Predicting the Acoustic Performance of Mufflers using Transmission Line Theory

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ABSTRACT

The effectiveness of a reactive muffler results from the destructive interference of acoustic waves travelling within the device. Cross-sectional area changes within the device produce the interfering acoustic waves. In this paper, plane wave transmission line theory is used to predict the acoustic performance of simple expansion chamber mufflers. The performance is quantified by the frequency-dependent sound transmission loss. The presence of dissipative materials has also been considered by the inclusion of a simple model describing wave propagation in porous materials. The predicted acoustic performance is obtained for various mufflers and compared with results obtained experimentally.

INTRODUCTION

Although the study of mufflers, particularly those used with reciprocating combustion engines, has been of interest for many years, the increased public awareness of noise issues has provided an impetus to improve the performance of these devices. The current paper describes work which enables small mufflers of a type commonly used with motorcycles to be easily modelled. These mufflers, which are often called "straight through" mufflers, are of simple construction and provide only small obstruction to the mean flow. Such a muffler usually consists of a metal can which surrounds a perforated tube through which the exhaust gases flow. Porous material is often included in the volume surrounding the perforated tube.

Many factors such as geometry, porous material properties, and flow effects should be considered to predict the acoustic performance of a muffler. The acoustic performance is usually quantified by the frequency-dependent sound transmission loss. A variety of methods have been used to predict this quantity for mufflers, ranging from analytical and computational methods to experimental techniques. Much of the earlier literature to predict the performance of mufflers has been documented by Jones (1984) and Munjal (1987). The most common type of linear acoustic model uses classical electrical filter theory and is known as the transfer matrix method, also called the 4-pole parameter method (Jones 1984; Munjal 1987; Davies 1988). Other methods include the 3-point method (Wu and Wan 1996) and transmission line theory (Byrne 1980). A comparison of various analytical methods has been given (Bilawchuk and Fyfe 2003). Numerical approaches include the finite element method (FEM) (Young and Crocker 1975) and the boundary element method (BEM) (Cheng and Seybert 1991), and computational fluid dynamics (CFD) (Middelberg *et al.* 2004). The BEM has been used to predict the acoustic performance of muffler configurations with extended inlet/outlet tubes, or internal connecting tubes (Cheng and Seybert 1991). The BEM has also been used in conjunction with the 3-point method for evaluating transmission loss in acoustic ducts (Bilawchuk and Fyfe 2003). The CFD approach implemented by Middelberg *et al.* (2004) provides a very complete method of modelling both the acoustic and mean flow performance of mufflers. The acoustic performance of mufflers in the presence of mean flow can be considered as can the effect of the viscosity of the gas. Further, the effect of the muffler on non-plane waves entering it can be considered. The difficulty with the CFD approach is that it is computationally very intensive. While the setting up of the actual model is not particularly difficult, the computational effort is very substantial.

The problem described in the previous paragraph provides the motivation for the work described in this paper. The transmission line approach, which does not include the effects of mean flow and non-plane wave propagation, is easy to implement and it is shown to allow useful predictions to be made.

THE ACOUSTIC PERFORMANCE MEASURE

The commonly used acoustic performance measure for a muffler is the frequency-dependent sound transmission loss of the device. It is the sound power level of the acoustic wave incident on the muffler less that of the wave leaving the muffler. The attraction of this measure is that it depends only on the device and not on the device and its installation. Fortunately, because of the diameter of the inlet pipe and the exhaust frequency components of interest, it is usually the case that only plane wave propagation need be considered. Consider the simple expansion chamber muffler shown in Figure 1. The single frequency acoustic wave incident on the muffler has an amplitude P_i. At Point 1 in the inlet pipe, this incident wave generates a wave of amplitude P_r which is reflected back along the inlet pipe, and a wave which enters the expansion chamber. Similarly at Point 3, another reflected wave is produced. The wave which emerges from the muffler at Point 4 and travels away from the muffler has an amplitude P_t.

The acoustic intensity associated with the incident and transmitted plane waves can be readily derived from expressions of the form $P^2 / 2\rho c$ where P is the amplitude of the pressure of the wave while ρ and c are the density and velocity of sound for the gas. The acoustic powers are then simply given by multiplying the intensities by the relevant cross-sectional areas. Thus the acoustic power associated with the incident wave is simply $P_i^2 S_A / 2\rho c$, where S_A is the cross-sectional area of the inlet pipe as shown in Figure 1. When the sound powers are expressed in levels, simple manipulation of expressions such as the preceeding one give the sound power transmission loss as in equation (1):

$$TL = L_{pi} - L_{pt} + 10\log\frac{S_A}{S_C} \qquad (1)$$

L_{pi} and S_A are the sound pressure level and the cross-sectional area associated with the incident wave, and L_{pt}, S_C are the sound pressure level and cross-sectional area associated with Point 4.

