Nonlinear vibration of nonlocal strain gradient ultrasmall tubes with a geometric imperfection

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ABSTRACT

In the present article, a modified nonlinear continuum model is presented for the large-amplitude vibration of ultrasmall tubes taking into account the effect of a geometric imperfection. The higher-order size-dependent theoretical model is developed via help of a nonlocal strain gradient theory of elasticity in conjunction with the Euler–Bernoulli beam theory. The presented scale-dependent model is able to better capture the effect of the size of ultrasmall tubes on the vibration behaviour. According to the work/energy principle, size-dependent relations are presented for the elastic energy and motion energy of the micro/nanoscale tube as well as the work performed by external forces. Using these relations, the size-dependent differential equations of the described problem are then derived. Lastly, the Galerkin decomposition scheme and the continuation-based scheme are employed to extract the nonlinear vibration characteristics. It is found that geometric imperfections have an important influence on the nonlinear forced vibration of ultrasmall tubes. For small geometric imperfections, the nonlinear vibration is of hardening type whereas a combination of softening-type and hardening-type nonlinear responses is observed for large imperfections.

1 INTRODUCTION

Nanoscale tubes have notably been utilized as the fundamental parts of many nanoelectromechanical devises involving nanoresonators, mass nanosensors and nanoscale energy harvesters. To have a more reasonable design for fabricating these precious devices, it is important to improve the level of understanding of the forced vibration of nanoscale tubes since excitation mechanical loads are often applied to them.

Ultrasmall beams and tubes such as carbon nanotubes and silver microscale wires which are broadly used in the fabrication of small-scale electromechanical devices are rarely found in a geometrically perfect form. Performing a precise manufacturing process at small scales is difficult and thus geometric imperfections are usually found in ultrasmall beams and tubes. However, in many theoretical models and computational simulations, these imperfections have not been taken into consideration due the complexity of the equation of motion(Wang and Wang 2007, Arash and Wang 2012, Saadatnia and Esmailzadeh 2017, Gul and Aydogdu 2018, Zhang, Liew, and Hui 2018, Barretta et al. 2016).

In recent years, few research articles have been reported on the mechanical behaviour of geometrically imperfect ultrasmall beams and tubes employing modified elasticity theories. A nonlocal continuum model was proposed by Farshidianfar and Soltani (Farshidianfar and Soltani 2012) for investigating the dynamic response of fluid-conveying nanoscale tubes in a geometrically imperfect form; a multi-scale perturbation approach was used in order to obtain explicit expressions for the solution of governing equations. In another analysis, Wang et al. (Wang, Deng, and Zhang 2013) utilised a classical nonlocal theory for examining the forced vibrational response of a single imperfect carbon nanotube (CNT) with large deformations; a one-term Galerkin method and an integration approach were employed for the discretisation of derived partial differential equations and the solution of the ordinary equation, respectively. Furthermore, the nonlinear buckling of geometrically imperfect beams at nanoscale levels was also investigated by Mohammadi et al. (Mohammadi et al. 2014) on the basis of a nonlocal theory; for the buckling analysis, the nanoscale beam was assumed to be embedded in an elastic foundation. In addition, Barati and Zenkour (Barati and Zenkour 2017) studied the effect of a geometric imperfection on the large-amplitude stability of metal foam nanobeams; the effects of structural porosities were also taken into consideration. The influences of a small initial curvature on the mechanical response of CNTs (Arefi and Salimi 2015) and graphene sheets (Wang et al. 2013) have been also studied in the literature; it has been reported a small initial curvature has an important influence on the mechanics of ultrasmall CNTs and monolayer graphene sheets.

