\omega}) \mathbf{J}_m(k_{lmn}r).$$
(39)

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Inverting the Fourier and Bessel series specifies the amplitudes in terms of the initial acoustic pressure, viz.:

$$A_{lmn}(\boldsymbol{\omega}) = \frac{1}{\pi L a^2} \left\{ \left[ 1 - \frac{m^2}{(k_{lmn}a)^2} \right] \left[ \mathbf{J}_m(k_{lmn}a) \right]^2 \right\}^{-1} \\ \times \int_0^L e^{-i2\pi l z/L} dz \int_0^{2\pi} e^{-im\varphi} d\varphi \int_0^R r \mathbf{J}_m(k_{lmn}a) \, p(r,\varphi,z,0) \, dr.$$
(40)

The case (ii) concerns the convected wave equation, with uniform axial flow (1), in the case of an acoustic source distribution with frequency  $\omega$ . The solution may be sought in the form similar to (38) for a frequency  $\omega$ :

$$p(r,\varphi,z,t) = e^{-i\omega t} \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} \sum_{n=1}^{+\infty} D_{lmn} J_m(k_{lmn}r),$$
(41)

that satisfies the boundary condition at the duct wall, retains the source frequency  $\omega$  and has distinct amplitudes  $D_{lmn}$ ; these are determined by comparison with the source term also with single frequency:

$$S(r, \varphi, z, t) = \tilde{S}(r, \varphi, z) e^{-i\omega t}, \qquad (42)$$

with spatial dependence expanded in a Fourier-Bessel series:

$$\tilde{S}(r,\boldsymbol{\varphi},z) = \sum_{l=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}2\pi l z/L} \sum_{m=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}m\boldsymbol{\varphi}} \sum_{n=1}^{+\infty} \mathrm{J}_m(k_{lmn}r) S_{lmn};$$
(43)

the coefficients are given by:

$$S_{lmn} \equiv \frac{1}{2\pi L} \frac{2}{a^2} \left[ \left( 1 - \frac{m^2}{(k_{lmn}a)^2} \right) \{ \mathbf{J}_m(k_{lmn}a) \}^2 \right]^{-1} \\ \times \int_0^{2\pi} e^{-im\varphi} \, \mathrm{d}\varphi \int_0^L e^{-2\pi i lz/L} \mathrm{d}z \int_0^a r \, \mathbf{J}_m(k_{lmn}a) \, \tilde{S}(r,\varphi,z) \, \mathrm{d}r.$$
(44)

Substitution of (41) and (43, 42) in (1) yields

$$\frac{1}{r^2} \left[ r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} + r \frac{\mathrm{d}}{\mathrm{d}r} + (k_l)^2 r^2 - m^2 \right] D_{lmn} \mathbf{J}_m(k_{lmn}r) = S_{lmn} \mathbf{J}_m(k_{lmn}r), \tag{45}$$

where the Bessel differential operator appears.

Noting that the eigenfunctions satisfy:

$$\frac{1}{r^2} \left[ r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} + r \frac{\mathrm{d}}{\mathrm{d}r} + (k_{lmn})^2 r^2 - m^2 \right] \mathbf{J}_m(k_{lmn}r) = 0, \tag{46}$$

the relation (45) becomes:

$$\left[ (k_l)^2 - (k_{lmn})^2 \right] D_{lmn} = S_{lmn}.$$
(47)

In the non-resonant case  $k_l \neq k_{lmn}$ , the inversion of (47) specifies the coefficients  $D_{lmn}$  in (41), so that

$$p(r, \varphi, z, t) = e^{-i\omega t} \sum_{l=-\infty}^{+\infty} e^{i2\pi l z/L} \sum_{m=-\infty}^{+\infty} e^{im\varphi} \sum_{n=1}^{+\infty} J_m(k_{lmn}r) \frac{S_{lmn}}{(k_l)^2 - (k_{lmn})^2},$$
(48)

is the acoustic field forced by the source distribution (42, 43).