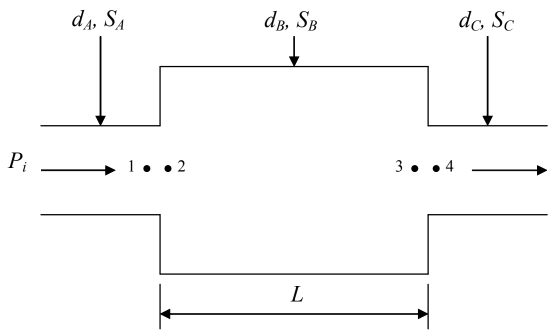

Figure 1. Simple expansion chamber muffler.

THE USE OF TRANSMISSION LINE THEORY

It can be seen from equation (1) that the sound power transmission loss at a particular frequency can be determined if the amplitude of the acoustic pressures associated with the incident and transmitted acoustic waves can be found. This can be readily done with transmission line theory using two simple formulae. The first relates the specific acoustic impedances at the ends of a column of gas and the second relates the acoustic pressures at the ends of the gas column. The use of these formulae, along with the fact that at changes of cross-sectional area there must be equality of acoustic pressure and acoustic impedance across the discontinuity, enables the required acoustic pressures to be found.

The development of the transmission line formulae is described in Appendix A. Appendix B contains similar formulae when the acoustic waves are travelling in a gas contained in a porous material. A simple model which describes wave propagation in such a a gas is used.

Application to a simple expansion chamber muffler

The use of the transmission line formulae is demonstrated by using them to determine the frequency-dependent sound transmission loss of the simple expansion chamber muffler shown in Figure 1. The following procedure is followed at each frequency where the sound transmission loss is to be found. At Point 4 on the diagram, the specific acoustic impedance in the direction shown, $\mathbf{z}_4$, is equal to the characteristic impedance ρc as the outlet pipe is assumed to be anechoically terminated. The acoustic impedance at Point 4, $\mathbf{Z}_4$, is then given by $\mathbf{Z}_4 = \frac{\mathbf{z}_4}{S_C}$. The acoustic pressures and volume velocities at Points 4 and 3 are equal, and so the acoustic impedances at these points are equal. The specific acoustic impedance at Point 3 is then given by $\mathbf{z}_3 = \mathbf{Z}_3 S_C$. Equation (A8) can be used to determine the specific acoustic impedance at Point 2, $\mathbf{z}_2$, from that at Point 3. The acoustic impedance at Point 2, $\mathbf{Z}_2$, is then given by $\mathbf{Z}_2 = \frac{\mathbf{z}_2}{S_B}$. The acoustic impedance $\mathbf{Z}_1$ and the specific acoustic impedance $\mathbf{z}_1$ at Point 1 are then respectively given by $\mathbf{Z}_1 = \mathbf{Z}_2$ and $\mathbf{z}_1 = \mathbf{Z}_1 S_A$.

The specific acoustic impedances at Points 4 to 1 have been established, and it is now possible by use of equation (A7) to find the acoustic pressure at these points in response to an incident wave, which has a pressure amplitude of unity. The complex representation of the acoustic pressure at Point 1 needs to account for the acoustic pressures associated with both the incident and reflected waves. Point 1 corresponds to the inlet of the expansion chamber at $x = 0$, at which the specific acoustic impedance and complex representation of the acoustic pressure are respectively given by:

$$\mathbf{z}_4 = \rho c\left(\frac{\mathbf{P}_i + \mathbf{P}_r}{\mathbf{P}_i - \mathbf{P}_r}\right) \qquad (2)$$

$$\mathbf{P}_1 = \mathbf{P}_i + \mathbf{P}_r \qquad (3)$$

Equations (2) and (3) can be rearranged to give the complex representation of the acoustic pressure at Point 1 by:

$$\mathbf{P}_1 = \frac{2\mathbf{P}_i}{1 + \frac{\rho c}{\mathbf{z}_1}} \qquad (4)$$

