Time-domain modelling of boundary reflection using an acoustic wave propagator and image source method

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ABSTRACT

The acoustic wave propagator method has been developed and extended since 2000 for investigating the propagation of acoustic and flexural waves. The boundaries are usually introduced by defining the mechanical properties in the wave transmission path. This was found to be highly efficient but slightly inaccurate, especially when a boundary-smoothing technique is used to weaken the discontinuity and maintain the numerical stability. In this paper, an image source method is implemented during modelling, which simulate the boundary reflection by placing virtual sources emits similar signals at the same time. Compared with the results of several previous papers, this is found to provide a more accurate solution for studying wave propagation under different boundary conditions.

1. INTRODUCTION

The image source method has become a widely used tool in many areas of engineering and acoustics, since a milestone paper by Allen and Berkeley was published in 1979. It was found to be an effective solution for modelling boundary conditions, and thus developed as a popular technique for the simulation of sound propagation, especially in the field of room acoustics(Lehmann, E.A., & Johansson, A.M., 2008, Mehta, P., & Bhadradiya, V., 2015). It is commonly used for calculating room impulse responses, predicting reverberation time (Lehmann, E.A et al, 2008) and speech transmission index(Li, K.M., & Lam, P.M. 2005.), and for the room acoustics auralisation (Rindel, J.H., & Christensen, C.L. 2003.). Some work has also been done to apply this method to modelling the vibration of beams and thin plates with various boundary conditions, and it showed good results (Gunda, R. 1995, Cuenca, J. et al, 2009, Cuenca, J. et al, 2014).

Moreover, in order to model wave propagation, an efficient numerical method, called the acoustic wave propagator scheme, was developed. After being proposed by Pan and Wang in 2000, this numerical scheme has been successfully developed and applied to investigate the propagation of acoustic and flexural waves in ducts, beams, plates, and two-dimensional rooms(Peng, S.Z. 2005., Sun, H.M.et al, 2003.), and it was found to be highly accurate and computationally effective. However, in these previous papers, the boundaries were usually introduced by defining the mechanical properties in the wave transmission path. This is a highly efficient approach, but if a very sharp boundary is present, the discontinuity is required to be smoothed by convoluting with a Gaussian function to obtain the numerical stability, thus may cause some negative effect on the accuracy of the result.

In this paper, the image source method and acoustic wave propagator method are combined for investigating the propagation of acoustic and flexural waves towards boundaries, in order to provide a more accurate solution for studying wave propagation under different boundary conditions.

2. THEORY

2.1 Acoustic Wave Propagator

The propagation of waves in air and in solids can be described by the following partial differential equation, which is known as the acoustic wave equation:

$$\frac{\partial}{\partial t}\phi = -\hat{H}\phi, \tag{1}$$

where $\hat{H}$ is defined as the wave propagator and ϕ represents the initial state vector.

For the simplest one-dimensional case, if ϕ represents the sound pressure and particle velocity in a duct, then:

$$\phi(x,t)=\begin{bmatrix} p \\ v \end{bmatrix}, \hat{H}=\begin{bmatrix} 0 & \rho_0 c_0^2 \frac{\partial}{\partial x} \\ \frac{1}{\rho_0}(\frac{\partial}{\partial x}) & 0 \end{bmatrix}, \tag{2}$$

where ρ_0 is the air density and c_0 is the sound speed. If a thin flexible beam is under investigation, then:

$$\phi(x,t)=\begin{bmatrix} V \\ M \end{bmatrix}, \hat{H}=\begin{bmatrix} 0 & -\frac{1}{\rho A}(\frac{\partial^2}{\partial x^2}) \\ EI(\frac{\partial^2}{\partial x^2}) & 0 \end{bmatrix}, \tag{3}$$

where $\mathrm{E}, \rho, A, I=\frac{Ah^2}{12}$ are the Young's modulus, density, cross-sectional area, and cross-sectional area moment of inertia of the beam, respectively. By integrating these equations with respect to time, the solution to the state vector can be obtained as:

$$\Phi(x,t)=e^{-(t-t_0)\hat{\mathbf{H}}}\Phi(x,t_0). \tag{4}$$

Thus, the state vector at any time and any position can be evaluated through the operation of the acoustic wave propagator acting on the initial state vector.

In previous papers, boundaries were usually regarded as a variation of the acoustical media, and thus could be readily included in the system operator. For example, the boundary condition of a semi-infinite duct can be described as a step function:

$$c=\begin{cases} c_0 & x<L \\ c_s & x\geq L \end{cases}, \rho=\begin{cases} \rho_0 & x<L \\ \rho_s & x\geq L \end{cases}, \tag{5}$$

where $x = L$ is the location of the boundary and c_0, ρ_0, c_s, ρ_s are the speeds of sound in and densities of the air and solid, respectively. The result in these papers showed that this is a highly efficient and easy-to-implement way to study the boundary reflection and scattering. But it can become slightly inaccurate, if a great discontinuity is assumed and a boundary smoothing technique is used to obtain the numerical stability. In this paper, the well-known image source method is applied to provide a more accurate solution for studying wave propagation under different boundary conditions.

