DESIGNING IDIOPHONES WITH TUNED OVERTONES

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The design of a musical instrument is normally the preserve of the specialist instrument maker using empirical techniques developed over centuries. The analysis of instruments and the sounds they produce is undertaken by mathematicians and scientists. Recent advances in computational power and numerical techniques have provided the opportunity for analysis of physically complex vibrating structures.
Moreover, these same numerical techniques can also be used to design vibrating physical structures. In this paper, we consider the application of constrained optimisation to the design of tuned beams, plates and bells.

1 INTRODUCTION

Tuned idiophones (struck instruments) such as marimbas, chimes and bells have undergone centuries of development, resulting in complex profiles, bumps and grooves, the purpose of which is usually to optimise the sound of the strike. The maker seeks to produce an instrument that responds with sounds that are pleasing to the ear, and for the most part this means that normal modes of vibration are appropriately tuned.

Structural engineers also optimise physical structures, such as a building or a bridge, to improve the efficiency of the design. In this case, the optimal design is generally one that minimises cost or the amount of construction material used, and the technique employed is numerical optimisation. To the engineer, a musical instrument is a physical structure, and as such its design can be undertaken using numerical optimisation techniques. This paper illustrates the use of a particular numerical technique, constrained optimisation, for the design of tuned bars, plates and bells. In this paper, we limit ourselves to outlining the approach and discussing the results. A more detailed discussion of the technique has been given elsewhere [1, 2].

2 CONSTRAINED OPTIMISATION

Numerical techniques such as finite element analysis have been used quite extensively to determine the mode shapes arising in musical structures. Less common, however, is the use of numerical techniques in the design of a musical instrument or part thereof. Where numerical optimisation strategies have been adopted, the temptation has been to emulate the maker in the choice of optimising function, namely to optimise the frequencies of the modes [3-5]. Such an approach can leave the designer with the dilemma of determining how close to the desired frequency each mode should be and which mode is more important to get right. In this paper, we outline a different approach.

The design goal is to produce a structure that responds with specified frequencies for specified modes of vibration. Hence, the required frequencies are critical for this design problem and we require a solution strategy where the frequencies are constraints. A general technique that allows maximum freedom in defining the shape, whilst constraining the frequencies, is constrained optimisation. The constrained optimisation problem is uniquely specified by three things, namely, the geometry parameters, the optimisation function and the constraint functions, and different solutions may be obtained by varying any of these. This approach also allows us significant freedom in the choice of optimising function.

2.1 MATHEMATICAL MODELS

The analysis of the motion of any vibrating structure begins with the governing equations. These equations depend on the model adopted, and the first decision is therefore the choice of model to be used to describe the structure.

For example, the dominant response of a struck xylophone bar is transverse to its longitudinal axis, and the motion is therefore beam-like in nature. Hence, a suitable model is a one-dimensional model that can account for shear deformation [1]. However, when the profile of the bar entails sudden jumps in height, the one-dimensional model fails to accurately account for the complex stress system set up around the sharp corners. In this case, a two-dimensional model is more accurate [6]. Similar considerations apply to any musical structure. A plate can generally be satisfactorily modelled as a two-dimensional structure, so long as its thickness is small in comparison to a typical plan dimension [7]. A bell is a three-dimensional structure. However, provided that the thickness is not too large in comparison with the radius, it may be modelled as a two-dimensional surface with appropriate stiffness and mass characteristics [8].

Analytical solutions of the governing equations are generally not available, except in the case of very simple structures. The structures we shall be designing are not simple, particularly given that they involve varying cross-sectional properties. Numerical techniques are therefore required, and finite element analysis is chosen as the most appropriate technique. Finite element analysis converts the governing differential equations into a set of eigenvalue equations that can be solved to give the natural frequencies and mode shapes of the structure.

2.2 STRUCTURAL GEOMETRY

The goal is to design a structure that responds with specified frequencies, that is we seek to determine the geometry of the structure that will respond in a pre-determined manner. Hence, we must describe the geometry by a number of parameters, *hi*. These parameters are the primary unknowns for the problem and will be varied to produce the required outcome.

2.3 OPTIMISATION FUNCTION

Optimisation implies one is seeking the best solution. Our goal is to design a structure that responds with specified frequencies. It is therefore tempting to optimise the frequency of each critical mode. However, we prefer to obtain a solution where the frequencies are accurate, and we choose some other function to optimise. This approach leaves us with a wide range of possible optimising functions, including the possibility of having no optimizing function.

