ANOMALOUS SOLUTIONS TO SIMPLE WAVE EQUATIONS

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Abstract: Standard fourth and sixth-order differential wave equations for beams and for pairs of fluid-coupled plates are shown to give rise to apparently anomalous solutions describing waves that grow in amplitude in the direction of propagation and thus violate conservation of energy when the propagating medium is semi-infinite, despite the fact that the original equations conform to this principle. The second-order wave equation for a string does not have these problems, nor do they occur in the other cases if the propagating medium is finite in extent and has simple boundary conditions at both ends. This apparent paradox can be resolved by consideration of the group velocity, which is shown to be negative in the case of the anomalous waves, thus preventing them from propagating into the half-space of the medium.

INTRODUCTION

This paper explores briefly the anomalous solutions that arise in the case of waves propagating on semi-infinite beams and plates for which the wave equation is of order higher than two. It is found that, in addition to the "normal" waves generally treated, which are attenuated exponentially as they propagate, there are formal solutions that correspond to waves that grow in amplitude as they propagate and, in the case of a equations of order higher than four, also evanescent waves that are localized very close to the source. While some of these solutions can be eliminated by consideration of the boundary condition at infinity, it appears that in each case at least one such anomalous solution remains. It is the purpose of the present paper to propose a resolution of this paradox. Three simple cases will be considered, corresponding to equations of order two, four, and six respectively.

WAVES ON A STRING

In the case of an ideally flexible string, assumed to have tension T and mass m per unit length, the one-dimensional wave equation has the form

$$m\frac{\partial^2 z}{\partial t^2} + \beta\frac{\partial z}{\partial t} = T\frac{\partial^2 z}{\partial x^2}\,, \tag{1}$$

where $\beta > 0$ is a damping coefficient. For the case to be considered here, we assume a semi-infinite string with $x \geq 0$ and a motion $z(0,t) = a\cos\omega t$ imposed at the point $x = 0$. Since a semi-infinite string is not physically realistic, this may be replaced by a very long string with a "radiation boundary condition" imposed at the far end. Such a boundary condition ensures that an incident wave continues to propagate past the end without reflection, and it is further appropriate to assume that there is no wave incident upon the string from beyond its far end.

To solve equation (1) it can be assumed that

$$z(x,t) = a\exp[i(kx-\omega t)],$$

where $\omega > 0$, the propagation constant k is a complex quantity, and the amplitude a is real, as required by the boundary condition at $x = 0$. The "physical" solution to the problem is then the real part of $z(x,t)$. Substituting (2) in (1) then gives the well-known formal result

$$k^2 = \frac{m\omega^2}{T} + i\frac{\beta\omega}{T} = \gamma\omega^2 + i\alpha\,, \tag{3}$$

where, in the version to the right, we have written $\gamma = m/T$ and $\alpha = \beta\omega/T$ for simplicity. The wave speed in the absence of damping is $c = (T/m)^{1/2}$. In the presence of damping that is other than simply viscous, α may have a different dependence upon frequency, but this is irrelevant to the present discussion.

Since $\alpha > 0$, the position of k^2 on a complex plot is as shown qualitatively by the point O in Fig. 1(a). For clarity, a large value has been assumed for the damping constant α.

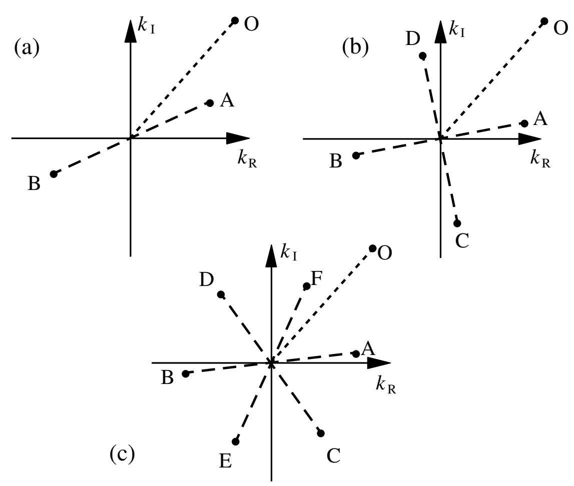

Figure 1: Solution diagram for (a) a taut string, (b) an ideally thin beam, and (c) a symmetric Lloyd-Redwood wave.