2.2 Implementation of the Propagator

To implement the exponential propagator in practice, it needs to be expanded as a series of Bessel functions and modified Chebyshev polynomials. Since the polynomials are defined in the range of [−1, 1], the matrix operator $\hat{H}$ needs to be normalised by its maximum eigenvalue:

$$\hat{H}'=\frac{\hat{H}}{\lambda_{\max}}. \tag{6}$$

Then, the so-called J-Chebyshev expansion is given as:

$$e^{-\Delta t\hat{H}}=\sum_{n=0}^{\infty} a_n J_n(R) T_n(\hat{H}'), \tag{7}$$

where T_n is the modified Chebyshev polynomial of the order *n*, J_n is a Bessel function of the first kind, $R=\Delta t\lambda_{\max}$, and the coefficient a_n is defined by:

$$a_n = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases}. \tag{8}$$

The spatial derivatives of the state vector are determined accurately by the fast Fourier transform F :

$$\frac{\partial^n}{\partial x^n}\phi(x,t) = F^{-1}\{(jk_x)^n F[\phi(x,t)]\}, \tag{9}$$

where F^{-1} is the inverse fast Fourier transform, and k_x is the wave number as in e^{jk_x}.

2.3 Image Source Method

The basic principle of the image source method is to replace the boundaries with virtual sources, which are conducted by mirroring the original source. The sound waves are assumed to propagate along straight lines from the original source to the receiver, and thus can be regarded as sound rays. When the rays reach the boundary, they will reflect from it and be caught by the receiver again. This can be considered as sound reaching the receiver from two sources, where the second source is the image source behind the boundary surface, as shown in Fig. 1. Every reflective boundary can create a first-order image source, while the combination of several boundaries would produce multiple higher-order image sources, mirrored by each of these boundaries in turn. The virtual sources are located opposite the boundary, with the same distance as their original source. Their weights are calculated by satisfying the different boundary conditions.

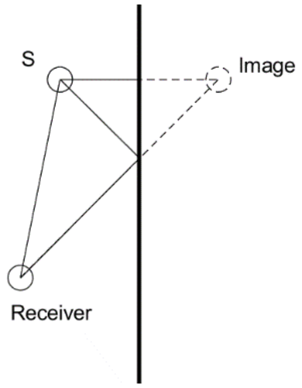

Figure 1: Diagram of the source, receiver, and an image source.

When an incident sound wave packet propagates into a rigid wall, it will be fully reflected. A zero normal velocity boundary condition can be satisfied by placing an image source emitting exactly the same signal, and the total sound pressure is as follows:

$$p_{incident} = p_i e^{ikx}, p_{total} = p_i(e^{ikx} + e^{-ikx}). \tag{10}$$

For a non-rigid boundary condition, the amplitude of the reflected wave is no longer to be the same as the incident wave as some part of the energy is transmitting through the boundary or is absorbed by the wall. Applying the boundary condition that the wave function and the first derivative of the wave function are both continuous, the reflection coefficient can then be calculated by the following equation:

$$r_p = \frac{p_r}{p_i} = \frac{\rho_1 c_1 - \rho_0 c_0}{\rho_1 c_1 + \rho_0 c_0}. \tag{11}$$

Thus, the amplitude weight of the image source is r_p and the total pressure is:

$$p_{total} = p_i (e^{ikx} + r_p e^{-ikx}). \tag{12}$$

Another topic of this paper is to investigate the propagation of flexural waves in a thin beam. A similar initial wave packet is considered:

$$w = Ae^{-i(rx+wt)}. \tag{13}$$

The simplest case, a pinned boundary, is supposed. The boundary condition can be described as Eqs. (14) and we can have the result as Eqs. (15):

$$w(0,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = 0. \tag{14}$$

$$w = Ae^{-i(rx+wt)} - Ae^{i(rx-wt)}. \tag{15}$$

It is easy to see that, for such boundaries, the weights of the image sources are −1.

3. RESULTS AND DISCUSSION

3.1 Duct

The components of the initial state vector are assumed to be:

$$p_s|_{t=0} = \exp\left[-\frac{(x-x_0)^2}{(2a)^2}\right], \quad v_s|_{t=0} = \frac{1}{\rho_0 c_0}\exp\left[-\frac{(x-x_0)^2}{(2a)^2}\right]. \tag{16}$$

For simulating a rigid wall, we could set the image source like so:

$$p_{image}|_{t=0} = \exp\left[-\frac{(x-x_0)^2}{(2a)^2}\right], \quad v_{image}|_{t=0} = -\frac{1}{\rho_0 c_0}\exp\left[-\frac{(x-x_0)^2}{(2a)^2}\right]. \tag{17}$$

Meanwhile, for a non-rigid wall with a reflection coefficient r_p, the amplitude of the image source should be multiplied by the coefficient. The parameters used in simulation are $a = 0.1,\ \rho_0 = 1.21,\ c_0 = 340$, and $r_p = 0.5$. The results are shown in Fig. 2.