If the amplitude of the incident wave is unity, then the complex representation of the acoustic pressure at Point 1 becomes $\mathbf{P}_1 = \frac{2}{1 + \frac{\rho c}{\mathbf{z}_1}}$. Equality of the acoustic pressures at Points 1 and 2 leads to $\mathbf{P}_1 = \mathbf{P}_2$. Equation (A7) can then be used to determine the complex representation of the acoustic pressure at Point 3, $\mathbf{P}_3$, from that at Point 2. The equality of the acoustic pressures at Points 3 and 4 leads to $\mathbf{P}_3 = \mathbf{P}_4$. The modulus of $\mathbf{P}_4$, P_4, is the required result and is equal to P_t shown in Figure 1.

Figure 2 shows the results of the acoustic analysis of a simple expansion chamber muffler. The muffler dimensions are $L = 540$ mm, $d_A = d_C = 52$ mm, $d_B = 104$, 156 and 208 mm. It can be seen that higher transmission losses can be achieved by increasing the area ratio.

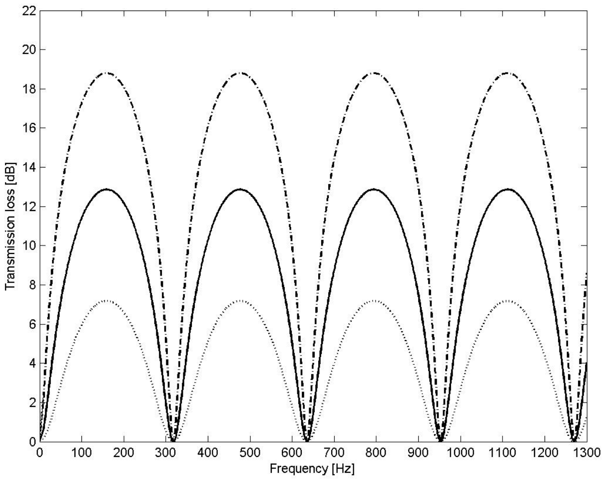

Figure 2. *TL* curves for different diameter ratios: $S_B / S_A = 4$ (dotted), $S_B / S_A = 9$ (solid), $S_B / S_A = 16$ (dash-dot).

Application to a muffler containing porous material and with extended inlet and outlet pipes

The principles applied to the simple expansion chamber muffler can be readily extended to a more complicated muffler such as the device shown in Figure 3, which contains porous material and has extended inlet and outlet pipes. As before, at each frequency the specific acoustic impedance and the acoustic impedance at various points through the muffler are initially found. The acoustic pressures at these points are then found in response to the incident wave, which is given a pressure amplitude of unity. The amplitude of the acoustic pressure in the outlet pipe of the muffler is the quantity of interest.

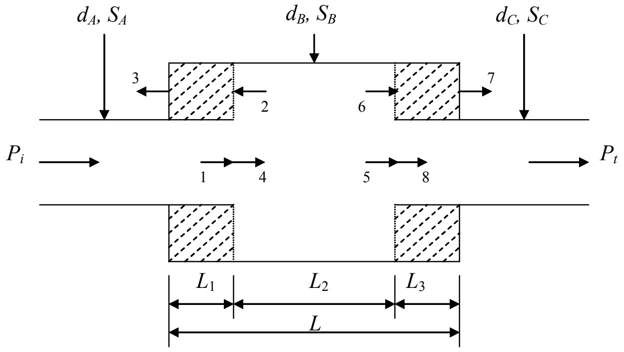

Figure 3. Expansion chamber muffler with extended inlet/outlet pipes and porous material.

