Relative performance of different strategies for wave attenuation by periodic structures

Darryl McMahon

Curtin University, Centre for Marine Science and Technology, Bentley, WA 6102, Australia

ABSTRACT
For a periodic structure of symmetric scatterers, such as a ribbed plate or cylinder, Bloch-Floquet waves (BFW) are well-known periodic structure waves (PSW). Conventional wisdom is to attenuate BFW propagation by reflections, energy absorption or stopping bands. Recent theoretical results show that periodic structures with asymmetric scatterers do not lead to BFW, revealing more possibilities for noise reduction through these different PSW. This paper compares the performance of reciprocal and nonreciprocal wave propagation as methods for PSW attenuation. It is found that nonreciprocal wave propagation, such as could be realized with asymmetric metamaterial scatterers, could achieve significantly better attenuation with relatively small deviations from symmetric scatterer properties.

1 INTRODUCTION

Previous papers showed that three types of PSW are possible in an infinite 1D periodic structure of idealized point scatterers (McMahon, 2016a, 2016b, 2017). Scattering theory is only accurate for periodic structures that are sufficiently sparse (i.e. spacing *d* between adjacent scatterers much larger than scatterer thickness). In contrast, many of the periodic structures analysed in the literature are not sparse, such as composites of periodic layers of different materials that are closely packed (Luan and Ye, 2001). Even further from a sparse system are continuous periodic domains where solutions are sought for periodic wave equations (PWE). Well known PWE are Mathieu's and Hill's equations (Brillouin, 1953). More recently Heun's equation has been solved to model waves in continuous periodic media (Bednarik and Cervenka, 2017). An increasing number of Fourier expansion terms in Hill's equation to model highly localized and widely spaced periodic inhomogeneities give PSW solutions the same as scattering theory, although scattering theory is more general since it is not constrained to scattering amplitudes from a superposition of Fourier terms related to spacing *d*.

A significant difference between solving PWE and the scattering approximation is that the former PSW are continuous whereas PSW from the scattering approximation vary due to two position independent complex factors $\gamma^{(\pm)}$ relating waves in adjacent "cells". Structure waves (SW) travelling in a uniform continuous medium between pairs of scatterers bookending the cells correlate with SW in adjacent cells, as quantified by $\gamma^{(\pm)}$, through backward and forward scattering. Consequently up to two different PSW modes are simultaneously possible for each propagation direction (McMahon, 2016a, 2016b, 2017).

Despite the limitations, scattering theory for an infinite structure is a useful approximation for novel periodic systems, such as asymmetric and nonreciprocal periodic structures, because these PSW properties have been derived analytically (McMahon, 2017). Scattering theory predicts features of PSW that may be more complicated to derive from PWE or other techniques appropriate to a particular periodic structure. An example is the effect of nonreciprocity for an asymmetric periodic structure. Nonreciprocity occurs in nonlinear media and spatiotemporal dependent media (Fleury et al, 2016). Suppose a lossless periodic structure has passing and stopping bands at different wavenumbers for opposite propagation directions. Then the passing bands for one direction overlaps the stopping bands for the opposite direction, thereby creating wave diodes in their overlap bands. This wave diode effect exists for a spatiotemporal periodic structure generated by a modulation wave (Trainiti and Ruzzene, 2016). McMahon 2017 showed that nonreciprocal point scatterers also give rise to a wave diode but in a single overlap band of infinite bandwidth.

The three possible types of PSW modes depend on scatterer symmetry properties. BFW are theoretically and experimentally well-known PSW that exhibit, for no energy losses, an infinite number of passing and stopping bands of finite bandwidth. BFW arise from symmetric scatterers where the difference of backward and forward phase shifts are $\delta = \pm\pi/2$. One might think that asymmetric scatterers for which $\delta \neq \pm\pi/2$ is just a simple extension to the theory for BFW. However PSW cell independence and $\delta \neq \pm\pi/2$ conflict with conservation of energy (CoE) (McMahon, 2015). Instead, if the magnitudes of the reflection and transmission coefficients are the same in both directions, $\delta \neq \pm\pi/2$ and CoE give rise to two modes (dubbed non-BFW modes) that unlike BFW have a single infinite bandwidth. For elastic scattering, one non-BFW is a passing mode and the other is a stopping mode (McMahon, 2016a, 2016b, 2017). This is an apparent discontinuous transition from one BFW mode to two non-BFW modes at the value $\delta \to \pm\pi/2$ but is qualitatively understandable for elastic scattering as randomization of PSW phase by a uniform distribution of scatterings with $\delta \neq \pm\pi/2$. Subsect. 4.2 argues that for $\delta \to \pm\pi/2$, BFW are not independent of the two non-BFW modes but are a superposition of them that avoids the discontinuity by interpreting the oscillation of BFW between finite passing and stopping bandwidths with SW wavenumber in a way consistent with infinite bandwidths for point scatterers.

If the magnitudes of the reflection and transmission coefficients are different for opposite SW directions, only one PSW mode exists for each direction, and in the case of elastic scattering a passing mode in one direction and a stopping mode in the opposite direction (McMahon, 2016a, 2016b, 2017). This is the abovementioned wave diode from nonreciprocal scattering for which the PSW solution is unique and corresponds to the incoherent energy wave (IEW) model (McMahon, 2017).

To demonstrate the potential advantages of nonreciprocal scattering for noise control, Sect. 4 compares the attenuation of the three types of PSW for similar magnitude scattering parameters and energy absorption. This shows that PSW modes for asymmetric scatterers have practical implications for noise reduction. For instance, metamaterial ribs could realize a wave diode where noise is much more attenuated in one propagation direction than the other, even for a small deviation from symmetry in the case of a large number of ribs.