In the above-mentioned papers, only one scale parameter is applied so as to describe the size influence on the mechanical behaviour of geometrically imperfect ultrasmall structures. Lately, it has been reported that incorporating only one scale parameter is not enough for developing a precis size-dependent continuum model. Lim et al. (Lim, Zhang, and Reddy 2015) introduced a novel modified elasticity theory which contains two different scale parameters; both nonlocal and strain gradient influences are taken into consideration in a refined scale-dependent elasticity theory called the nonlocal strain gradient theory. Recently, this theory has been utilised for analysing different ultrasmall structures such as CNTs (Li, Hu, and Ling 2016, Farajpour et al. 2018), nanorods (Zhu and Li 2017) and graphene sheets (Ebrahimi and Barati 2017, Farajpour et al. 2016). Nonetheless, in these valuable papers, the effect of geometric imperfection on the mechanical characteristics is not captured. In this analysis, a two-parameter size-dependent model is proposed for the large-amplitude forced vibration of ultrasmall tubes with a geometric imperfection. Incorporating both strain gradient and stress nonlocality, an accurate size-dependent model is developed. Hamilton's principle is then employed for deriving coupled motion equations. A solution procedure is presented via help of a continuation-based method as well as Galerkin's scheme. The influences of the geometrical imperfection, nonlocal and strain gradient parameters are studied.

2 NONLINEAR SIZE-DEPENDENT CONTINUUM MODELLING

In Fig. 1, a geometrically imperfect ultrasmall tube with length L is shown. It is assumed that the tube is clamped at both ends. This type of boundary conditions is assumed since in practical applications, especially in nanoresonators, carbon nanotubes are fixed at both ends. The internal and external radii of the ultrasmall tube are denoted by R_i and R_o, respectively. Furthermore, in the present formulation, h denotes the tube thickness. A, E, ρ, v and I are respectively the cross-sectional area, the Young modulus, the mass density, Poisson's ratio and the inertia moment.

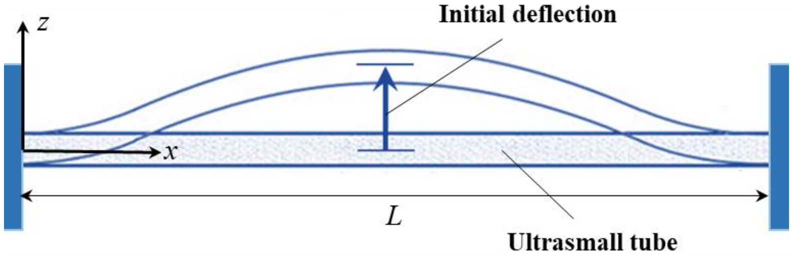

Figure 1: An ultrasmall tube with a geometric imperfection.

Let us use u, w and w_0 in order to indicate the axial, transverse and initial displacements of the imperfect ultrasmall tube, respectively. Applying the nonlocal strain gradient theory, one can write as (Lim, Zhang, and Reddy 2015, Li et al. 2017)

$$t_{xx} - (e_0 a)^2 \nabla^2 t_{xx} = E\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{dw_0}{dx} - z\frac{\partial^2 w}{\partial x^2}\right]$$
$$-El_{sg}^2 \nabla^2 \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial w}{\partial x}\frac{dw_0}{dx} - z\frac{\partial^2 w}{\partial x^2}\right], \tag{1}$$

Here t_{xx}, e_0, a, ∇^2 and l_{sg} stand for the total stress, the calibration coefficient, the internal characteristic length, the Laplace operator and the strain gradient parameter, respectively (Zhang, Zhang, and Liew 2017, Mercan and Civalek 2017). Employing molecular dynamic simulations, the applicability of the nonlocal strain gradient constitutive equation to carbon nanotubes has recently been shown (Li, Hu, and Ling 2016). It should be noticed that when the scale parameters are set to zero, the present model is reduced to that of the classical theory for macroscale tubes. Using Eq. (1), the force stress resultant (N_{xx}) and the moment stress resultant (M_{xx}) are obtained

$$N_{xx}-\left(e_0a\right)^2\nabla^2N_{xx}=EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{dw_0}{dx}\right]$$

$$-EAl_{sg}^2\nabla^2\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial w}{\partial x}\frac{dw_0}{dx}\right],$$

