# Radial vibration theory for thick-walled Hookean elastic hollow spheres with large strains and immersed in an incompressible fluid

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*Abstract* - This paper exploits the results of a previous paper that develops the theory of thick-shelled Hookean elastic hollow spheres not limited by small strain approximations. Large strain elasticity theory may give some insights into soft bodies subjected to high pressures such as deep underwater. The previous paper solved analytically the radial vibration (breathing mode) problem for a Poisson ratio equal to zero and an incompressible inner core. This paper extends to cases of nonzero Poisson ratio and derives an effective spring constant and radial vibrational mass (RVM) for harmonic oscillator equivalent to the sphere. For simplicity in this paper, the sphere's inner hollow does not exchange energy with the elastic material which applies to two hollow types, an incompressible core and a vacuum. The spring constant only needs to consider displacements at the outer surface whereas the RVM requires integrating the kinetic energy over the thickness of the shell. A consequence of a nonzero Poisson ratio is extreme strains have a transition to unphysical results interpreted as being a rupture. At slightly lower strains the spring constant may be negative meaning the shell stores less potential energy as it expands and is a possible precursor towards a rupture.

## 1 INTRODUCTION

An interesting question explored in a previous paper (McMahon, 2023) is what physical limits would exist for highly compressed or inflated hollow spheres with a shell made of a hypothetically indestructible Hookean material defined by two elastic constants, Young's modulus $E$ and Poisson's ratio $\nu$. For mathematical completeness, both positive and negative inner and outer pressures were considered. Negative pressures for instance could occur from electrically charged surface exposed to an attractive electric field. For simplicity either the inner hollow or the outer infinite volume was a vacuum defined by a pressure of zero. The main modification to the elasticity equations for unlimited strains was to rewrite them in terms of displaced radii $r'$ (rather than the usual un-displaced radii $r$ ) and to define the balance between radial and tangential (also called hoop) stresses to be in terms of displaced radii. It was found that $\nu \le 0$ allows compression to zero volume at a maximum outer pressure proportional to $E$ and depending on $\nu$. Unlimited inflation by an inner positive pressure or outer negative pressure is allowed. $\nu < 0$ also admits solutions with discontinuous transitions to unphysical negative radii. For $\nu > 0$, solutions transition continuously with pressure between physical and unphysical domains. Inflation for example, where the elastic shell radii expand, becomes unphysical when the thickness of spherical layers reach zero and extrapolate to negative values. This was likened to the inflation of a rubber balloon that reaches a maximum inner pressure, then decreases pressure as a balloon expands faster due to increasing gas, then bursts (Mangan and Destrade, 2015). The beginning of an unphysical domain for highly inflated elastic shells might lead to a different explanation for bursting than currently published for balloons.

This paper investigates highly strained hollow elastic shells in more detail, particularly how the Poisson ratio affects radial (breathing) mode vibration. This analysis is limited to $\nu \ge 0$ showing the effects close to unphysical domains. For simplicity there is no energy transfer between the elastic shell and material within the hollow, restricting the analysis to two cases, an inner vacuum defined by a pressure of zero and an incompressible spherical core defined by a radius that is constant. Energy is only transferred between the elastic shell and the outer incompressible fluid. So, with only one surface for energy exchanges, it is found that the radial mode is equivalent to a single harmonic oscillator requiring only the derivation of the spring constant and vibrational mass and hence the resonant frequency.

The system spring constant $\kappa_{system}$ , is the sum of two parts, one from the shell elastic potential energy (PE) and the other from incompressible fluid PE caused by its pressure and outer radius movement. The net PE of the system is proportional to the square of the outer radius deviation from its equilibrium value, so $\kappa_{system}$ is obtained from its proportionality factor.

The system vibrational mass $M_{system}$ is obtained from the total kinetic energy (KE) of radial motion. The KE is the sum of elastic shell and outer fluid parts. Deriving the shell part of the KE is complicated by the dependence of the displaced density $\rho'$ on the radius $r'$ of each spherical layer within the shell. Also, the shell radial speed depends on $r'$ . The elasticity equations show that this radial speed is proportional to the radial speed at the outer radius $r'_{out}$ so the sum of KE over all $r'$ is proportional to the square of $\partial r'_{out} / \partial t$ [1]. The elastic shell part of $M_{system}$ is a proportional factor of the shell KE. The surrounding fluid part is included in $M_{system}$ , often called the added mass, which is easily calculated for an incompressible fluid as used in Minnaert's bubble model (Leighton, T. G., 1997). The resonant frequency of radial vibrations is just the well-known formula for a harmonic oscillator with spring constant $\kappa_{system}$ and mass $M_{system}$ . This frequency is only valid if the radial wavelength of pressure waves is much larger than the thickness of the shell.