2 PERIODIC STRUCTURE WAVES

2.1 Characteristic equation and conservation of energy for PSW

In the case of an infinite periodic structure the PSW are defined to be identical in all cells since there are no boundaries and hence no boundary effects. Figure 1 shows how a PSW mode arises from equally spaced and generally asymmetric scatterers spread along the *x* axis. The SW between any two adjacent scatterers (a cell) has a waveform $e^{ik_s x}A^{(+)}$ for the $+k_s$ wavenumber and $e^{-ik_s x}B^{(-)}$ for the $-k_s$ wavenumber. In the next cell, $A^{(+)}$ is modified to $A^{(+)}\gamma^{(+)}$ and $B^{(-)}$ is modified to $B^{(-)}\gamma^{(-)}$. The two functions $\gamma^{(\pm)}$ are waveforms of two PSW generally corresponding to opposite propagation directions. Because $\gamma^{(+)}$ and $\gamma^{(-)}$ are correlated, the pair together define a PSW mode. The derivation of a "characteristic" equation (CE) for $\gamma^{(\pm)}$ in the case of an infinite periodic structure only needs to consider continuity at the scatterer bordering two adjacent cells (McMahon, 2017). $\gamma^{(+)}$ and $\gamma^{(-)}$ are linked in the CE by scattering defined by complex reflection and transmission coefficients $R^{(\pm)}$ and $T^{(\pm)}$ respectively. $R^{(\pm)}$ and $T^{(\pm)}$ separate out phase shifts and energy absorption by defining

$$
\begin{aligned}
R^{(\pm)} &= \sigma^{(\pm)}\left|R_0^{(\pm)}\right|e^{i\chi^{(\pm)}} \\
T^{(\pm)} &= \sigma^{(\pm)}\left|T_0^{(\pm)}\right|e^{i\phi^{(\pm)}}
\end{aligned}
\tag{1}
$$

$\chi^{(\pm)}$ and $\phi^{(\pm)}$ are the reflection and transmission phase shifts for the two SW wavenumbers $\pm k_s$. Energy absorption by the scatterers is taken into account by $\sigma^{(\pm)}$ where $1-\sigma^{(+)2}$ is the fraction of energy a scatterer absorbs from a $+k_s$ wavenumber SW and $1-\sigma^{(-)2}$ is the fraction of energy a scatterer absorbs from a $-k_s$ wavenumber SW. Details of the CE derivation are in previous papers (McMahon, 2015, 2016a, 2017). Note the normalization property $\left|R_0^{(\pm)}\right|^2 + \left|T_0^{(\pm)}\right|^2 = 1$. The scattering parameters are generally wavenumber dependent but is left implicit.

CoE is a fundamental requirement for all acceptable PSW solutions of the CE. CoE is derived in terms of PSW parameters by considering the time-averaged energy fluxes towards and away from any scatterer. Omitting the SW phase speed factor because it cancels out of flux equations, the $+k_s$ wavenumber fluxes each side of a scatterer are $\left|A^{(+)}\right|^2$ and $\left|A^{(+)}\right|^2\left|\gamma^{(+)}\right|^2$ whereas for the $-k_s$ wavenumber these fluxes are $\left|B^{(-)}\right|^2$ and $\left|B^{(-)}\right|^2\left|\gamma^{(-)}\right|^2$. CoE is depicted in Fig. 2 where the green arrows show the fluxes incident onto the scatterer and the red arrows are the fluxes reflected or forward transmitted by the scatterer. The equation on the top RHS is CoE and shows how energy absorption is taken into account by the factors $\sigma^{(\pm)2}$ multiplying the green arrow fluxes. CoE leads to a relationship between the reflectivity and "persistence" which, in Fig. 2 for the case of an energy source $x \to -\infty$, are denoted $\overline{\mu}^{(+)}$ and $\overline{\xi}^{(+)}$ respectively. For an energy source $x \to +\infty$ the relevant reflectivity and persistence are $\overline{\mu}^{(-)}$ and $\overline{\xi}^{(-)}$. For the most general case of an asymmetric periodic structure $\overline{\mu}^{(+)} \neq \overline{\mu}^{(-)}$ and $\overline{\xi}^{(+)} \neq \overline{\xi}^{(-)}$ representing asymmetric or nonreciprocal wave energy propagation.

3 DIFFERENT TYPES OF PSW FOR AN INFINITE 1D PERIODIC STRUCTURE

A key parameter determining the solutions of the CE and different types of PSW is the scattering phase shift difference δ given by (McMahon 2016a, 2017)

$$\delta = \frac{1}{2}\left(\left(\chi^{(-)} + \chi^{(+)}\right) - \left(\phi^{(-)} + \phi^{(+)}\right)\right) \tag{2}$$

It is also important to distinguish symmetric and asymmetric scattering coefficients. A scatterer is fully symmetric if $\left|R_0^{(+)}\right| = \left|R_0^{(-)}\right| \equiv \left|R_0\right|$, $\left|T_0^{(+)}\right| = \left|T_0^{(-)}\right| \equiv \left|T_0\right|$ and $\delta = \pm\pi/2$. Two versions of scatterer asymmetry are defined. One has symmetric moduli $\left|R_0\right|$ and $\left|T_0\right|$ but asymmetric scattering phase shifts where $\delta \neq \pm\pi/2$. The other asymmetry includes asymmetric moduli $\left|R_0^{(+)}\right| \neq \left|R_0^{(-)}\right|$ and $\left|T_0^{(+)}\right| \neq \left|T_0^{(-)}\right|$.