Solving (35) for the eigenvalues of the radial wavenumber leads to

$$k_{lmn} = \left| \frac{1}{c^2} \left( \omega_{lmn} - \frac{2\pi l U}{L} \right)^2 - \left( \frac{2\pi l}{L} \right)^2 \right|^{1/2}; \tag{49}$$

comparing with (5), it follows that

$$(k_l)^2 - (k_{lmn})^2 = \frac{1}{c^2} \left[ (\omega - 2\pi l U/L)^2 - (\omega_{lmn} - 2\pi l U/L)^2 \right],$$
(50)

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that can be simplified:

$$[(k_l)^2 - (k_{lmn})^2] = \frac{1}{c^2} (\omega - \omega_{lmn}) (\omega + \omega_{lmn} - 4\pi l U/L).$$
(51)

Therefore resonances can occur only if  $k_l = \pm k_{lmn}$ , at two frequencies:

$$\omega = \omega_{lmn} \tag{52a}$$

$$\omega + \omega_{lmn} = 4\pi l U/L, \tag{52b}$$

that may be distinct (simple resonance) or coincide (double resonance).

The non-resonant mode is a general term

$$p(r, \varphi, z, t) = \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{+\infty} P_{lmn}(t),$$
(53)

or (48), thus is given by

$$P_{lmn}(t) = e^{i(m\varphi + 2\pi lz/L - \omega t)} J_m(k_{lmn}r) \frac{c^2 S_{lmn}}{(\omega - \omega_{lmn})(\omega + \omega_{lmn} - 4\pi lU/L)}.$$
(54)

#### 6. **DISCUSSION**

The Figure 3 compares the modulus (left) and phase (right) of the acoustic pressure as a function of the radial distance r divided by the duct radius. The sound source is an axial dipole. The case (solid line) of a uniform wall specific impedance (55a) is compared (dotted line) with a non-uniform wall impedance (55b) that is a small perturbation (55c):

$$Z_0 = 1 + i, \tag{55a}$$

$$Z(\theta) = Z_0 \left[ 1 + \varepsilon \cos\left(\frac{2\pi l}{L}\right) \right], \tag{55b}$$

$$\varepsilon = 0.3 + 0.2i. \tag{55c}$$

The case (a) at the top corresponds to an axially uniform mode l = 0 in which case there is little difference between the sound pressure in the presence of uniform or non-uniform liners; in the case l = 1 (in the middle) of the first non-uniform axial mode again the non-uniform liner is poorly matched to the sound field and has little effect. The non-uniform liner provides significant attenuation relative to the uniform liner in the case l = 2 (at the bottom) of the second longitudinal acoustic mode. Similar analysis can be made in the presence of mean flow and for other non-uniform liner impedances.

## REFERENCES

- 1. Howe MS. The attenuation of sound in a randomly lined duct. Journal of Sound and Vibration. 1983;87(1):83–103.
- 2. Fuller CR. Propagation and radiation of sound from flanged circular ducts with circumferentially varying wall admittances, I: semi-infinite ducts. Journal of Sound and Vibration. 1984;93(3):321–340.
- 3. Fuller CR. Propagation and radiation of sound from flanged circular ducts with circumferentially varying wall admittances, II: finite ducts with sources. Journal of Sound and Vibration. 1984;93(3):341–351.
- 4. Vaidya PG. The propagation of sound in ducts lined with circumferentially non-uniform admittance of the form  $\eta_0 + \eta_q \exp(iq\theta)$ . Journal of Sound and Vibration. 1985;100(4):463–475.
- 5. Campos LMBC, Oliveira JMGS. On the optimization of non-uniform acoustic liners on annular nozzles. Journal of Sound and Vibration. 2004;275(3–5):557–576.
- 6. Campos LMBC, Oliveira JMGS. On the acoustic modes in a cylindrical nozzle with an arbitrary impedance distribution. Journal of the Acoustical Society of America. 2004;116(6):3336–3347.



Figure 3 – Amplitude or modulus (l.h.s.) and phase or argument (r.h.s.) of radial eigenfunction as a function of r/a radial distance r divided by duct radius a, for modes with axial order l = 0 (top), l = 1 (middle) and l = 2 (bottom), comparing the case of a uniform (solid line) and non-uniform (dotted line) liner for an axial dipole, when the duct is carrying a flow with uniform Mach number M = 0.3.

- 7. Campos LMBC, Oliveira JMGS. On sound generation in cylindrical flow ducts with non-uniform wall impedance. International Journal of Aeroacoustics. 2013;12(4):309–348.
- 8. Campos LMBC, Oliveira JMGS. On Sound Radiation from an Open-Ended Non-Uniformly Lined Cylindrical Nozzle. Acta Acustica United with Acustica. 2014;100(5):795–809.
- 9. Nayfeh AH, Kaiser JE, Telionis DP. Acoustics of aircraft engine-duct systems. AIAA Journal. 1975 Feb;13(2):130–153.
- 10. Campos LMBC. Complex Analysis with application to flows and fields. CRC Press; 2011.