Taking the square root of this quantity to find the propagation vector k then leads to the two possibilities indicated qualitatively by the points A and B. These might be represented by the expression $a\exp(ik_{\rm R}x - k_{\rm I}x - i\omega t)$ where $k_{\rm R}$ and $k_{\rm I}$ are respectively the real and imaginary parts of k. Point A has $k_{\rm R}$ positive, so that the wave is propagating in the $+x$ direction and, since $k_{\rm I}$ is also positive, the wave is attenuated as it propagates. Similarly, the solution at point B represents a wave propagating in the $-x$ direction and again being attenuated as it propagates. In the domain $x \geq 0$ this second wave can be eliminated, since it must be an external influence that has entered the string through the radiation termination at its far end-point, and it has been assumed that there is no such influence.

All of this is perfectly straightforward and leads to no difficulties at all. It is only when more complex physical situations are considered that problems arise.

WAVES ON A BEAM

Consider next an ideally thin elastic beam under no tension, often referred to as an Euler-Bernoulli beam. The governing differential equation is then[1]

$$m\frac{\partial^2 z}{\partial t^2} + \beta\frac{\partial z}{\partial t} = -K\frac{\partial^4 z}{\partial x^4}\,, \tag{4}$$

where K now measures the elastic stiffness of the beam. A very simple form has been assumed for the damping term, corresponding to damping by an external viscous fluid of negligible mass; internal damping in the beam would lead to a more complicated function. Assuming a solution of the form (2) then leads to the result

$$k^4 = \frac{m\omega^2}{K} + i\frac{\beta\omega}{K} = \gamma\omega^2 + i\alpha\,, \tag{5}$$

where now $\gamma = m/K$ and $\alpha = \beta/K$. The resemblance between this equation and (3) should be noted. If damping is neglected by setting $\beta = 0$, then from (5) the wave speed is

$$c = \frac{\omega}{k} = \left(\frac{K}{m}\right)^{1/4}\omega^{1/2}\,, \tag{6}$$

as is well known. The behavior of such beams has been discussed many times before, most recently by Pavić[2], but the anomalies to be discussed here have not been commented upon.

For the simple case of a finite beam with end conditions specifying both $z(0,t)$ and $\partial z(0,t)/\partial x$, (4) leads to an expression for $z(x,t)$ in terms of trigonometric and hyperbolic functions, and thence to specification of the vibrational mode frequencies. The problem arises only when a semi-infinite beam with $x \geq 0$, or a long beam with a radiation boundary condition at the remote end, is considered. The point O in Fig. 1(b) is the value of k^4 given by (5) plotted in the complex plane, and with again a large value of the damping parameter α for clarity. The points A, B, C, and D then represent the four values of k given by (5). Point A represents a wave propagating in the $+x$ direction and being progressively damped as it propagates, just as before, and point B represents a similar wave propagating in the $-x$ direction that can be eliminated by the boundary condition at the far end of the beam. The problem arises with the solutions at points C and D.

Consider point C, for which $k_{\rm R} > 0$ and $k_{\rm I} < 0$. This represents a wave propagating in the $+x$ direction but being amplified as it propagates. Similarly point D represents a wave propagating in the $-x$ direction and being amplified as it propagates. This second case, D, can be ruled out for the domain $x \geq 0$ by the boundary conditions at infinity, since it requires a non-zero amplitude of input wave in the $-x$ direction at the far end of the beam. But what about the wave at point C? It can be matched to the boundary conditions at $x = 0$, and does not violate the radiation condition at the distant end of the beam. It does, however, appear to violate the physical principle of conservation of energy, since a small wave input at $x = 0$ leads to an arbitrarily large input of energy to the radiation resistance at the remote termination of the beam.

