The n^2-exponential sound speed profile and its use for modelling sound propagation in an isothermal surface duct

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ABSTRACT

The present author showed previously that, if the sound speed profile (SSP) in a vertically stratified fluid half-space is described by a specific formula, the exact solution of the depth-separated Helmholtz equation is a Bessel function with the argument exponentially depending on depth. As, according to this formula, the square of the refraction index "*n*" changes exponentially with depth, it is suggested here that this SSP be called the "n^2-exponential SSP". It is shown that the n^2-exponential SSP in a limiting case can approximate the linear SSP in the isothermal surface duct. This approximation is compared with a commonly used approximation by means of the n^2-linear SSP. It is shown that the equation for the horizontal wavenumbers in a medium with the n^2-exponential SSP in its approximate form coincides with the corresponding approximate equation for the n^2-linear SSP. Green's function for the acoustic field trapped in the duct with the n^2-exponential SSP is obtained from a known equation. The acoustic field calculated with the use of the Green's function is compared with the acoustic field obtained by means of numerical models BELLHOP and RAMGeo. It is shown that the spatial distribution of the acoustic field described by the Green's function is close to the one calculated by these models.

1 Introduction

Formation of an acoustic duct in the mixed layer of water below the ocean surface is a well-known phenomenon. It occurs when the temperature in the layer is uniform due to mixing and the sound speed increases linearly with depth due to rising hydrostatic pressure. As the sound waves are refracted towards the surface within the duct, successive reflections from the surface lead to significant increase in the propagation range.

The exact analytical solution for the acoustic field within the duct with the linear SSP is not known. Instead, various numerical models are used for prediction of acoustic field in such a duct. In addition, the linear SSP is often approximated by the n^2-linear SSP, where the inverse of the sound speed squared changes linearly with depth (Jensen et al., 2011). In this case, the corresponding solution for the acoustic pressure is represented via Airy functions with their argument being a linear function of depth.

The main purpose of this paper is to suggest an alternative technique to model the sound field in the duct with the linear SSP. The technique is based on the solution derived previously by the present author (Zinoviev, 2016) for a special case of the "transitional" layer described by Brekhovsikh (1960).

In Section 2, the previously derived SSP and the corresponding solution for the acoustic field are presented. It is suggested that this SSP be called the "n^2-exponential SSP", as, for this SSP, the inverse sound speed squared exponentially tends to an asymptotic value. In Section 3, the n^2-exponential SSP and the corresponding solution for the acoustic field are adapted for the modelling of the linear SSP. In Section 4, the approximation by means of the n^2-exponential SSP is compared with the commonly used approximation utilising the n^2-linear SSP in terms of its accuracy and mathematical compactness. Section 5 is devoted to the derivation of the Green's function of the duct based on the obtained solution for the n^2-exponential SSP and to the comparison of the results of calculations of the acoustic field using the Green's function with the results obtained with the use of numerical models BELLHOP and RAMGeo.

2 The n^2-exponential SSP and the corresponding equation for the acoustic field

Consider an acoustic medium that is a fluid half-space where the sound speed values at the surface and far below the surface, c_s and c_0, are such that $c_0 > c_s$. The boundary of the half-space is considered to be pressure-release and smooth. As the harmonic time dependence is assumed and the medium is range-independent, in the cylindrical coordinates the acoustic pressure in the far field, $P(r,z,t)$, is considered to be of the form

$$P(r,z,t) = \frac{A}{\sqrt{r}} e^{i(k_r r - \omega t)} p(z), \tag{1}$$

where r is the horizontal range, z is depth, t is time, A is the complex pressure amplitude, k_r is the horizontal wavenumber, $\omega = 2\pi f$, f is the acoustic frequency, and $p(z)$ satisfies the depth-separated Helmholtz equation for the medium (Jensen et al., 2011).

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c^2(z)}\right] p(z) = 0, \tag{2}$$

where $c(z)$ is the depth-dependend sound speed.