## 2 LARGE STRAIN ELASTICITY THEORY FOR A HOOKEAN SPHERICAL SHELL

This section summarises the notation and basic relationships previously defined and derived for large strain elastic theory (McMahon, 2023). The radial strain at a thin spherical shell of stress loaded thickness $z'$ and unloaded thickness z is defined by $\varepsilon_r = z'/z - 1$ which in radial differential form becomes $\varepsilon_r = \partial r'/\partial r - 1$. The tangential strain of a spherical shell of stress loaded circumference $2\pi r'$ and unloaded circumference $2\pi r$ is defined by $\varepsilon_\theta = r'/r - 1$. The equations for large strains are greatly simplified by defining functions $\gamma_r$ and $\gamma_\theta$ , and a constant $\eta$ , where $\eta\gamma_r = 1 + \varepsilon_r$ and $\eta\gamma_\theta = 1 + \varepsilon_\theta$ . $\eta$ is a calibration factor that takes care of relating displaced radii to inner and outer pressures at the shell surfaces. For small strains, this method gives the same results as Lame′′s for a hollow sphere (McMahon, 2023). The displaced radius is $r' = \eta\gamma_\theta r$ and its radial derivative is $\partial r'/\partial r = \eta\gamma_r$ . The relationship between $\gamma_r$ and $\gamma_\theta$ is derived from solving the differential strain and stress equations that are coupled through Hooke's law. This leads to

$$\gamma_r = \gamma_\theta^{1-3\alpha} - \frac{1}{\alpha}(1-\alpha)\gamma_\theta, \qquad \alpha = \frac{1}{3}\frac{3-5\nu}{1-\nu} \tag{1}$$

In terms of $r$ and $\alpha$ , $\gamma_\theta$ is

$$\gamma_\theta = \left(\alpha + \psi\frac{1}{r^3}\right)^{\frac{1}{3\alpha}} \tag{2}$$

where $\psi$ is a variable independent of radius $r$ . Varying $\psi$ in $\gamma_\theta$ allows surface radii and pressures to be scanned but has no obvious physical interpretation except it is analogous to the amount of gas that relates radii and pressures for an inflated balloon.

[1] Partial derivative notation $\partial$ is used as in A.E.H. Love's treatise on the mathematical theory of elasticity.

Radial and tangential stresses are denoted $\sigma_r$ and $\sigma_\theta$ respectively. For convenience they are made dimensionless by normalising with $E$ so defining $\hat{\sigma}_r = \sigma_r / E$ and $\hat{\sigma}_\theta = \sigma_\theta / E$. The connection between strains and stresses is through Hooke's law which gives for stresses

$$\begin{aligned}\hat{\sigma}_r &= \frac{1}{1-2\nu}(\eta R - 1),\ R = \gamma_\theta + \frac{1-\nu}{1+\nu}(\gamma_r - \gamma_\theta) \\ \hat{\sigma}_\theta &= \frac{1}{1-2\nu}(\eta S - 1),\ S = \gamma_\theta + \frac{\nu}{1+\nu}(\gamma_r - \gamma_\theta)\end{aligned} \tag{3}$$

Solutions are found as a result of boundary conditions. The "soft" boundary condition this paper uses is the $E$-normalised pressure at the inner shell surface is fixed at $\hat{P}_{in} = -\hat{\sigma}_{r,in} = 0$ giving the calibration factor $\eta = \left(1 - (1-2\nu)\hat{P}_{in}\right) / R_{in}$ where $\gamma_{r,in}$ and $\gamma_{\theta,in}$ at the inner surface give $R_{in}$ from Eqn. (3). The "hard" boundary condition constrains the inner radius of the shell to be fixed hence $r'_{in} = \eta\gamma_{\theta,in} r_{in} = r_{in}$ and the calibration factor is $\eta = 1/\gamma_{\theta,in}$. Unphysical domains arise where a layer thickness goes negative such as $\gamma_{\theta,in} \le 0$.

## 2.1 Comparison of "hard" and "soft" core elastic spherical shell radii versus surface pressure and pressure differences

*Table 1 – Parameters for demonstration of high strain effects on Hookean elastic spherical shells vibration*

| Parameter | Symbol | Units | Value |
|---|---|---|---|
| Shell inner radius | $r_{in}$ | metres | 0.5 |
| Shell outer radius | $r_{out}$ | metres | 1 |
| Young's modulus | $E$ | kg/ms$^2$ | 10000 |
| Shell density | $\rho$ | kg/m$^3$ | 1000 |
| First fluid density | $\rho_1$ | kg/m$^3$ | 1000 |
| Second fluid density | $\rho_2$ | kg/m$^3$ | 0 |