At Point 8 on Figure 3, the specific acoustic impedance in the direction shown, $\mathbf{z}_8$, is equal to the characteristic impedance ρc as the outlet pipe is considered to be anechoically terminated. The acoustic impedance at Point 8 is then given by $\mathbf{Z}_8 = \dfrac{\mathbf{z}_8}{S_C}$. The specific acoustic impedance at Point 7 is infinite. Substituting this result in equation (B3), the specific acoustic impedance at Point 6, $\mathbf{z}_6$, can be found. The acoustic impedance at Point 6 is then given by $\mathbf{Z}_6 = \dfrac{\mathbf{z}_6}{S_B - S_C}$. The acoustic impedance at Points 6 and 8 form parallel impedances and so the acoustic impedance at Point 5, $\mathbf{Z}_5$, is given by $\mathbf{Z}_5 = \dfrac{\mathbf{Z}_6\mathbf{Z}_8}{\mathbf{Z}_6 + \mathbf{Z}_8}$. The specific acoustic impedance at Point 5 is then $\mathbf{z}_5 = \mathbf{Z}_5 S_B$. Equation (A8) can be used to determine the specific acoustic impedance at Point 4, $\mathbf{z}_4$, from that at Point 5. The acoustic impedance at Point 4 can then be found from $\mathbf{Z}_4 = \dfrac{\mathbf{z}_4}{S_B}$. The same procedure as that used for finding the acoustic impedance at Point 6, can be used to find $\mathbf{Z}_2$. The acoustic impedances at Points 4 and 2 form parallel impedances, and so $\mathbf{Z}_1 = \dfrac{\mathbf{Z}_4\mathbf{Z}_2}{\mathbf{Z}_4 + \mathbf{Z}_2}$. The specific acoustic impedance at Point 1 is then $\mathbf{z}_1 = \mathbf{Z}_1 S_A$.

The specific acoustic impedances at Points 1 to 8 have been established. It is now possible to find the acoustic pressures at these points in response to an incident wave, which has a pressure amplitude of unity. When the incident wave has a pressure amplitude of unity, the complex representation of the acoustic pressure at Point 1 is given by $\mathbf{P}_1 = \dfrac{2}{1 + \dfrac{\rho c}{\mathbf{z}_1}}$. The equality of the acoustic pressures at Points 1 and 4 leads to $\mathbf{P}_1 = \mathbf{P}_4$. The complex representation of the acoustic pressure at Point 5, $\mathbf{P}_5$, can be found from that at Point 4 by use of equation (A7). The equality of the acoustic pressures at points 8 and 5 leads to $\mathbf{P}_8 = \mathbf{P}_5$. The modulus of $\mathbf{P}_8$, P_8, is the required result, and is equal to P_t shown in Figure 3.

An example of the transmission loss for a muffler computed by the previously described procedure is given in Figure 4. The porous material is assumed to have a flow resistivity, R_1 of 40,000 rayls/m and the wave propagation associated with this material is described by equations (B1) and (B2). These equations have been developed from what is probably the simplest model to describe the propagation of acoustic waves in gases in porous materials.

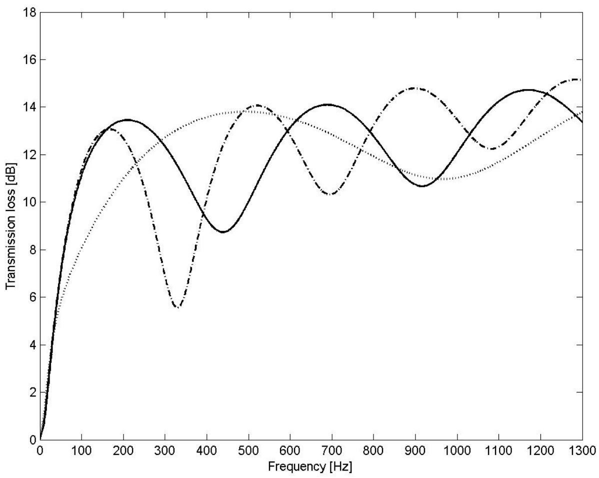

Figure 4. *TL* curves for a muffler containing porous material and extended inlet and outlet pipes. $L = 540$ mm, $d_A = d_C = 52$ mm, $d_B = 156$ mm, $L_1 = L_3 = 50$ mm (dash-dot), $L_1 = L_3 = 100$ mm (solid), $L_1 = L_3 = 200$ mm (dotted).

EXPERIMENTAL VERIFICATION

It has been shown in the previous section that a simple procedure can be used to compute the transmission loss of simple expansion chamber mufflers. An obvious question is how accurate are the predictions. In this section an experimental technique to measure transmission loss is described (Seybert and Ross 1977), and the results it produces are compared with predicted results.

Measurement Principle

Figure 5 shows a block diagram of the arrangement used here to measure the transmission loss of the muffler. A sound source formed by a group of horn drivers was used to generate a transient acoustic pulse which propagated down the inlet pipe and whose pressure was measured by Microphone M_1. The pressure of the acoustic pulse emerging from the muffler was measured by Microphone M_2. The portions of the time histories measured by the two microphones associated with the initial positive travelling waves were selected by rectangular windowing of the total time histories. These windowed time histories were then Fourier Transformed and the transmission loss was then calculated using equation (5).

$$TL = 10 \log \frac{\mathrm{FFT}_1}{\mathrm{FFT}_2} \tag{5}$$

where FFT_1 and FFT_2 are the FFT results from M_1 and M_2, respectively.