Based on Brekhovskikh's (1960) equations for the "transitional" layer, Zinoviev (2016) showed that, if $c(z)$ is determined by the following equation:

$$c(z) = \frac{c_0 c_s}{\sqrt{(c_0^2 - c_s^2)e^{-mz} + c_s^2}}, \quad z > 0, \ m > 0, \tag{3}$$

then the exact solution of Eq. (2) can be described via a Bessel function of the first kind, $J_{2\kappa}(x)$, of the order 2κ with the argument exponentially depending on z:

$$p(z) = J_{2\kappa}\left(2\phi e^{-mz/2}\right). \tag{4}$$

In Eq. (4), ϕ is the non-dimensional frequency defined as

$$\phi = \frac{\omega\sqrt{c_0^2 - c_s^2}}{m c_0 c_s}, \tag{5}$$

and the order of the Bessel function, 2κ, can be determined from conditions at $z = 0$ and $z \to \infty$.

Zinoviev (2016) showed that, if the difference $\Delta_c = c_0 - c_s$ is small in comparison with c_0, then the SSP described by Eq. (3) can be reduced to an important limiting case, the exponential SSP, where the sound speed exponentially tends to its asymptotic value, c_0, with increasing depth:

$$c_{exp}(z) = c_0\left(1 - \frac{\Delta_c}{c_0} e^{-mz}\right), \quad \Delta_c / c_0 \ll 1. \tag{6}$$

The solution for pressure in this case is determined by Eq. (4) with the non-dimensional frequency, ϕ, being in the form of

$$\phi = \frac{\omega}{m c_0}\sqrt{\frac{2\Delta_c}{c_0}}. \tag{7}$$

It can be shown that Eq. (3) can be rewritten in terms of the refraction index, $n(z) = c_0 / c(z)$:

$$n^2(z) = 1 + \left(\frac{c_0^2}{c_s^2} - 1\right) e^{-mz}. \tag{8}$$

Eq. (8) shows that the refraction index exponentially tends to its asymptotic value, one, with increasing z. Therefore, it is suggested here that the SSP described by Eqs. (3) and (8) be called the "n^2-exponential SSP".

3 Modelling the SSP and the acoustic field in an isothermal surface duct

3.1 Linear SSP

In an isothermal surface duct the sound speed increases linearly with depth due to rising hydrostatic pressure. If the sound speed gradient in the duct is g and the sound speed at the surface is c_s then the linear dependence of the sound speed on depth can be defined as

$$c_{lin}(z) = c_s + gz. \tag{9}$$

As an exact analytical solution for the acoustic pressure in the linear SSP is not known, to enable analytical derivations for a surface duct the "n^2-linear" sound speed profile is commonly utilised (Jensen et al., 2011). This technique is briefly described in the next subsection.

3.2 The n^2-linear SSP

The n^2-linear sound speed profile is determined by an equation for sound speed where the refraction index squared, or the inverse square of sound speed, depends linearly on depth:

$$\frac{1}{c_{nsl}^2(z)} = az + b, \quad a = \text{const}, \quad b = \text{const}. \tag{10}$$

To model a linear SSP described by Eq. (9), the constants a and b are usually assumed to be in the following form:

$$a = -\frac{2g}{c_s^3}, \ \ b = \frac{1}{c_s^2}. \tag{11}$$

Eqs. (10) and (11) lead to the following equation for the sound speed in a medium with the n^2-linear SSP:

$$c_{nsl}(z) = \frac{c_s}{\sqrt{1 - \frac{2g}{c_s} z}}. \tag{12}$$

Eq. (12) can be reduced via Taylor series to Eq. (9) if $2gz/c_s \ll 1$. As the typical values for g and c_s are 0.016 s^{-1} and 1500 m/s respectively, it is clear that this condition is satisfied even for the deepest of realistic ducts.