(2)

$$M_{xx}-\left(e_0a\right)^2\nabla^2M_{xx}=-EI\frac{\partial^2 w}{\partial x^2}+EIl_{sg}^2\nabla^2\frac{\partial^2 w}{\partial x^2},$$

(3)

The force and moment stress resultants are defined by

$$N_{xx}=\int_A t_{xx}dA,\quad M_{xx}=\int_A zt_{xx}dA.$$

(4)

Let us use m to denote the mass per unit length of the ultrasmall tube. It is assumed that the tube is under a harmonic force as $F_1\cos\left(\omega t\right)$ along the transverse direction; F_1 and ω are respectively the forcing amplitude and frequency. Formulating the kinetic energy, potential energy and the external work, and then substituting the resultant relations into the following work/energy principle

$$\int_{t_1}^{t_2}\left(\delta W_F+\delta K-\delta U\right)\mathrm{d}t=0,$$

(5)

the motion equations of nonlocal strain gradient ultrasmall tubes are obtained as

$$\frac{\partial N_{xx}}{\partial x}=m\frac{\partial^2 u}{\partial t^2},$$

(6)

$$\frac{\partial^2 M_{xx}}{\partial x^2}+\frac{\partial N_{xx}}{\partial x}\left(\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\right)+N_{xx}\left(\frac{\partial^2 w}{\partial x^2}+\frac{d^2w_0}{dx^2}\right)$$

$$+F\left(x\right)\cos\left(\omega t\right)=m\frac{\partial^2 w}{\partial t^2}.$$

(7)

Applying Eqs. (2) and (3) as well as Eqs. (6) and (7), one obtains the nonlinear equations for nonlocal strain gradient ultrasmall tubes with a geometric imperfection. The equation of motion along the transverse direction is as follows

$$EI\left(-\frac{\partial^4 w}{\partial x^4}+l_{sg}^2\frac{\partial^6 w}{\partial x^6}\right)+EA\left[\frac{\partial^2}{\partial x^2}\left(w_0+w\right)-\left(e_0a\right)^2\frac{\partial^4}{\partial x^4}\left(w_0+w\right)\right]\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{dw_0}{dx}\frac{\partial w}{\partial x}\right]$$

$$+EA\left[\frac{\partial}{\partial x}\left(w_0+w\right)-3\left(e_0a\right)^2\frac{\partial^3}{\partial x^3}\left(w_0+w\right)\right]\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\frac{\partial^2 w}{\partial x^2}+\frac{d^2w_0}{dx^2}\frac{\partial w}{\partial x}\right)$$

$$-EA\left\{\left[l_{sg}^2+3\left(e_0a\right)^2\right]\frac{\partial^2}{\partial x^2}\left(w_0+w\right)-\left(e_0a\right)^2l_{sg}^2\frac{\partial^4}{\partial x^4}\left(w_0+w\right)\right\}\times$$

$$\left[\frac{\partial^3 u}{\partial x^3}+\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{\partial^3 w}{\partial x^3}\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\frac{\partial^3 w}{\partial x^3}+2\frac{d^2w_0}{dx^2}\frac{\partial^2 w}{\partial x^2}+\frac{d^3w_0}{dx^3}\frac{\partial w}{\partial x}\right]$$

$$-EA\left\{\left[l_{sg}^2+\left(e_0a\right)^2\right]\frac{\partial}{\partial x}\left(w_0+w\right)-3\left(e_0a\right)^2l_{sg}^2\frac{\partial^3}{\partial x^3}\left(w_0+w\right)\right\}\times$$

$$\left(\frac{\partial^4 u}{\partial x^4}+3\frac{\partial^3 w}{\partial x^3}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial x^4}\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\frac{\partial^4 w}{\partial x^4}+3\frac{d^2w_0}{dx^2}\frac{\partial^3 w}{\partial x^3}+3\frac{d^3w_0}{dx^3}\frac{\partial^2 w}{\partial x^2}+\frac{d^4w_0}{dx^4}\frac{\partial w}{\partial x}\right)$$