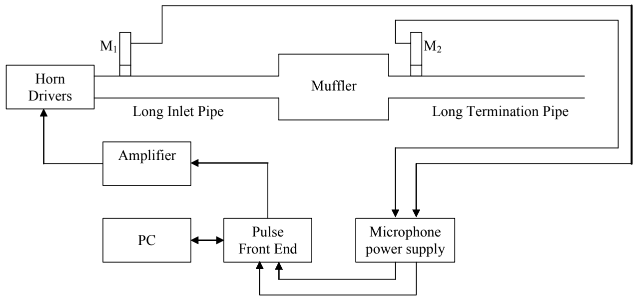

Figure 5. Block diagram of an experimental rig to measure the transmission loss of an expansion chamber.

Experimental Details

The mufflers were manufactured from PVC pipe and aluminium discs. The required long pipes attached to the inlet and outlet of the muffler were made from 50mm nominal diameter plastic pipe. Photographs of a typical muffler and the long pipe to which the sound source was attached are shown in Figures 6 and 7.

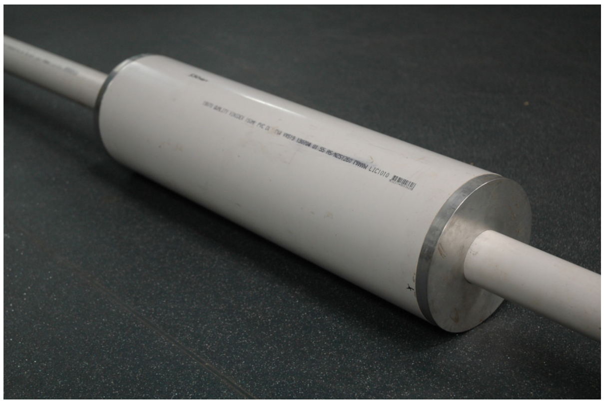

Figure 6. Experimental rig set-up.

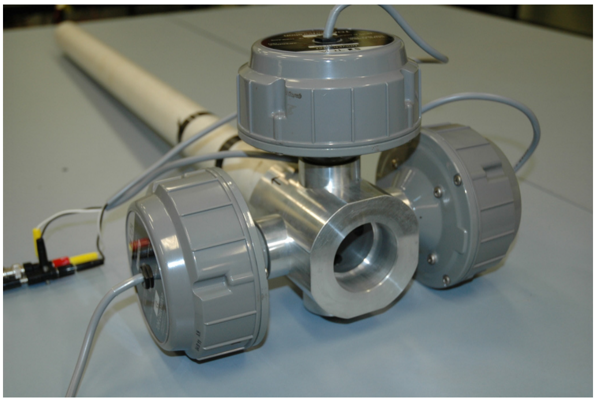

Figure 7. The three horn drivers used in the experiments.

The electrical pulse which was fed to the horn drivers was a single, full sinusoid pulse with a duration of 0.3ms. This duration was selected on the basis of the frequency at which the first non-plane wave could exist in the expansion chamber. The cut-off frequency for a circular pipe of diameter d is given by $f_{co} = 0.586c / d$, where c is the sonic velocity (Morse and Ingard 1968).

An example of the time histories of the pressures measured by Microphones M_1 and M_2 is shown in Figure 8. The decaying pulse train evident in the pressure time history of the outlet pressure is associated with reflections within the expansion chamber. Testing revealed that it was necessary to include more than fourteen reflections from the expansion chamber in the pressure time history of Microphone M_2. The outlet pipe had to be at least seven times as long as the expansion chamber so that Microphone M_2 did not sense waves reflected from the open end of the outlet.

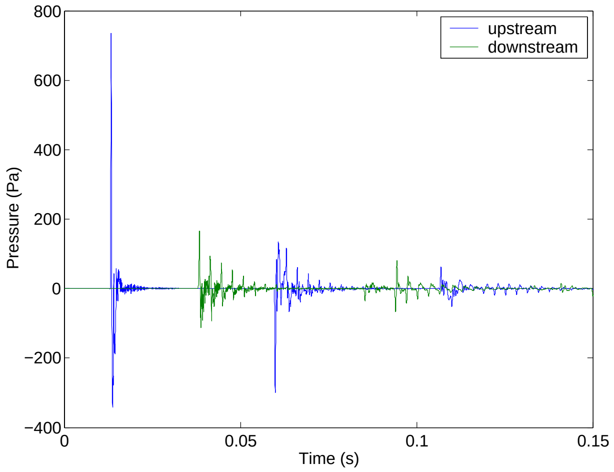

Figure 8. Recorded time histories of the upstream and downstream signals.