In a medium with SSP in the form of Eq. (12), the dependence of the acoustic pressure on z is determined via Airy functions, $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$. As shown by Jensen et al. (2011), the solution for the pressure via the function $\mathrm{Ai}(\zeta)$ is:

$$p_{nsl}(z) = \mathrm{Ai}\left(\frac{k_r^2 - \omega^2(az + b)}{(\omega^2 a)^{2/3}}\right), \quad k_s = \frac{\omega}{c_s}. \tag{13}$$

Substitution of a and b in the form of Eq. (11) into Eq. (13) leads to the following solution for the acoustic pressure in the n^2-linear SSP:

$$p_{nsl}(z) = \mathrm{Ai}\left(\frac{k_r^2 - k_s^2\left(1 - \frac{2g}{c_0} z\right)}{\left(-k_s^2 \frac{2g}{c_0}\right)^{2/3}}\right). \tag{14}$$

The function $\mathrm{Ai}(-x)$ can be represented via Bessel functions $J_{\pm 1/3}(x)$:

$$\mathrm{Ai}(-x) = \frac{1}{3}\sqrt{x}\left(J_{-1/3}\left(\frac{2}{3}|x|^{3/2}\right) + J_{1/3}\left(\frac{2}{3}|x|^{3/2}\right)\right). \tag{15}$$

The pressure-release boundary conditions at $z = 0$ lead to the following equation for horizontal wavenumbers, $k_{r,n}$:

$$k_{r,n} = k_s\sqrt{1 + \left(\frac{2g}{\omega}\right)^{2/3}\zeta_n}, \qquad n = 1, 2, 3 \ldots, \tag{16}$$

where n are duct mode numbers and ζ_n are zeros of the Airy function $\mathrm{Ai}(\zeta)$, which can be approximated as (Jones et al., 2016):

$$\zeta_n \approx -\left(\frac{3\pi(n - 1/4)}{2}\right)^{2/3}. \tag{17}$$

As the value of $\left(\frac{2g}{\omega}\right)^{2/3}$ is small, the horizontal wavenumbers $k_{r,n}$ for the infinite duct under consideration are real and positive.

Eq. (16) can be simplified via Taylor series as follows:

$$k_{r,n} \approx k_s\left[1 - \frac{1}{2}\left(\frac{3\pi g}{\omega}\right)^{2/3}\left(n - \frac{1}{4}\right)^{2/3}\right]. \tag{18}$$

Eqs. (14) – (18) are commonly used to model the linear sound speed profile and the acoustic field in an isothermal surface duct. The next subsection describes an alternative technique for such modelling by means of the n^2-exponential SSP.

3.3 The n^2-exponential SSP

When deriving Eqs. (3) – (5) for the n^2-exponential SSP no assumptions have been made about the asymptotic value of sound speed, c_0. Therefore, it can be considered to be very large. If the sound speed gradient at the surface $(z = 0)$ is g_s and $c_0 \to \infty$, then Eqs. (3) and (4) can be simplified and reduced to the following equations:

$$c_{nse}(z) = c_s e^{\frac{g_s z}{c_s}}, \tag{19}$$

$$p_{nse}(z) = J_{\frac{k_r c_s}{g_s}}\left(\frac{\omega}{g_s} e^{-\frac{g_s z}{c_s}}\right). \tag{20}$$

In Eq. (20), $J_{\frac{k_r c_s}{g_s}}(x)$ is Bessel function of the first kind of the order $k_r c_s / g_s$.

It is clear that Eq. (19) can be reduced to Eq. (9) for the linear SSP if $g_s z / c_s \ll 1$, which is the case in realistic ocean ducts. Therefore, Eqs. (19) and (20) can be used for modelling the sound speed profile and the pressure field in the surface duct with the linear SSP.

The pressure-release conditions at $z = 0$ lead to the following equation for the horizontal wavenumbers, $k_{r,n}$:

$$J_{\frac{k_{r,n}c_s}{g_s}}\left(\frac{\omega}{g_s}\right) = 0. \tag{21}$$

Eqs. (19) – (21) represent one of the results of this work. Eq. (19) describes the sound speed profile in a medium with the n^2-exponential SSP with the condition $c_0 \to \infty$, whereas Eq. (20) is the exact solution for the vertical pressure profile in such a medium. Eq. (21) is the exact equation for finding the horizontal wavenumbers, $k_{r,n}$. The use of these equations for modelling sound propagation in the surface duct with the linear SSP is demonstrated further in this paper. A detailed derivation of these equations is considered to be outside the scope of this work.