$$+3EAl_{sg}^2\left(e_0a\right)^2\left[\frac{\partial^2}{\partial x^2}\left(w_0+w\right)\right]\left[\frac{\partial^5 u}{\partial x^5}+3\left(\frac{\partial^3 w}{\partial x^3}\right)^2+4\frac{\partial^4 w}{\partial x^4}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^5 w}{\partial x^5}\frac{\partial w}{\partial x}+\frac{dw_0}{dx}\frac{\partial^5 w}{\partial x^5}\right.$$
$$\left.+4\frac{d^2w_0}{dx^2}\frac{\partial^4 w}{\partial x^4}+6\frac{d^3w_0}{dx^3}\frac{\partial^3 w}{\partial x^3}+4\frac{d^4w_0}{dx^4}\frac{\partial^2 w}{\partial x^2}+\frac{d^5w_0}{dx^5}\frac{\partial w}{\partial x}\right]$$
$$+EAl_{sg}^2\left(e_0a\right)^2\left[\frac{\partial}{\partial x}\left(w_0+w\right)\right]\left(\frac{\partial^6 u}{\partial x^6}+10\frac{\partial^4 w}{\partial x^4}\frac{\partial^3 w}{\partial x^3}+5\frac{\partial^5 w}{\partial x^5}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^6 w}{\partial x^6}\frac{\partial w}{\partial x}\right.$$
$$\left.+\frac{dw_0}{dx}\frac{\partial^6 w}{\partial x^6}+5\frac{d^2w_0}{dx^2}\frac{\partial^5 w}{\partial x^5}+10\frac{d^3w_0}{dx^3}\frac{\partial^4 w}{\partial x^4}+10\frac{d^4w_0}{dx^4}\frac{\partial^3 w}{\partial x^3}+5\frac{d^5w_0}{dx^5}\frac{\partial^2 w}{\partial x^2}+\frac{d^6w_0}{dx^6}\frac{\partial w}{\partial x}\right)$$
$$-m\left(e_0a\right)^2\left[\frac{\partial}{\partial x}\left(w_0+w\right)\right]\left[\frac{\partial^4}{\partial x^2\partial t^2}\left(\left(e_0a\right)^2\frac{\partial^2 u}{\partial x^2}-u\right)\right]-3m\left(e_0a\right)^4\frac{\partial^4 u}{\partial x^2\partial t^2}\frac{\partial^3}{\partial x^3}\left(w_0+w\right)$$
$$-m\left(e_0a\right)^2\left[\frac{\partial^2}{\partial x^2}\left(w_0+w\right)\right]\left[\frac{\partial^3}{\partial x\partial t^2}\left(3\left(e_0a\right)^2\frac{\partial^2 u}{\partial x^2}-u\right)\right]-m\left(e_0a\right)^4\frac{\partial^3 u}{\partial x\partial t^2}\frac{\partial^4}{\partial x^4}\left(w+w_0\right)$$
$$=m\frac{\partial^2}{\partial t^2}\left[w-\left(e_0a\right)^2\frac{\partial^2 w}{\partial x^2}\right]-F_1\cos\left(\omega t\right). \tag{8}$$

The following dimensionless parameter are now proposed in order to prepare the nonlinear motion equations for a standard solution procedure

$$x^*=\frac{x}{L},\ \left\langle u^*,\ w^*,\ w_0^*\right\rangle=\left\langle\frac{u}{r},\frac{w}{r},\frac{w_0}{r}\right\rangle,\ \left\langle\chi_{nl},\chi_{sg}\right\rangle=\left\langle\frac{e_0a}{L},\frac{l_{sg}}{L}\right\rangle,$$
$$\left\langle\beta,r,F_1^*\right\rangle=\left\langle\frac{L}{r},\sqrt{\frac{I}{A}},\frac{F_1L^3}{EI}\right\rangle,\ t^*=t\phi,\ \Omega=\frac{\omega}{\phi},\ \phi=\sqrt{\frac{EI}{L^4m}} \tag{9}$$