SYMMETRIC LLOYD-REDWOOD WAVES

An even more complex situation arises in the propagation of mirror-symmetric waves on two thin plates separated by a layer of dense fluid.[3] These waves have generally been studied in an ultrasonics context, but have also recently received attention in relation to a possible "second-filter" mechanism in human hearing [4, 5]. The propagation speed of these waves can be extremely small if the plates are thin, because a small displacement of the plates towards each other results in a large "squirting" motion of the liquid enclosed between the plates. As has been shown elsewhere [3, 4], the propagation of these waves obeys a sixth-order differential equation so that, if simple damping is assumed, the propagation vector k is given in analogy with the previous results by an expression of the form

$$k^6 = \gamma\omega^2 + i\alpha\,, \tag{7}$$

with $\alpha > 0$ and γ a positive constant depending upon the system parameters. Once again the resemblance of this equation to the string equation (3) and the beam equation (5) should be noted. In the absence of damping ($\alpha = 0$), the wave speed c is proportional to $\omega^{2/3}$. Equation (7) indicates that there are now six roots for the propagation vector k, and these are illustrated in Fig. 1(c).

The solutions at points A and B represent normal damped waves propagating in the $+x$ and $-x$ directions, as before. Points C and D similarly represent the anomalous waves associated with points with the same identifiers in Fig. 1(b). Points E and F represent a new phenomenon. Point F is a wave propagating in the $+x$ direction but with an extremely high damping, and similarly point E is a wave propagating in the $-x$ direction, again with very high damping. Solution E can once more be ruled out because it would require an input wave at the remote termination $x = \infty$, while solution F can be regarded as an "evanescent wave" which is confined to the immediate vicinity of the wave source at $x = 0$ and so presents no problems.

PARADOX RESOLUTION

While second-order wave equations display no anomalous features, wave equations of higher order give rise to at least two solutions C and D for waves that appear to grow in amplitude as they propagate. These are in addition to evanescent waves such as E and F that are localized near the wave source. In the case of a semi-infinite domain with $x \geq 0$, some of these waves can be ruled out since they require a finite incident wave from outside the system at $x = \infty$, but at least one such as C always remains, and this appears to violate the principle of conservation of energy despite the fact that the original differential equation is in each case based upon simple Newtonian principles.

The solution to this apparent paradox can be found by examining the group velocity $v = \partial\omega/\partial k$ in each case. Suppose that the solution to the general wave equation has the form

$$k^n = \gamma\omega^2 + i\alpha \tag{8}$$

where n is an even integer and α is a positive function of ω. Differentiating with respect to k gives the result

$$\frac{\partial\omega}{\partial k} = \frac{1}{2\gamma\omega}\left(nk^{n-1} - i\frac{\partial\alpha}{\partial k}\right). \tag{9}$$

Suppose that $k = k_R + ik_I$, then

$$k^{n-1} = k_R^{n-1} + i(n-1)k_R^{n-2}k_I - \frac{(n-1)(n-2)}{2}k_R^{n-3}k_I^2 + \ldots. \tag{10}$$

In the case of the anomalous solution for an elastic beam, $n = 4$ so that there are only three terms in the expansion. As shown in Fig. 1(b) for the anomalous solution point C, $|k_I| \gg |k_R|$ so that the third term in (10) is much larger than the first term and the real part of k^{n-1} is therefore negative. When the expression (10) is substituted in (9) the result is therefore that the group velocity for this wave is negative so that it effectively vanishes into the source at $x = 0$.

This result means that the envelope of the anomalous wave propagates in the $-x$ direction, so that a disturbance originating at $x = 0$ is unable to propagate into the domain $x \geq 0$ despite the fact that its phase velocity is positive. If there is a pre-existing wave of this type at time $t = 0$, then it will collapse towards the origin and deposit its energy there. This resolves the apparent anomaly and there is no violation of energy conservation.

The case of symmetric Lloyd-Redwood waves is mathematically more complicated, since now $n = 6$ so that there are three real terms to be considered in (10) and two anomalous waves in the total solution for $x \geq 0$. Resolution of the paradox in this case requires more detailed mathematical analysis, but it is almost certain that the answer is similar: point F corresponds to an evanescent wave localised near the wave source at $x = 0$ while point C describes a wave of which the envelope collapses into the source.

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