4 Comparison of approximations of the linear SSP by means of the *n*²-linear and *n*²-exponential SSPs

4.1 Equations for the horizontal wavenumbers

Since the order of the Bessel function in Eq. (21) is large, an approximation of the Bessel function can be utilised to simplify this equation. As a result, it is possible to obtain an approximate equation for $k_{r,n}$ and show that it coincides with Eq. (18). Therefore, although the exact equations for finding the horizontal wavenumbers $k_{r,n}$ (Eqs. (16) and (21)) are different for the n^2-linear and n^2-exponential sound speed profiles, their approximations are the same in the form of Eq. (18). This confirms that these two SSPs are approximations of the same linear SSP with some degree of accuracy. Differences between the two approximations are discussed in the next two subsections.

4.2 Accuracy

Accuracy of an approximation of the linear SSP by another SSP can be represented by a measure of deviation of the sound speed gradient in the approximate SSP from the constant gradient g in the linear SSP. This measure, Δ_g, can be calculated by the following equation:

$$\Delta_g = \frac{c'(z) - g}{g} = \frac{c'(z)}{g} - 1, \tag{22}$$

where $c'(z)$ is the derivative of the approximate sound speed profile over z.

Figure 1 shows the value of Δ_g for the two approximate sound speed profiles discussed here. Figure 1a with the maximum depth of 30 km shows that the n^2-linear SSP deviates much more significantly from the linear SSP at very large depths. Although it is obvious that such depths are not realistic for ocean surface ducts, this Figure demonstrates that the approximation of the linear SSP by the n^2-exponential SSP is much more accurate than the approximation by the n^2-linear SSP.

The better accuracy of the approximation by means of the n^2-exponential SSP can be also seen in Figure 1b for a duct with the maximum depth of 1000 m, which can be observed in polar regions (Jensen et al., 2011). This Figure shows that the difference between the sound speed gradient in the n^2-linear SSP and the one in the linear SSP is about three times larger than the difference between the gradients in the n^2-exponential and the linear SSPs at any given depth.

4.3 Mathematical compactness

The depth dependence of the pressure field in the n^2-linear SSP is described by Eqs. (14) and (15). Eq. (14) determines the pressure field via an Airy function of an argument x depending linearly on depth. The equation for the Airy function (Eq. (15)) shows that this function is a sum of two Bessel functions of the order $\pm 1/3$ multiplied by square root of x. The argument of the Bessel functions is proportional to $|x|^{3/2}$. In the case of the n^2-exponential SSP the pressure field is determined by a single Bessel function with the argument exponentially depending on depth (Eq. (20)). It is clear that this representation is simpler and can lead to more convenient analytical derivations and better approximations. Also, it may benefit numerical modelling by facilitating faster and more accurate calculations.