2.4 CONSTRAINTS

The constraint functions describe what must be obtained for an acceptable solution. Obvious constraints are those that retain the integrity of the structure. Idiophones for instance should not be so thin that they permanently distort or break when struck. In our approach, the required frequencies are critical to the design, and we therefore include them as constraints.

2.5 CONSTRAINED OPTIMISATION

Constrained optimisation involves solving the governing equations for the problem and minimising the optimisation function f(hi) whilst satisfying the constraint functions gj(hi). The approach can be written as

Minimise $f(hi)$ subject to $gj(hi) = 0$; $i = 1, 2 \ldots N$; $j = 1, 2 \ldots Nc$

where N is the number of variables and Nc is the number of constraints.

In our case, an optimisation procedure is used to determine the values of the parameters describing the geometry of the structure, such that the structure has the desired frequency characteristics and satisfies other specified criteria.

3 EXAMPLES

3.1 MARIMBAS AND XYLOPHONES

Marimbas are simple instruments consisting of an array of transverse beams or bars. Each bar sits on two supports placed at the nodes of its fundamental transverse mode. The bar is struck at its centre when played. Simple linear analysis shows that when a uniform beam is struck between its two supports the transverse modes generated are not harmonic. Hence, there is not a simple integer relationship between their frequencies. From a musical point of view, this non-harmonic response is not desirable. To tune the beam so that at least the three lowest modes are harmonically related, the instrument maker carves a parabolic arch cut on the underside.

Reduction in thickness at the centre of the bar's length results in a significant reduction in the stiffness of the bar and hence the fundamental frequency, and a lesser effect on the higher modes. Thus, the profile of the bar determines the relationship between the modes.

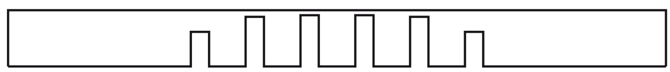
The exact dimensions of the undercut are an empirical design. There has been significant interest in determining the effect of the profile on the frequencies of the natural modes [9-12]. However, we turn the problem around and ask, what must the shape of the undercut be to produce a particular frequency regime?

3.1.1 Numerical solution

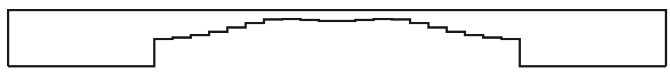
We begin by formulating the equations that governs the motion. A one-dimensional model that can account for shear deformation, such as Timoshenko's beam theory, is usually sufficient [1]. The problem is formulated in terms of the amplitudes of the displacement and rotation of the beam as it vibrates, and results in a fourth-order system of coupled ordinary differential equations.

In the case of a xylophone or marimba bar, it is the undercut that is used to tune the appropriate vibrational modes, so the primary unknowns are the heights of the bar at various locations. Hence, we define the height of the bar as a function of the position x along the length of the bar, *h(x)*, restrict the cut section to be between the two supports and choose a geometry with which to work. There are a wide range of geometries that can be adopted, and we previously considered three examples, namely piecewise-continuous heights, piecewise-linear heights and sinusoidal functions [2].

We begin with a solid bar with defined physical and geometrical parameters and request an undercut that will produce a particular fundamental and overtones in the ratio 1:4:10. Figure 1 illustrates three different solutions. The solution for case (a) was obtained by minimising the amount volume of material removed to produced the profile. However, this profile has sudden jumps in height, and the one-dimensional model is less accurate in this case [1]. The solutions for cases (b) and (c) were obtained by requesting smoother profiles. Case (b) was obtained by minimising the height differences between adjacent sections while case (c) used a smoother profile geometry, namely sine curves.



(a)



(b)

(c)

Figure 1: Profiles of xylophones (vertical scale distorted for clarity).

3.2 PLATES

Musical plates are somewhat rarer than the xylophone or marimba. Nevertheless, they are occasionally used as substitutes for bells [13, 14]. Simple linear theory again shows that the transverse modes generated on a flat rectangular or round plate are not generally harmonic [13]. However, a similar approach to that applied to the marimba bar can be used to design the plate.

If the plate is made to vary in thickness along its profile, then there should be some designs whereby the modes are harmonically related. Hence, we set ourselves the goal of designing a plate with a specified geometry, such as rectangular or circular, so that it responds with specified frequencies when struck in the transverse direction.