The modulii of the Fourier Transforms of the windowed time histories shown in Figure 8 are shown in Figure 9.

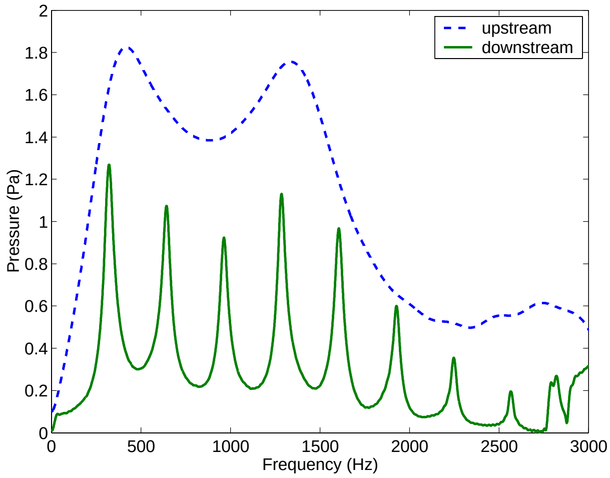

Figure 9. Upstream and downstream FFT signals.

EXAMPLES OF PREDICTED AND MEASURED RESULTS

Experimental results are presented for a simple expansion chamber muffler, and an expansion chamber muffler with extended inlet and outlet pipes.

Simple expansion chamber muffler

Figure 10 shows the results of the acoustic analysis of a simple expansion chamber muffler of the type shown in Figure 1. The muffler dimensions are $L = 540$ mm, $d_A = d_C = 52$mm and $d_B = 156$ mm. The results obtained using the transmission line theory approach are shown as are the experimental results.

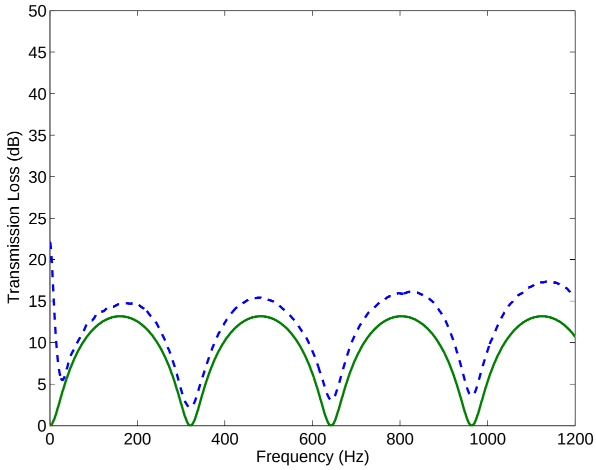

Figure 10. Comparison of *TL* curves obtained using transmission line theory (solid) and experimentally (dashed) for a simple expansion chamber.

Figures 11 and 12 present analytical and experimental results for an expansion chamber muffler in which the inlet and outlet pipes have the same extended lengths. The overall dimensions of the muffler are the same as for the simple expansion chamber. The inlet and outlet pipes extend 100 mm (Figure 11) and 200 mm (Figure 12) into the expansion chamber. As expected, by increasing the extended length, the maximum transmission loss peak is shifted to a lower frequency. There is reasonable agreement between the transmission line and experimental results.

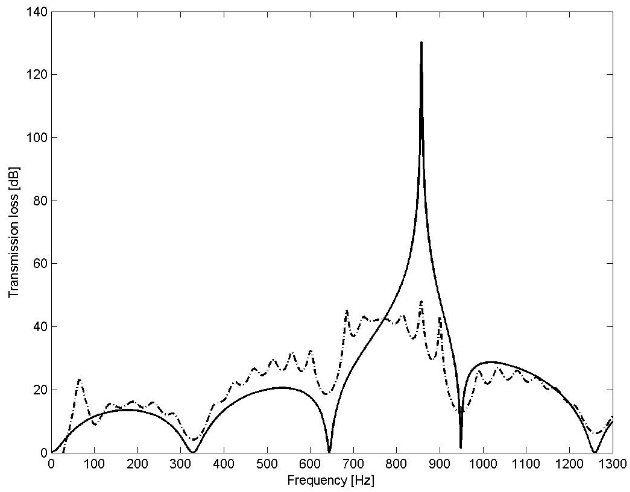

Figure 11. *TL* curves for 100 mm extended inlet/outlet pipes: analytical results (solid), experimental results (dash-dot).