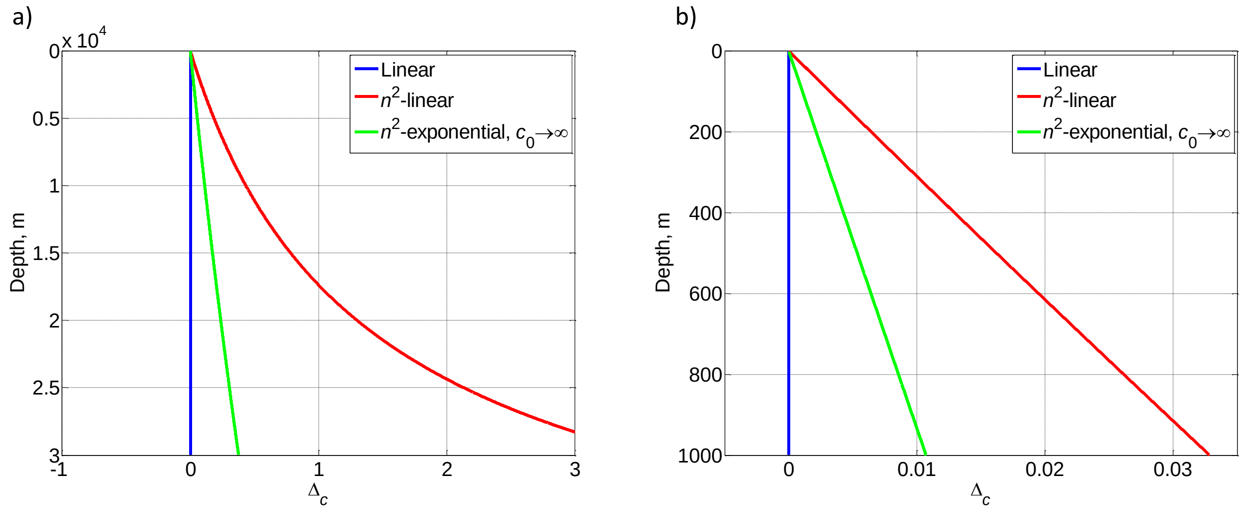

Figure 1: Relative deviation Δ_g (Eq. (22)) of the vertical derivative of the sound speed in the n^2-linear and n^2-exponential SSPs from the constant gradient in the linear SSP.

5 Green's function for the duct with the n^2-exponential SSP

5.1 Formulation

Knowledge of the Green's function of a medium is crucial for modelling acoustic propagation in the medium, since this function represents the field of a single monopole source. Ye (1995) considered a surface layer where gas bubbles generated by breaking waves lead to the exponential vertical sound speed profile described by Eq. (6). He demonstrated that an *approximate* solution for the pressure field in such a layer has the form of Eqs. (4) and (7). Based on this solution, he derived the Green's function for such a layer as a series with terms representing modes trapped in the layer. The field propagating below the layer was excluded from his consideration. Since the solution determined by Eqs. (4) and (7) differ from the solution described by Eq. (19) only by the appearance of the order and the argument of the Bessel function, it is justified to utilise the Green's function obtained by Ye (1995) in the case considered here. As a result, the Green's function for the fluid half-space with the linear SSP approximated by the n^2-exponential SSP can be written as the following series where each term represents a mode of the layer:

$$G(z,r) = \sum_{n=1}^{\infty} A_n J_{\frac{c_s \kappa_{r,n}}{g_s}}\left(\frac{\omega}{g_s} e^{-\frac{g_s z}{c_s}}\right) H_0^{(1)}(k_{r,n} r). \tag{23}$$

The modal coefficients, A_n, are determined by

$$A_n = \pi^2\sqrt{\pi}(-1)^{n+1} \frac{k_{r,n} c_s}{2 g_s \arccos\left(\frac{c_s k_{r,n}}{\omega}\right)} \left(\frac{\omega^2 - c_s^2 k_{r,n}^2}{4 g_s^2}\right)^{1/4} J_{\frac{c_s \kappa_{r,n}}{g_s}}\left(\frac{\omega}{g_s} e^{-\frac{g_s z_s}{c_s}}\right) H^{(1)}_{\frac{c_s \kappa_{r,n}}{g_s}}\left(\frac{\omega}{g_s}\right). \tag{24}$$

In Eqs. (23) and (24), $H_\nu^{(1)}(x)$ is the Hankel function of the first kind of the order ν and argument x, z_s is the source depth, and the modal horizontal wavenumbers, $k_{r,n}$, are calculated by means of either the exact Eq. (21) or the approximate Eq. (18). It can be noted that the summation in Eq. (23) is infinite, as the n^2-exponential SSP extends to infinite depths and, as a result, all modes are considered to be trapped. Eqs. (23) and (24) represent another result of this work.