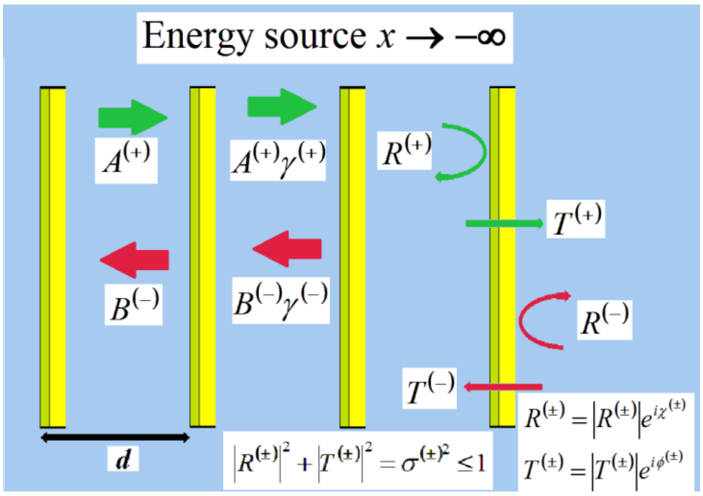

Figure 1: Parameters that define the characteristic equation for the PSW and SW of a periodic structure.

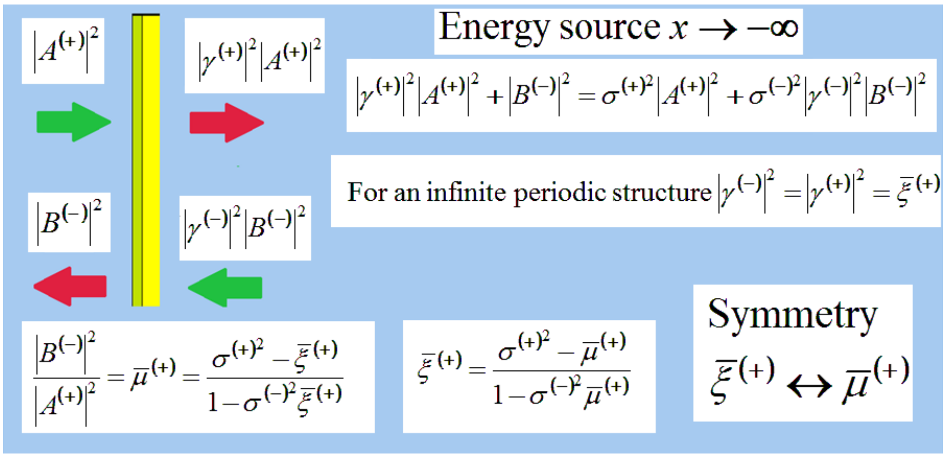

Figure 2: Parameters that define conservation of energy for PSW and SW of an infinite periodic structure.

An example of asymmetric scattering phase shifts can be demonstrated with scattering by a noncentric composite of two refractive media immersed in a third refractive medium. Text book acoustic theory can be applied to this case (Kinsler et al, 1999) and Fig. 3 illustrates the geometry of such a scatterer. For no energy absorption, the phase shifts have the form

$$\begin{aligned} &\phi^{(\pm)} = \pm\phi \\ &\chi^{(+)} = \phi + \varepsilon, \chi^{(-)} = -\phi + \varepsilon + (\pm)\pi \\ &\delta = \frac{1}{2}\left(\chi^{(+)} + \chi^{(-)} - \phi^{(+)} - \phi^{(-)}\right) = \varepsilon + (\pm)\frac{\pi}{2} \end{aligned} \quad (3)$$

Equation (3) shows that the deviation from symmetry gives a contribution to the reflection phase shift $\varepsilon \neq 0$ that does not change sign for opposite SW travel directions. Figure 4 plots δ/π versus dimensionless wavenumber $k_s b/2\pi$ where the impedances are defined only by the phase speeds (i.e. all media have the same density). Curve 0 plots the symmetric scatterer case with a single internal medium and shows that δ switches discontinuously from $\pi/2$ to $-\pi/2$ and back at wavenumbers corresponding to integer numbers of half wavelengths within the scatterer. For two different media inside the scatterer δ is a complicated function of SW wavenumber.

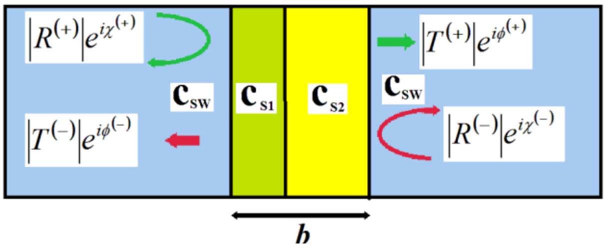

Figure 3: Parameters for a noncentric composite refractive scatterer embedded in a refractive medium. The parameters c_{SW}, c_{S1} and c_{S2} denote the phase speeds of acoustic waves in the media.

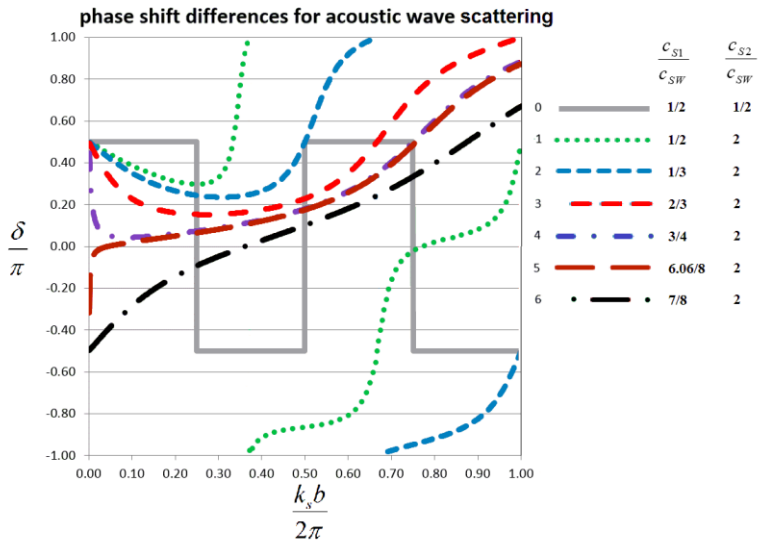

Figure 4: The difference between the backward reflection and forward transmission phase shifts of acoustic waves for a noncentric composite scatterer embedded in a refractive medium.