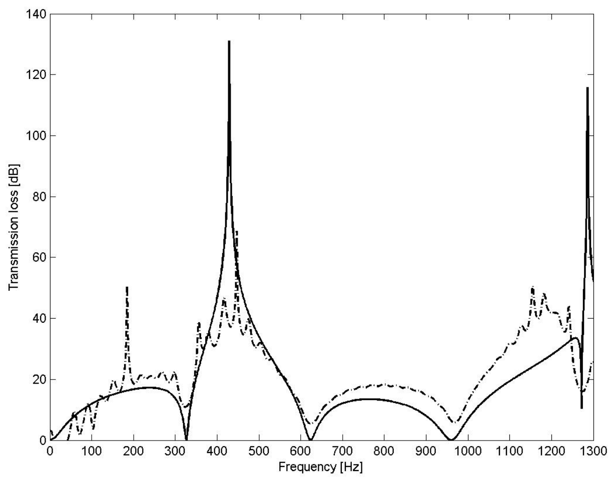

Figure 12. *TL* curves for 200 mm extended inlet/outlet pipes: analytical results (solid), experimental results (dash-dot).

CONCLUSIONS

It has been shown in this paper how the transmission line approach can be easily applied to make predictions of the acoustic performance of simple expansion mufflers. Further, the results have been shown to provide reasonable agreement with measured results. Thus the transmission line approach provides a low cost tool to aid in the design of these mufflers.

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APPENDIX A

Impedance and pressure formulae for undamped harmonic plane waves in a gas column

A column of gas, which has a density of ρ and a sonic velocity of c, is terminated at the end $x = L$ by an impedance, which has a specific acoustic impedance of $\mathbf{z}_t$ as shown in Figure A1. The impedance formula for undamped harmonic plane acoustic waves in a gas column allows the specific acoustic impedance $\mathbf{z}_i$ of the inlet impedance to be found in terms of the characteristic impedance of the gas, the length of the column, and the specific acoustic impedance of the terminating impedance.

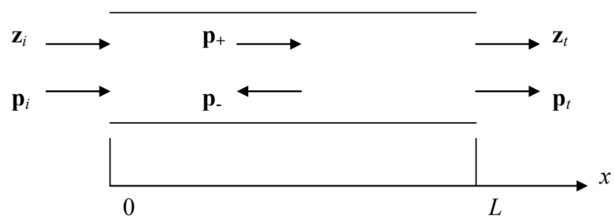

Figure A1. Model for developing the impedance and pressure formulae for undamped waves in a gas column.

The acoustic pressures associated with positive and negative travelling one dimensional harmonic plane acoustic waves of angular frequency ω and wave number $k = \omega / c$ have the following complex representations:

$$\mathbf{p}_+ = \mathbf{P}_+ e^{j(\omega t - kx)} \tag{A1}$$

$$\mathbf{p}_- = \mathbf{P}_- e^{j(\omega t + kx)} . \tag{A2}$$

The amplitude and phase information has been grouped as $\mathbf{P} = Pe^{j\phi}$. The corresponding particle velocities $\mathbf{u}_+$ and $\mathbf{u}_-$, associated with these positive and negative travelling waves, are related to the acoustic pressures $\mathbf{p}_+$ and $\mathbf{p}_-$ by the characteristic impedance ρc . The specific acoustic impedance is defined as the ratio of the complex representation of the acoustic pressure $\mathbf{p}$ to the complex representation of the particle velocity $\mathbf{u}$. Figure A1 shows the specific acoustic impedance and complex representation of the harmonic acoustic pressure, at $x = 0$ ($\mathbf{z}_i , \mathbf{p}_i$) and at $x = L$ ($\mathbf{z}_t , \mathbf{p}_t$). At $x = 0$ and $x = L$, the specific acoustic impedances $\mathbf{z}_i$ and $\mathbf{z}_t$ are then respectively given by:

$$\mathbf{z}_i = \rho c \left(\frac{\mathbf{P}_+ + \mathbf{P}_-}{\mathbf{P}_+ - \mathbf{P}_-} \right) \tag{A3}$$

$$\mathbf{z}_t = \rho c \left(\frac{\mathbf{P}_+ e^{-jkL} + \mathbf{P}_- e^{jkL}}{\mathbf{P}_+ e^{-jkL} - \mathbf{P}_- e^{jkL}} \right) \tag{A4}$$