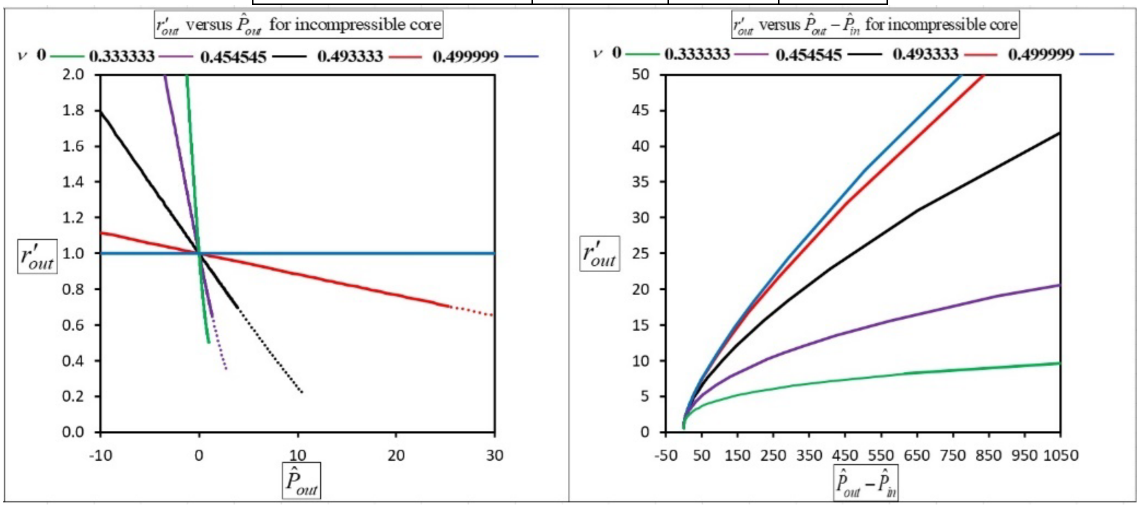


*Figure 1 – Outer radii vs outer pressure and pressure difference for an incompressible core and different shell $\nu$.*

The left-side plots of Fig. 1 show that outer radii versus outer pressure reach higher pressures for larger $\nu$. The right-side plots cover larger radii. The lowest pressure difference corresponds to the highest pressures hence lowest outer radius. Dotted lines are unphysical domains where $\gamma_{\theta,in} \le 0$. Increasing the outer pressure decreases the outer radius as expected for a positive spring constant. The outer pressure is larger than inner pressure, even when both pressures are negative.

Incompressibility of the core together with ideal rubber $\nu = 1/2$ makes outer radius changes extremely small for medium pressures. Owing to the constant $1-2\nu$ appearing in the elasticity equations, if the inner core is incompressible and shell close to $\nu$=1/2 of idealised rubber, it takes very large pressures $\hat{P}_{out}$ and $\hat{P}_{in}$ but only moderate differences between them to show the $r'_{out}$ changes seen in Fig. 1.

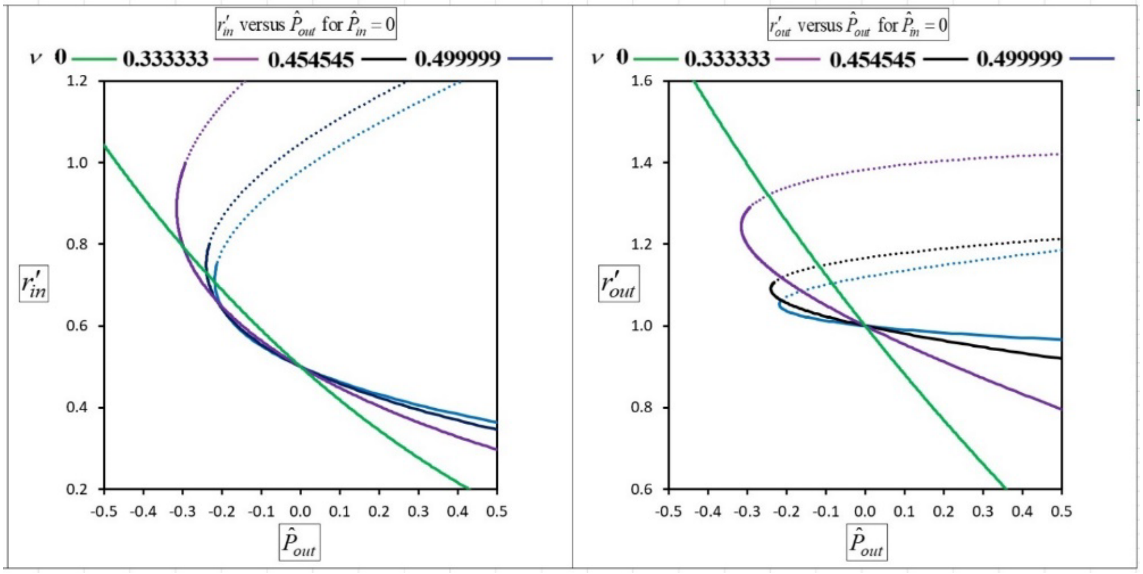


*Figure 2 – Left and Right plots show respectively inner radii and outer radii vs outer pressure for several shell $\nu$ in the case of a vacuum spherical core. Dotted lines are unphysical domains where $\gamma_{\theta,in} \leq 0$ .*