3.1 Bloch-Floquet and non-Bloch-Floquet waves

For elastic scattering BFW are the result of SW forward and backward scattered any number of times such that the phase shifts δ cannot randomize the BFW phase. Depending on SW wavenumber the effect of δ either creates a travelling wave (passing band) or a standing wave (stopping band) within every cell. Note that an extension of the scattering model for BFW by taking into account energy absorption weakens the distinction of stopping and passing bands (McMahon 2015). In the inelastic case it not correct to refer to different bands as "passing" and "stopping". For clarity the evolved passing and stopping bands are denoted damped passing bands (DPB) and damped stopping bands (DSB). For sufficiently large energy losses, DPB and DSB are indistinguishable. This effect of energy absorption on passing and stopping bands has been observed in earlier studies such as one on damped Bloch waves (Hussein, 2009).

The assumptions of the theory for BFW fail when $\delta \neq \pm\pi/2$ because, although it is a simple extension to the CE, it does not conform to CoE. A solution to this problem is found by assuming that $\delta \neq \pm\pi/2$ requires opposite travelling SW to be coupled by scattering such that $\gamma^{(-)} = e^{-i\psi}\gamma^{(+)}$ where ψ is determined by the solution of the

CE constrained by CoE (McMahon 2016a, 2017). Introducing $\tilde{\gamma} = e^{\mp i\psi/2}\sqrt{T^{(-)}/T^{(+)}}\gamma^{(\pm)}$ the CE is symmetrized which, together with CoE, gives two solutions for ψ. The correlated pair $\gamma^{(\pm)}$ define the BFW mode when $\psi = 0$ and non-BFW modes when $\psi \neq 0$. Table 1 contrasts the BFW and non-BFW with each other.

Table 1: Distinctions between BFW and non-BFW modes.

BFW	**Non-BFW**
Same single mode in both directions $\psi = 0$	Same two modes in both directions $\psi \neq 0$
Symmetric modulus of SW scattering coefficients & symmetric phase shifts $\delta = \pm\pi/2$	Symmetric modulus of SW scattering coefficients but asymmetric phase shifts $\delta \neq \pm\pi/2$
Lossless propagation oscillates with SW wavenumber between passing and stopping bands	For lossless propagation, one mode is a passing band and the other is a stopping band

3.2 Incoherent energy waves (IEW)

The CE for the IEW mode is derived by considering only the energy fluxes back reflected and forward transmitted by the scatterers of an infinite periodic structure (McMahon 2015). The scattering phase shifts need not be explicitly considered although two correlated pairs $\gamma^{(\pm)}$ still need to exist with a phase ψ relating them. The CE for IEW only involves $|\tilde{\gamma}|^2 \equiv \tilde{\xi}$ which in terms of the more general PSW theory coincides with the energy equation (EE) (see Eq. (5)). IEW properties such as periodic structure reflectivity depend on the symmetry of the scattering coefficients. Symmetric scattering IEW results are derived in previous papers (McMahon 2015, 2016a) and asymmetric scattering IEW results are derived in (McMahon 2017). It can be shown that the IEW for symmetric scatterers gives the same results as both non-BFW for $\delta = 0, \pm\pi$. For asymmetric scattering $|R_0^{(+)}| \neq |R_0^{(-)}|$ and $|T_0^{(+)}| \neq |T_0^{(-)}|$, IEW is the only solution that satisfies all the PSW equations of an infinite periodic structure (McMahon 2017). An important conclusion is that an infinite periodic structure of asymmetric (nonreciprocal) elastic scatterers is a wave diode (i.e. energy only propagates in one direction). Table 2 summarises the differences of IEW modes for symmetric and asymmetric scattering.

Table 2: Distinctions between IEW modes for symmetric (reciprocal) and asymmetric (nonreciprocal) scatterers.

IEW - symmetric SW scattering	**IEW - asymmetric SW scattering**
The same single mode in each direction	A different single mode in each direction
For lossless propagation, the mode is a stopping band	For lossless propagation, one mode is a passing band and the other is a stopping band

4 ATTENUATION OF PERIODIC STRUCTURE WAVES

The theory of PSW in periodic structures with asymmetric scatterers (McMahon 2017) defines two parameters pertinent to wave attenuation, persistence $\bar{\xi}^{(\pm)}$ and reflectivity $\bar{\mu}^{(\pm)}$. These parameters are related as depicted in Fig. 2 which from conservation of energy encapsulates that energy penetration down a periodic structure is reduced by reflection and energy absorbed. For $|R_0^{(+)}| \neq |R_0^{(-)}|$ and $|T_0^{(+)}| \neq |T_0^{(-)}|$, we find $\bar{\xi}^{(+)} \neq \bar{\xi}^{(-)}$ and $\bar{\mu}^{(+)} \neq \bar{\mu}^{(-)}$ meaning energy propagation is asymmetric. For an energy source at $x \to -\infty$, across N scatterers the ratio of energy U_N at an arbitrary N^{th} cell to U_0 at the 0th cell is given by

$$\frac{U_N}{U_0} = \bar{\xi}^{(+)N} \tag{4}$$

Evaluation of $\bar{\xi}^{(\pm)}$ requires solving the symmetrized EE which is (McMahon 2017)

$$\tilde{\xi}^2 - 2\tilde{\Delta}\tilde{\xi} + 1 = 0 \tag{5}$$

where $\tilde{\Delta}$ is a function of the scattering parameters. Also

$$\bar{\xi}^{(+)} = \tilde{\xi}|T^{(+)}|/|T^{(-)}| \tag{6a}$$

and

$$\bar{\xi}^{(-)} = \bar{\xi}^{(+)}|T^{(-)}|^2/|T^{(+)}|^2 \tag{6b}$$

Equations (6a, b) take into account asymmetry when $\left|T^{(+)}\right| \neq \left|T^{(-)}\right|$. The reflectivity can be derived from CoE as shown in Fig. 2 and gives

$$\bar{\mu}^{(\pm)} = \frac{\sigma^{(\pm)2} - \bar{\xi}^{(\pm)}}{1 - \sigma^{(\mp)2}\bar{\xi}^{(\pm)}} \tag{7}$$

4.1 Attenuation of BFW energy

Fully symmetric scattering coefficients give rise to BFW that, in practical periodic systems such as ribbed structures, may cause unwanted noise propagation in the passing wave bands. This Subsection considers symmetric scatterers that absorb energy and estimates the energy attenuation in the DPB and DSB.