Galerkin's scheme is now applied for the discretisation of the motion equations; hence, the axial and transverse displacements are expressed as

$$u\left(x,t\right)=\sum_{l=1}^{N_u}r_l(t)\Psi_l(x), \tag{10}$$

$$w\left(x,t\right)=\sum_{l=1}^{N_w}q_l(t)\Phi_l(x), \tag{11}$$

Here N_w denotes the number of generalised coordinates in the z axis while the number of generalised coordinates in the x axis is denoted by N_u. Moreover, r_l and q_l are respectively the generalised coordinates in the x and z axes whereas Ψ_l and Φ_l indicate the shape functions in the x and z axes, respectively. In the present analysis, the geometric imperfection is assumed as $w_0=A_0\Phi_1(x)$ where A_0 is the imperfection amplitude. Using Eqs. (10) and (11), implementing appropriate shape functions, and finally applying a continuation-based technique, the nonlinear vibration characteristics of geometrically imperfect nonlocal strain gradient ultrasmall tubes are determined. It should be noticed that eight base functions are chosen along each axis in the solution procedure.

3 NUMERICAL RESULTS

Numerical results are presented in this section for an ultrasmall tube of length L=100 nm, internal radius R_i=0.5 nm and external radius R_o=0.84 nm. The tube elastic properties are set to E=1.0 TPa and v=0.19. In addition, the mass density, the slenderness ratio and the modal damping ratio (used in the numerical method) are taken as ρ=2300 kg/m^3, β=204.5935 and ζ=0.005, respectively.

Figure 2 shows the amplitude-frequency behaviour of geometrically imperfect nonlocal strain gradient ultrasmall tubes for the transverse motion. The nonlocal coefficient is set to $\chi_{nl}=0.1$ while the strain gradient coefficient is $\chi_{sg}=0$. A value of A_0=0.8 is chosen for the imperfection amplitude while the forcing amplitude is F_1=0.1. The nonlinear vibrational response of the ultrasmall tube is of hardening type. Moreover, two distinct saddle points are observed for the imperfect ultrasmall system.

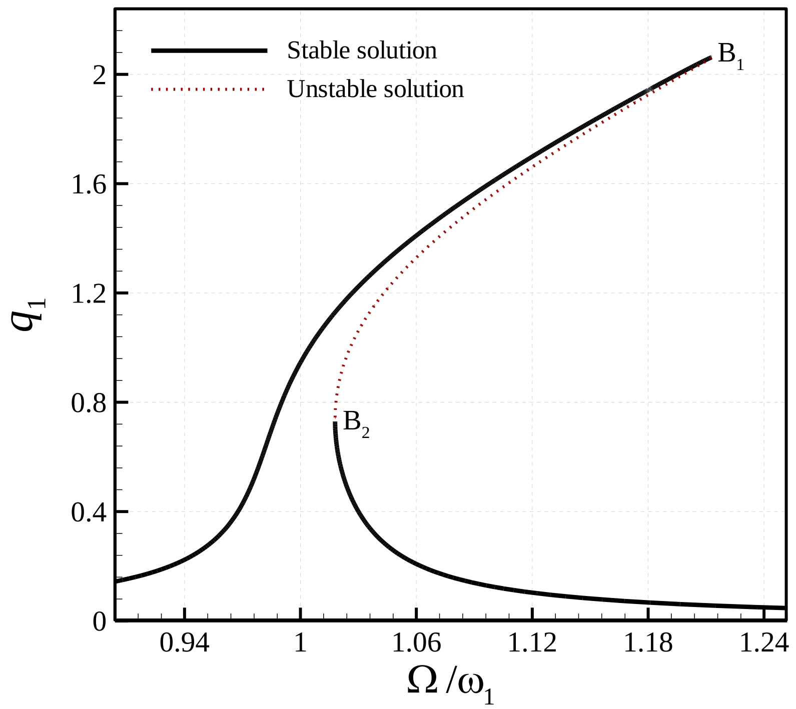

Figure 2: Amplitude-frequency behaviour of imperfect ultrasmall tubes for the transverse motion; A_0=0.8.