3.2.1 Numerical solution

While a plate is a three-dimensional structure, it may be accurately modelled as a two-dimensional structure if its thickness is small compared to a typical plan dimension. Several plate theories are available depending on the assumptions made [15]. We have adopted Mindlin's theory [16] as the underlying theoretical model. This theory is applicable to both thin and moderately thick plates. As an example, we consider an initially flat circular plate with a central hole. The plate is divided into eight concentric rings and the task is to vary the thickness of each ring until the first three distinct natural modes are in the ratio 2:3:4. Figure 2 illustrates two possible profiles. The solution for case (a) was obtained by having no optimising function whereas the solution for case (b) was obtained by requesting a smooth profile.

Figure 2: Profiles of circular plates with central hole (vertical scale distorted for clarity).

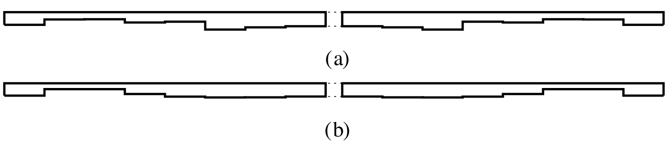


Figure 2: Profiles of circular plates with central hole (vertical scale distorted for clarity).

3.3 BELLS

The sound produced by a large church or carillon bell is the result of the vibrational modes of a particular profile, the decay rates of the natural modes and the sound perceived by the listener [5]. In this paper, we will look only at the design of the bell profile such that important modes are in tune.

Since the 1600s, western church bells have been manufactured to exhibit a clear, recognisable pitch through the design of a bell profile. The five important modes for a tuned bell are generally considered to be the first five extensional modes. These modes are labelled Hum, Prime, Tierce, Quint and Nominal, and are generally tuned so that their frequencies are in the ratio 1:2:2.4:3:4 [13]. Thus, we set ourselves the problem of determining a suitable profile for a bell such that the five lowest modes are tuned according to the ratios above. As with the bar and the plate, the constraints are simply the desired frequencies of these five lowest modes of vibration, together with the usual structural integrity issues. However, in this case an added constraint can arise from the construction technique for a large bell, namely that the bell should be readily removable from its mould after casting.

3.3.1 Numerical Solution

There are various options for describing the profile of the bell. The simplest option is to describe the profile using piecewise-linear variations for the radii, coupled with piecewise-constant variations of thickness. Smoother profiles can be obtained by using higher-order polynomial functions for the variables. For the purpose of this paper, we have limited ourselves to the first option.

We began with a truncated cone with geometry crudely following a standard bell described by Lehr [17]. The structure is then divided into sixteen concentric truncated conical rings and capped with a nearly horizontal piece. The task is then to vary the radius and thickness of each ring until the desired frequencies regime is obtained. Figure 3 depicts two possible profiles. The first was obtained with a zero optimising function and the second included the construction constraint of no reversed radii, thus allowing easy removal from its mould after casting. In both cases, the thickness of each ring also varies, although this is not shown in the Figure.

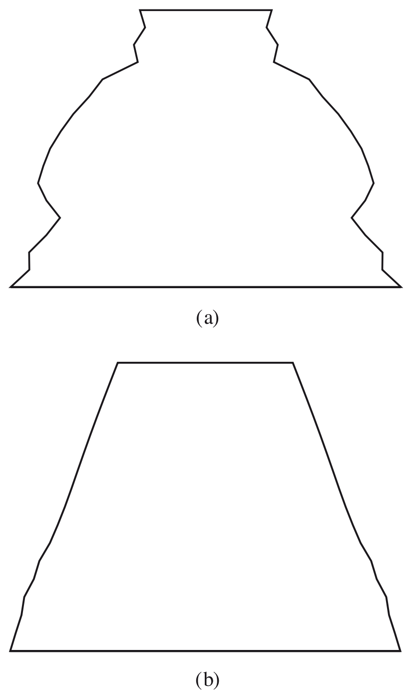


Figure 3: Profiles of bells.

4 CONCLUSIONS

The optimisation of any musical instrument is usually the task of the maker. The design generally follows empirical techniques that attempt to optimise the tuning of the natural modes of the instrument. In this paper, we have outlined a mathematical approach to design that constrains the design to the required frequency regime and allows other parameters to be optimised. The technique is limited only by the adequacy of the model used to describe the structure and the ability to describe the constraint and optimising functions mathematically.

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