BFW energy attenuation is accounted for using Eq. (5) with $\tilde{\Delta}$ given by

$$\tilde{\Delta} = \frac{1}{2|T_0|^2}\left[\left(\frac{\left(1-\tilde{\sigma}^2\right)^2}{2\tilde{\sigma}^2}+1\right) + \cos(k_s d) + \sqrt{\left(\frac{\left(1-\tilde{\sigma}^2\right)^2}{2\tilde{\sigma}^2}+1\right)^2 + \left(2|T_0|^2-1\right)^2 - \sin^2(k_s d)}\right] \tag{8}$$

where $\tilde{\sigma}^2 = \sigma^{(+)}\sigma^{(-)}$ takes into account energy absorption that may depend on SW direction. Figure 5 plots three repeating cycles of energy reduction for BFW by 100 scatterers and several values of reflection coefficient. The energy absorption factor per scatterer is 1% which is the sole reason why U_{100}/U_0 peaks at -4.36 dB for $|R|^2 = 0$. Energy is much more attenuated in and near the DSB that only arise when $|R|^2 \neq 0$ however for small $|R|^2 << 1$ DSB are much narrower than DPB and contribute little to the average noise reduction. The attenuation at the centre of the DPB is a reasonable measure of effective attenuation but is a weak function of the reflection coefficient (e.g. $|R|^2 = 0.1$ only increases attenuation to 4.6 dB).

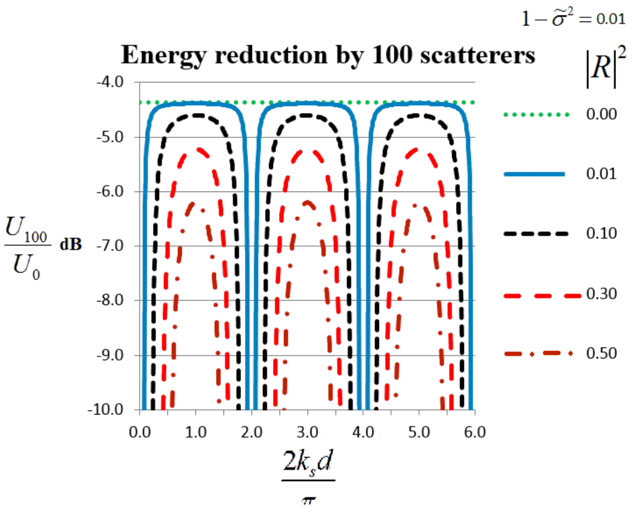

Figure 5: BFW energy attenuation by 100 scatterers for 1% energy absorption and five reflection coefficients.

4.2 Attenuation of non-BFW energy

As discussed in Sect. 3 non-BFW arise from asymmetric scattering phase shifts $\delta \neq \pm\pi/2$ but symmetric moduli of the reflection and transmission coefficients $\left|R_0^{(+)}\right| = \left|R_0^{(-)}\right| \equiv \left|R_0\right|$ and $\left|T_0^{(+)}\right| = \left|T_0^{(-)}\right| \equiv \left|T_0\right|$. Periodic structures with scatterers made of ordinary (non-metamaterial) material but noncentral geometry exhibit reciprocal wave propagation and attenuation but asymmetric phase shifts (see Figs. 3 and 4). There are two non-BFW modes, with one damped passing band mode and the other a damped stopping band mode, in contrast to BFW that have DPB and DSB in a single mode. However in the limit $\delta \to \pm\pi/2$ the two non-BFW modes equate to the centres of the DPB and DSB of BFW. This can be seen from Eq. (8) and the analytical expressions for $\tilde{\Delta}$ of the two non-BFW modes (McMahon 2017):

$$\tilde{\Delta}_{(\pm)} = 1 + \frac{\left(1-\tilde{\sigma}^2\right)^2 + 2\tilde{\sigma}^2\left(1-\tilde{\sigma}^2\cos^2(\delta) - (\pm)\sin(\delta)\sqrt{1-\tilde{\sigma}^4\cos^2(\delta)}\right)|R_0|^2}{2\tilde{\sigma}^2|T_0|^2} \quad (9)$$

Figures 6a and 6b compares attenuation for the BFW damped passing and stopping bands with the non-BFW damped passing and stopping modes. Figure 6a clearly shows that the non-BFW damped passing mode is identical to the centre of the BFW DPB. The largest attenuation occurs for the damped stopping bands and Fig. 6b shows that the non-BFW damped stopping mode is identical to the centre of the BFW DSB. The damped passing bands and modes are of most practical significance since they determine attenuation of the most penetrating waves. For phase shifts close to $\delta = \pm\pi/2$, Fig. 6a shows that attenuation is fairly insensitive to the reflection coefficient but very sensitive to energy absorption.