5.2 Comparison with numerical modelling results

As a verification of the Green's function in the form of Eqs. (23) and (24), a comparison has been carried out between the acoustic field predicted with the use of the Green's function as well as two numerical models. Two frequencies, 600 Hz and 6000 Hz, were considered. A high-frequency beam tracing model BELLHOP was used in the latter case, whereas a parabolic equation model RAMGeo was used for the lower frequency. The source depth, z_s, was considered to be 5 m in the high-frequency case and 50 m in the low-frequency case. Other duct parameters were the same for both cases. Their values were as follows: the sound speed at the surface, $c_s = 1500$ m/s, the duct depth, $H = 150$ m, the sound speed gradient in the duct, $g = 0.016$ s^{-1}. The environment below the duct was considered to be an isovelocity half-space with the sound speed of 1502.4 m/s.

In the analysis provided in the previous sections, the fluid layer extends to infinite depth, so that the number of trapped modes in Eq. (23) is infinite. As is well-known, for a duct of finite depth only the lowest modes are trapped in the duct. Based on the equation for the modal trapping frequency provided by Jones et al. (2017), the number, N, of the modes taken into account in Eq. (23) is determined as follows:

$$N \leq \frac{4\sqrt{2g}}{3} f \left(\frac{H}{c_s}\right)^{3/2} + \frac{1}{4}. \tag{25}$$

In the modelling described here, N was equal to 45 in the high-frequency case and 4 in the low-frequency case. For the acoustic field determined by the Green's function (Eq. (23)), there was no coupling between the modes trapped in the layer and the isovelocity space below.

Figure 2 shows the results of calculations of transmission loss (TL) in decibels versus range and depth obtained by the three methods. The Figures 2a and 2b are for the frequency $f = 6000$ Hz. It is clear from the Figures that the spatial TL distributions calculated by means of the Green's function obtained here and by means of the BELLHOP model are very similar both at large and small scales. The main difference between the distributions is in the area just below the source. This difference can be explained by the fact that the Green's function takes account of the modes propagating in the layer only, whereas the BELLHOP model describes the total acoustic field including its part propagating towards the sea floor. The same explanation is also valid for the small-scale variations of TL calculated using the Green's function in the bottom half of the duct at ranges up to 15 km.

Figures 2c and 2d show TL distributions for the lower frequency $f = 600$ Hz produced by means of the RAMGeo parabolic equation model and the Green's function obtained here. The difference between the two figures is more noticeable than in the high-frequency case due to lower number of propagating modes. However, spatial distributions of TL in these two figures are still quite similar.

Overall, the comparison of TL calculated using the Green's function with TL obtained by means of the two numerical models based on different principles confirms the validity of the representation of the acoustic field described by Eqs. (23) and (24). The observed differences with the results of numerical modelling due to a limited number of trapped modes can be overcome in the future by taking into consideration leakage of the acoustic energy from the duct.