Figure 3 depicts the amplitude-frequency behaviour of geometrically imperfect nonlocal strain gradient ultrasmall tubes for a larger imperfection amplitude (A_0=1.5). The numerical results are plotted for the first generalised co-ordinate in the transverse direction. The forcing amplitude, nonlocal and strain gradient coefficients are assumed as F_1=0.14, χ_{nl}=0.0 and χ_{sg}=0.1, respectively. This time the nonlinear vibrational behaviour of the imperfect ultrasmall system is a combination of softening-type and hardening-type nonlinear responses. In addition, four saddle points (i.e. B_i for i=1,2,3,4) are found in the nonlinear behaviour. It is found that the imperfection amplitude has a critical role to play in the nonlinear vibrational behaviour of ultrasmall tubes.

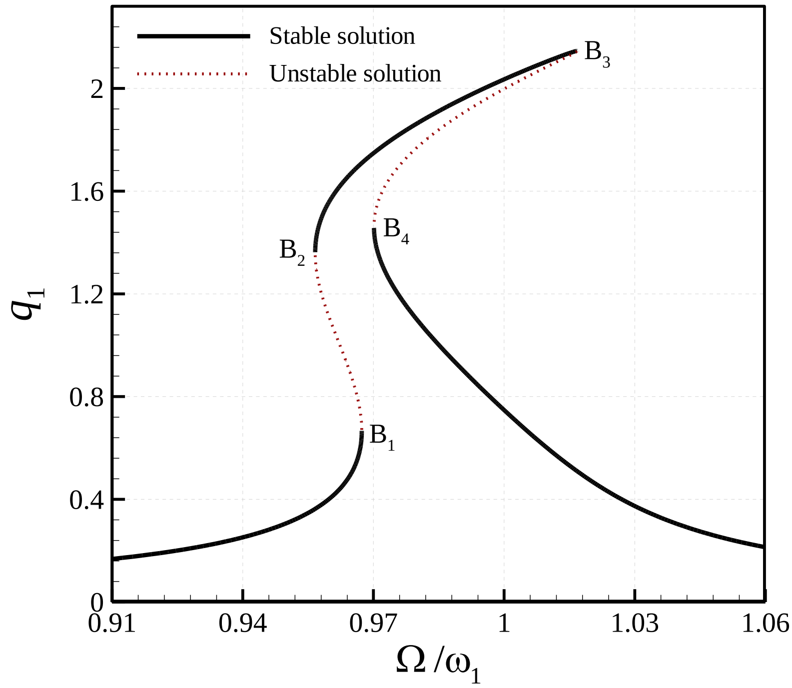

Figure 3: Amplitude-frequency behaviour of imperfect ultrasmall tubes for the transverse motion; A_0=1.5.

4 CONCLUSIONS

A modified scale-dependent nonlocal model was developed for the nonlinear vibrational response of ultrasmall tubes capturing the influence of geometric imperfections. The nonlocal strain gradient theory of elasticity was utilised in order to develop the modified model. Strain gradients and stress nonlocality at ultrasmall levels were taken into account to obtain more reliable results. Hamilton's principle as an approach for the equation derivation was used. Furthermore, a combination of Galerkin's approach and the continuation scheme was employed for the equation solving step. It was observed that the geometric imperfection has a significant influence on the nonlinear vibrational response of ultrasmall nonlocal strain gradient tubes. For small imperfection amplitudes, the nonlinear vibrational response is of hardening type while a combination of softening-type and hardening-type nonlinear responses is found for higher imperfection amplitudes.

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