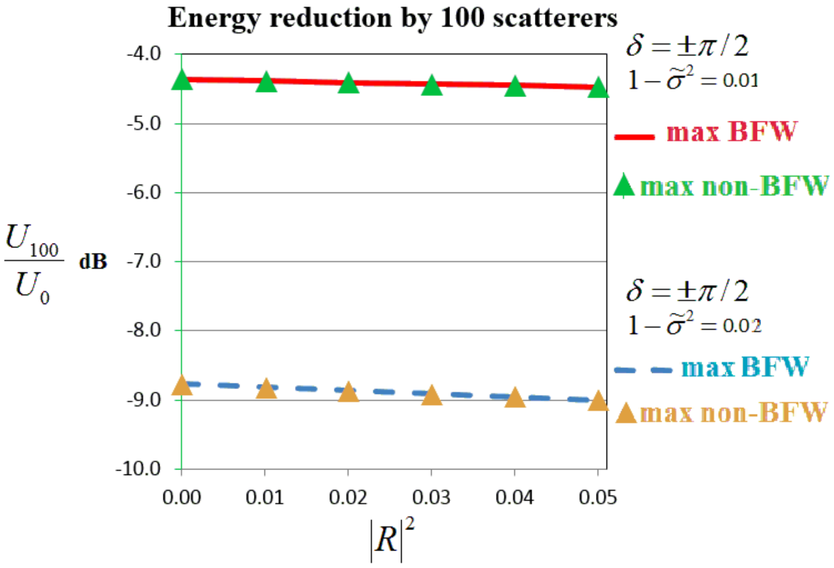

Figure 6a: Comparison of BFW centre DPB and non-BFW DPB energy attenuation by 100 scatterers for 1% & 2% energy absorption for the same phase shift difference $\delta = \pm\pi/2$.

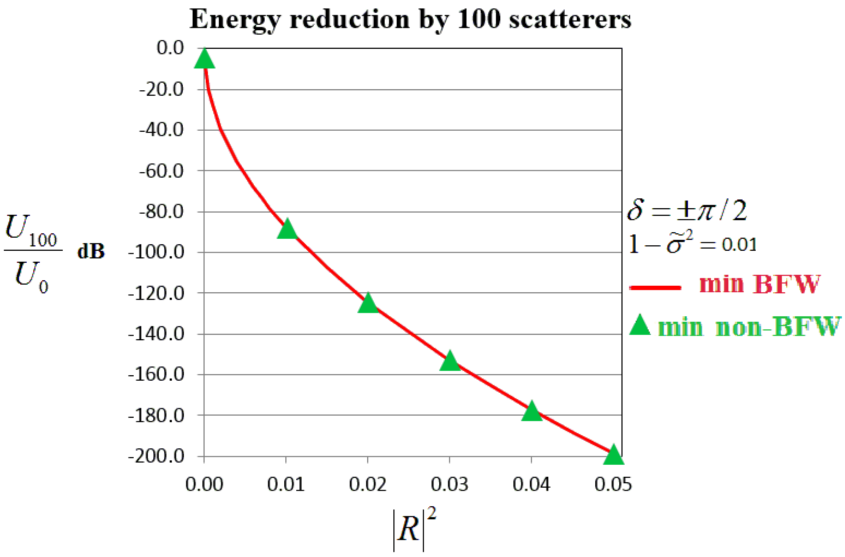

Figure 6b: Comparison of BFW centre DSB and non-BFW DSB energy attenuation by 100 scatterers for 1% energy absorption for the same phase shift difference $\delta = \pm\pi/2$.

The coincidence of $\tilde{\Delta}_{(\pm)}$ for the two non-BFW in the limit $\delta \to \pm\pi/2$ with $\tilde{\Delta}$ at the centre of the DPB and DSB of the BFW suggests that a BFW may be equivalent to a superposition of the two non-BFW modes. If true this would eliminate an apparent discontinuous transition from one BFW mode to two non-BFW modes at the value $\delta = \pm\pi/2$. Such a superposition reinterprets the k_s dependent BFW oscillation between finite width damped passing and stopping bands in a way consistent with infinite PSW bandwidths expected from point scatterers. Then BFW and non-BFW for $\delta \to \pm\pi/2$ are not independent and the BFW k_s dependence comes from the coefficients of a superposition. Further investigation of the superposition concept, and its possible generalization for any $\delta \neq \pm\pi/2$, is needed to provide an actual proof.

Figure 7 demonstrates the effect of deviations of δ from $\pm\pi/2$ for the non-BFW damped passing mode. Attenuation increases strongly for $\delta \to \pm\pi$. It can be shown that non-BFW damped passing and stopping band modes become identical for $\delta = 0, \pm\pi$ and give the same attenuation as IEW for symmetric scattering (McMahon 2017).

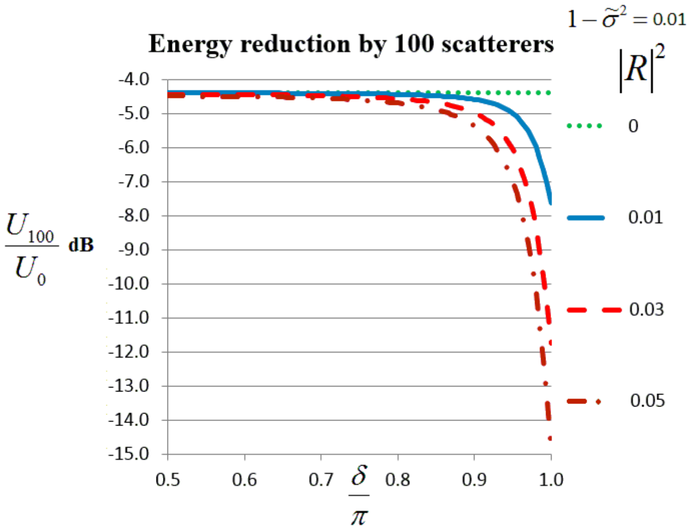

Figure 7: Attenuation of the non-BFW DPB by 100 scatterers with 1% energy absorption.