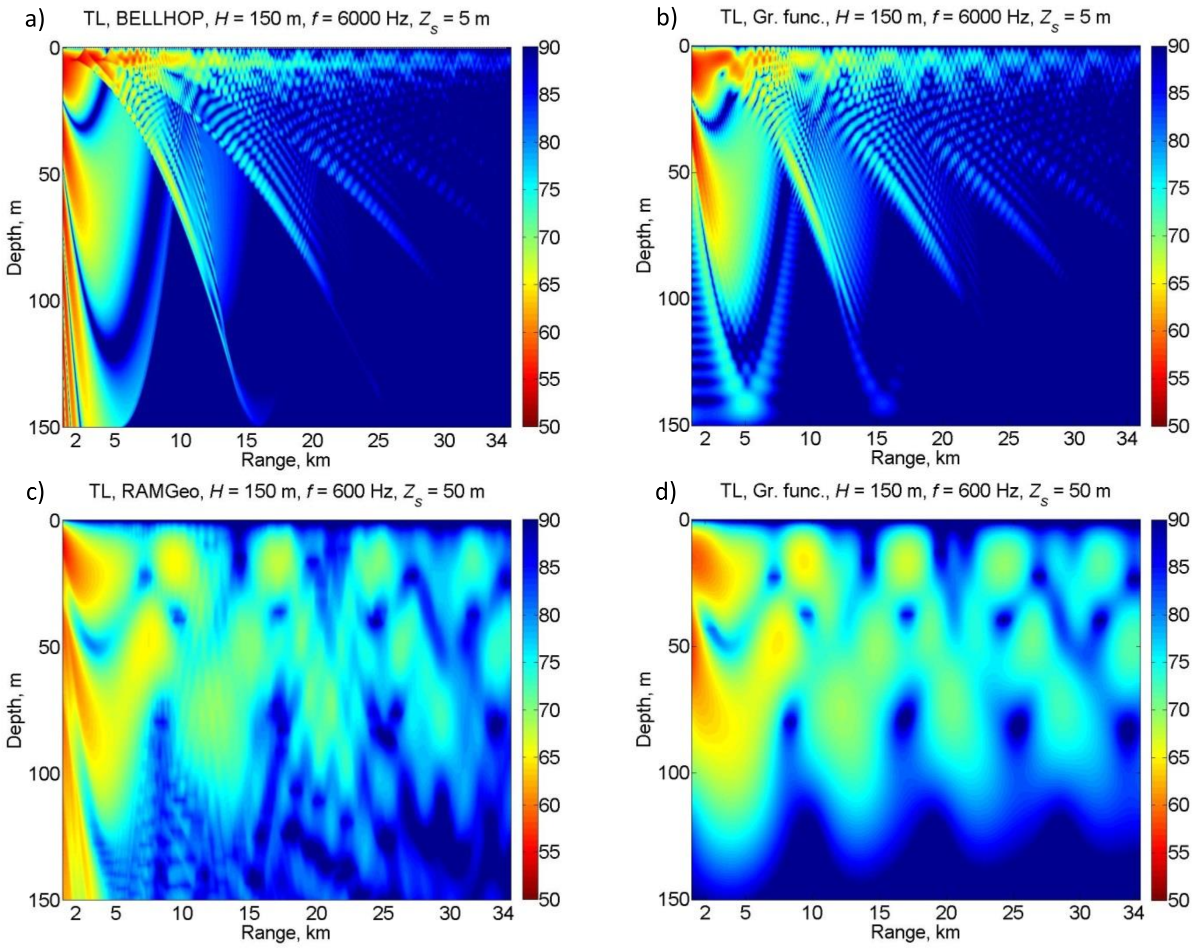

Figure 2: Transmission loss calculations by means of the Green's function (Eqs. (23) and (24)) as well as the numerical models BELLHOP and RAMGeo. Top row: $f = 6000$ Hz, bottom row: $f = 600$ Hz, the left column: numerical models, the right column: Green's function. H is the duct depth, f is frequency, z_s is the source depth.

6 Conclusions

In this paper, the n^2-exponential sound speed profile (SSP) is considered. It is shown that this profile in a limiting case can approximate well the linear SSP in the isothermal surface layer. A general solution for the acoustic pressure in such a layer is obtained. The obtained approximation for the linear SSP is compared with the well-known approximation by means of the n^2-linear SSP and shown to be more accurate and more compact mathematically.

The Green's function for the layer with the approximated linear SSP is obtained as a series of modes trapped in the layer. This Green's function is used to calculate the transmission loss (TL) distributions vs range and depth for two frequencies: 600 Hz and 6000 Hz. The comparison of these results with the ones obtained using two numerical models shows overall agreement. The observed differences can be explained by the limited number of trapped modes taken into account in the Greens' function. It is proposed that these differences can be reduced in the future research by considering energy leakage from the layer to the space below.

In general, as the n^2-exponential SSP describes a family of curves with various vertical dependencies, it can be suggested that the obtained solution for the acoustic pressure as well as the corresponding Green's function can become the basis of a layered model describing sound propagation in realistic sound speed profiles.

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