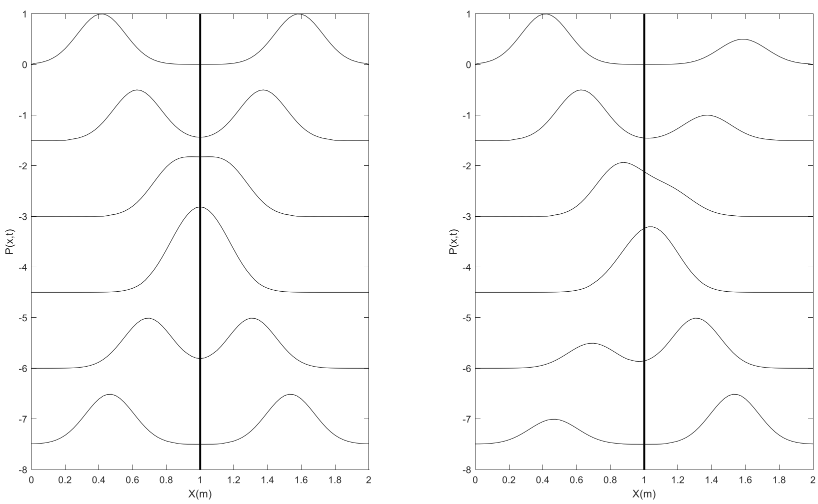

Figure 2: Sound pressure wave propagation in a duct with (a) a rigid boundary and (b) a non-rigid boundary at x = 1 m (vertical black line). Each curve is offset by 1.5 for clarity.

The wave packets reach the boundary at the same time, which thus produces the largest value of the sound pressure. Then the waves emitted by the image sources continue propagating in the opposite direction, and thus can be seen as the reflected waves. For the rigid boundary, the amplitude of the reflected wave is the same as that of the incident wave, while some incident waves will be absorbed by the non-rigid boundary and therefore reduce the magnitude.

3.2 Beam

Similarly, we assume an initial condition as follows:

$$W|_{t=0} = \exp\left[-\frac{(x-x_0)^2}{(2a)^2}\right], \qquad (18)$$

while the amplitude of the image source will be multiplied by −1. The properties are as follows: $a = 0.05$, $\mathrm{h} = 0.005$, $\rho_0 = 7700$, $E = 19.5\times10^{10}$, υ=0.28, and $A = 10^{-4}$. The simulation results are shown in Fig. 3.

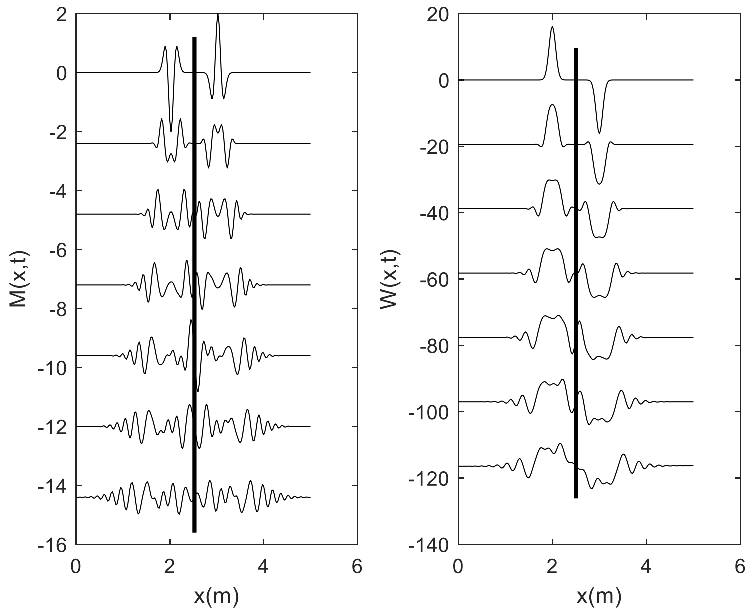

Figure 3: (a) Bending moment and (b) displacement at different times with a pinned boundary at *x* = 2.5 m (vertical black line). For clarity, each curve is offset by (a) 2 and (b) 20, respectively.

In previous papers, the rigid boundary was simulated by setting the characteristic impedance to be far greater than that of air, usually 10 times greater, or by changing the thickness and cross-section of the beam, and then smoothing by convoluting with a Gaussian function. In comparison with the results of the previous papers, the image source method is believed to be a more accurate way of simulating boundary reflection.

4. CONCLUSION

In this paper, the acoustic wave propagator method and the widely used image source method were combined to simulate the one-dimensional propagation of acoustic and flexural waves in front of different boundary conditions. It should be easy to extend this method to two-dimensional cases, so it could be applied to investigate some interesting topics of room acoustics, such as the prediction of reverberation time, and to acoustic auralisation technology.

5. ACKNOWLEDGEMENTS

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