4.3 Attenuation of IEW energy

Scatterers with asymmetric moduli of the reflection and transmission coefficients do not satisfy the reciprocity properties of linear acoustic wave theory (Kinsler et al, 1999). Acoustical nonreciprocity has been recognised to have applications such as noise control and has led to various concepts for acoustic metamaterials (Fleury et al 2015, Haberman and Norris, 2016). The IEW model is the unique PSW solution for an infinite periodic structure of asymmetric scatterers (McMahon 2017). Calculations of energy attenuation for asymmetric periodic structures can be made using Eqs. (5) and (6a, b) with $\tilde{\Delta}$ given by

$$\tilde{\Delta} = 1 + \frac{\left(1-\tilde{\sigma}^2\right)^2 + 2\tilde{\sigma}^2\left(1-\tilde{\sigma}^2\right)\left(1-\left|T_0^{(+)}\right|\left|T_0^{(-)}\right|\right) + \tilde{\sigma}^4\left(\left|T_0^{(+)}\right| - \left|T_0^{(-)}\right|\right)^2}{2\tilde{\sigma}^2\left|T_0^{(+)}\right|\left|T_0^{(-)}\right|} \qquad (10)$$

Figure 8 demonstrates the wave diode effect of an asymmetric periodic structure with 0.0001% energy absorption by the scatterers. For an energy source at $x \to -\infty$ the net energy flux in the positive *x* direction is proportional to $\overline{F}^{(+)} = 1 - \overline{\mu}^{(+)}$, and for a source at $x \to +\infty$ the flux is proportional to $\overline{F}^{(-)} = 1 - \overline{\mu}^{(-)}$. For a single asymmetric scatterer these fluxes are proportional to $\overline{F}^{(+)} = 1 - \left|R^{(+)}\right|^2$ and $\overline{F}^{(-)} = 1 - \left|R^{(-)}\right|^2$. A measure of the asymmetry is the ratio $\left|T_0^{(+)}\right| / \left|T_0^{(-)}\right|$ of transmission coefficients for opposite direction SW wavenumbers. Figure 8 plots $\overline{F}^{(+)} / \overline{F}^{(-)} \leq 1$ for three different $\left|T_0^{(+)}\right| / \left|T_0^{(-)}\right| < 1$, and shows that energy travelling in the +*x* direction is more strongly reflected than for the -*x* direction. A small $\left|R^{(-)}\right|$ exhibits a much stronger diode effect for a periodic structure than a single scatterer, but the periodic structure diode effect weakens as $\left|R^{(-)}\right|$ increases towards 1.0. Even a very small scatterer asymmetry is able to create a large periodic structure diode effect for $\left|R^{(-)}\right|$ sufficiently small.

Figure 9 repeats the diode effect comparison of asymmetric periodic structure and single scatterer but with 1% energy absorption. Even this small amount of energy absorption dramatically diminishes the diode effect of a periodic structure although it is still larger than for a single scatterer.

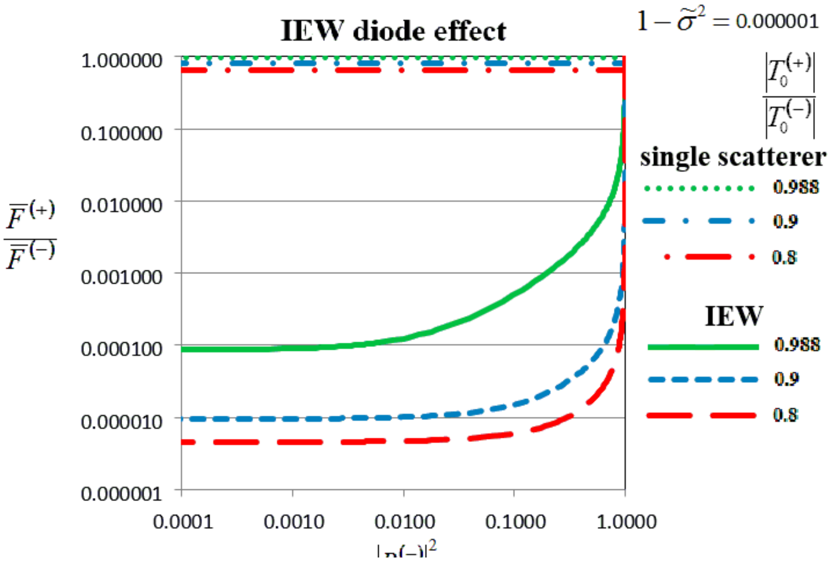

Figure 8: Enhancement of the wave diode effect by an asymmetric periodic structure compared to a single asymmetric scatterer with 0.0001% energy absorption

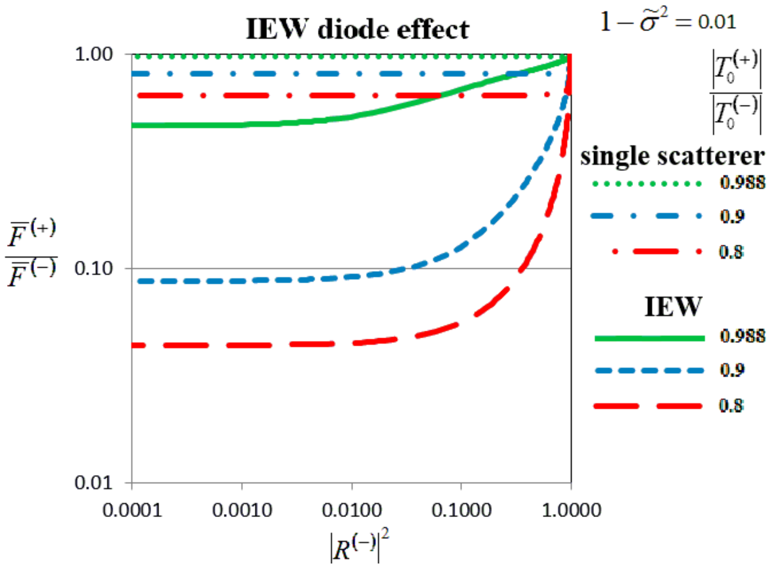

Figure 9: Enhancement of the wave diode effect by an asymmetric periodic structure compared to a single asymmetric scatterer with 1% energy absorption