The amplitudes of the inlet pressure and the terminating pressure can be expressed in terms of the positive and negative travelling waves by:

$$\mathbf{P}_i = \mathbf{P}_+ + \mathbf{P}_- \tag{A5}$$

$$\mathbf{P}_t = \mathbf{P}_+ e^{-jkL} + \mathbf{P}_- e^{jkL} \tag{A6}$$

Substituting equations (A3) and (A5) into (A6) yields an expression for the complex terminating pressure amplitude $\mathbf{p}_t$ in terms of the inlet pressure and inlet specific acoustic impedance:

$$\mathbf{P}_t = \frac{\mathbf{P}_i}{2}\left(\left(1+\frac{\rho c}{\mathbf{z}_i}\right)e^{-jkL} + \left(1-\frac{\rho c}{\mathbf{z}_i}\right)e^{jkL}\right) \tag{A7}$$

where $\mathbf{p}_t = \mathbf{P}_t e^{j(\omega t - kx)}$. Using transmission line theory, an expression for the specific acoustic impedance at the inlet in terms of the terminating impedance can be found. It is given by:

$$\mathbf{z}_i = \rho c\left(\frac{\left(1+\frac{\rho c}{\mathbf{z}_t}\right)e^{jkL} + \left(1-\frac{\rho c}{\mathbf{z}_t}\right)e^{-jkL}}{\left(1+\frac{\rho c}{\mathbf{z}_t}\right)e^{jkL} - \left(1-\frac{\rho c}{\mathbf{z}_t}\right)e^{-jkL}}\right) \tag{A8}$$

APPENDIX B

Impedance and pressure formulae for damped harmonic plane waves in a gas in a column of porous material

Porous material such as mineral wool, fibreglass and sintered materials are widely used for noise control. Wave propagation in gas contained in such materials is characterised by the complex wave number $\mathbf{k}$ and the complex characteristic impedance $\mathbf{Z}_0$. The simplest model for estimating these quantities involves the flow resistivity of the material R_1. Consider a layer of porous material of thickness *t*. The flow resistivity R_1 is the specific flow resistance per unit thickness of the material, and is given by $R_1 = R_f / t$. The specific flow resistance is defined by $R_f = \Delta p / u$, that is, the pressure drop Δp produced when the gas of interest flows with a nominal velocity *u* through a layer of the material. The complex wavenumber is given by:

$$\mathbf{k} = k(1 - jR_1/\rho\omega)^{1/2} \tag{B1}$$

The characteristic impedance $\mathbf{Z}_0$ is the ratio of the complex representation of the acoustic pressure to the particle velocity, and is given by:

$$\mathbf{Z}_0 = \rho c(1 - jR_1/\rho\omega)^{1/2} \tag{B2}$$

An expression for the complex terminating pressure can be found using a procedure which is identical to that used for the undamped case. Equation (B3) is the result.

$$\mathbf{P}_t = \frac{\mathbf{P}_i}{2}\left(\left(1+\frac{\mathbf{Z}_0}{\mathbf{z}_i}\right)e^{-j\mathbf{k}L} + \left(1-\frac{\mathbf{Z}_0}{\mathbf{z}_i}\right)e^{j\mathbf{k}L}\right) \tag{B3}$$

The impedance formula can be found by a procedure which is identical to that described in the derivation of the impedance formula for undamped waves. Equation (B4) is the result.

$$\mathbf{z}_i = \mathbf{Z}_0\left(\frac{\left(1+\frac{\mathbf{Z}_0}{\mathbf{z}_t}\right)e^{j\mathbf{k}L} + \left(1-\frac{\mathbf{Z}_0}{\mathbf{z}_t}\right)e^{-j\mathbf{k}L}}{\left(1+\frac{\mathbf{Z}_0}{\mathbf{z}_t}\right)e^{j\mathbf{k}L} - \left(1-\frac{\mathbf{Z}_0}{\mathbf{z}_t}\right)e^{-j\mathbf{k}L}}\right) \tag{B4}$$