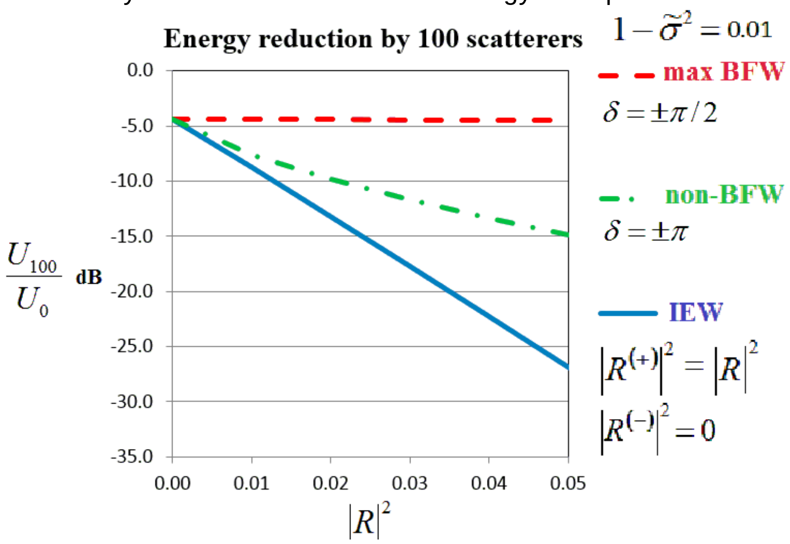

Figure 10: Comparison of BFW, non-BFW and IEW attenuation by 100 scatterers with 1% energy absorption

Figure 10 compares the theoretical performance of BFW, non-BFW and IEW for energy attenuation along a periodic structure sufficiently long to omit boundary effects. BFW correspond to the centre of the damped passing band (i.e. lowest attenuation), non-BFW are for $\delta = 0, \pm\pi$ which is the same as IEW for symmetric scattering, and IEW are for asymmetric scattering in this case with no reflections of SW travelling in the *-x* direction. IEW for asymmetric scattering clearly has superior attenuation, essentially because the *–x* direction flux is not attenuated and energy builds up less near the 100th scatterer than for symmetric scattering.

5 DISCUSSION

This paper models the attenuation of energy propagated along a periodic structure approximated as equally spaced point scatterers. Applying the theory for an infinite periodic structure assumes that the boundary effects of a finite structure are small. This reproduces the well-known BFW that are a special case where the scattering phase shift difference δ is constrained to $\pm\pi/2$. BFW are only weakly attenuated in the damped passing bands for realistic conditions, such as low energy losses and low reflection coefficients for physical systems such as ribbed plates and cylinders. However for the same energy losses and similar magnitude reflection coefficients, attenuation becomes much stronger when the scatterers are asymmetric, especially when propagation is nonreciprocal. The nonreciprocal IEW mode achieves the strongest attenuation of all the PSW types, physically because energy that propagates into the least penetrable direction is less attenuated when it is reflected back in the opposite direction.

Increasing wave attenuation via asymmetry is no doubt possible but further theoretical developments are needed for practical applications. The theory needs to take into account the boundary effects of finite structures. Boundaries would weaken decoherence and its contribution to nonreciprocal propagation which relies on a structure extending to infinity. Further, boundaries would remove the discontinuity in the sensitivity of BFW to deviations of δ from $\pm\pi/2$, and continuously relate BFW, non-BFW and IEW. Energy absorbers emulating a continuation of a finite structure to infinity might approximate the advantages of nonreciprocal propagation.

This paper estimated the relative performance of different concepts for passively attenuating unwanted noise that propagates along periodic structures. These concepts are a combination of energy absorption and asymmetric scattering, and showed that deviations of only a few percent from existing symmetric designs can have a large effect over many scatterers. Although the concepts of this paper are derived from theoretical analysis of an infinite one-dimensional structure of scatterers, they scope out potentially greater passive noise reduction and may motivate an in-depth analysis of realistic periodic structures that are finite, incorporate metamaterials and predict the amplitudes of different PSW modes excited by energy sources coupled to a structure.

REFERENCES

Bednarik, M. and Cervenka, M., 2017. 'Description of waves in inhomogeneous domains using Heun's equation', *Waves in Random and Complex Media*, 12 June, 1-17.

Brillouin, L., 1953, *Wave Propagation in Periodic Structures*, Second Edition, Dover, Inc.

Fleury, R., Sounas, D. L., Haberman, M. R., Alù, A., 2015. 'Nonreciprocal Acoustics', *Acoustics Today*, 11 (3), 14 – 21.

Haberman, M. R. and Norris, A. N**.,** 2016. 'Acoustic Metamaterials', *Acoustics Today*, 12 (3), 31 – 39.

Hussein, M.I., 2009. 'Theory of damped Bloch waves in elastic media', *Physical Review B*, 80, 212301, 1 – 4.

Kinsler, L. E., Frey, A. R., Coppens, A. B. and Sanders, J. V., 1999. *Fundamentals of Acoustics*, Fourth Edition, John Wiley and Sons, Chap. 6.

Luan, P. and Ye, Z., 2001. 'Acoustic wave propagation in a one-dimensional layered system', *Physical Review E,* 63, 066611

McMahon, D 2015, 'Wave propagation in infinite periodic structures taking into account energy absorption', *Proceedings of Acoustics 2015*, Hunter Valley, Australia.

McMahon, D 2016a, 'Wave propagation in infinite periodic structures', *Proceedings of the 23rd International Congress on Sound and Vibration* (ICSV23), Athens, Greece.

McMahon, D 2016b, 'Symmetry of waves in an infinite 1D periodic structure under the interchange of forward and backward scattering coefficients', *Proceedings of Acoustics 2016*, Brisbane, Australia.

McMahon, D 2017, 'Wave propagation and attenuation in an infinite periodic structure of asymmetric scatterers', *Proceedings of the 24th International Congress on Sound and Vibration* (ICSV24), London, UK.

Trainiti, G. and Ruzzene, M., 2016, 'Non-reciprocal elastic wave propagation in spatiotemporal periodic structures', *New Journal of Physics*, 18, 1 – 22.