Figures 1 and 2 contrast "hard" and "soft" cores and both show an overall decrease the outer radius with pressure but have their unphysical domains at opposite ends of the pressure range. Also, from the theory of Subsection 3.1 and Fig. 2, the vacuum core case has a negative spring constant domain.

## 3 HARMONIC OSCILLATOR PARAMETERS OF AN ELASTIC SPHERICAL SHELL

This section explains how the harmonic oscillator parameters $\kappa_{system}$ and $M_{system}$ for Hookean elastic shell radial vibration are derived.

### 3.1 Spring constant for an elastic shell with an incompressible inner core and surrounded by an outer incompressible fluid

An incompressible inner hard core is defined by an immovable inner radius hence $r'_{in} = r_{in}$ . Then $\eta = 1/\gamma_{\theta,in}$ determines $\eta$ as a function of $\psi$ and $r$ . It also determines the inner and outer pressures, the internal stresses and displaced radii as functions of un-displaced radii $r$ .

Starting at any stable condition of outer radius $r'_{out}$ with outer *E*-normalised pressure $\hat{P}_{out}$ , deviations $\delta_{r'_{out}}$ such as from radial vibration cause the net sphere – fluid system PE, denoted $U_{system}$ , to shift from its local minimum by $\delta_{U_{system}}$ which is a quadratic function of $\delta_{r'_{out}}$ i.e. $\delta_{\hat{U}_{system}} = \hat{\kappa}_{\text{system}} \delta^2_{r'_{out}} / 2$ where $\hat{\kappa}_{\text{system}}$ is the *E*-normalised spring constant. $\delta_{U_{system}}$ is the sum of shell elastic linear and quadratic $\delta_{r'_{out}}$ terms and likewise outer fluid linear and quadratic $\delta_{r'_{out}}$ terms as shown in McMahon (2023) Eqns. (25) and (26). The linear terms cancel leaving only the quadratic terms from which we find $\hat{\kappa}_{system}$ is given by

$$\hat{\kappa}_{system} = -4\pi r_{out}'^{2} \left( \frac{\partial \hat{P}_{out}''}{\partial r_{out}''} \right)_{r_{out}'} \tag{4}$$

where $\hat{P}_{out}''$ is the normalised pressure that makes the outer radius $r_{out}''$ stable. $\hat{\kappa}_{system}$ is normally expected to be positive which corresponds to a decrease in outer radius when the outer pressure increases. Indeed Fig. 1, where the system is an incompressible core with elastic shell, shows $\partial \hat{P}_{out}'' / \partial r_{out}'' < 0$ hence $\hat{\kappa}_{system} > 0$ in the physical domain $\gamma_{r,in} > 0$ for every $\nu$. However for an inner vacuum $\hat{P}_{in} = 0$ and close to the transition to an unphysical domain $\gamma_{r,in} \le 0$, Fig. 2 shows $\partial \hat{P}_{out}'' / \partial r_{out}'' > 0$ so that $\hat{\kappa}_{system}$ becomes negative. This means that the system PE reduces with increasing outer radius approaching the unphysical domain so that in a dynamical situation the system KE must increase which would be a precursor to shell rupture.

### 3.2 Analytical solutions for spring constants of an elastic shell with incompressible core

The subscript "system" refers to the combined elastic shell Poisson ratio $\nu$ and radii, the type of core material, and the density of the surrounding incompressible fluid. For $\hat{\kappa}_{system}$, sufficient details of "system" are $\nu$, *core* where "*core*" is either "*vac*" for vacuum or "*inc*" for an incompressible spherical core. The outer fluid density is only included in "system" to indicate the added mass contribution to the radial vibrations mass $M_{system}$.

Analytical evaluation of $\hat{\kappa}_{\nu,inc}$ from Eqn. (4) is easiest done using the expressions for pressure and radii in terms of $\psi$ then employing $\partial \hat{P}_{out}'' / \partial r_{out}'' = \left( \partial \hat{P}_{out}'' / \psi \right) / \left( \partial r_{out}'' / \psi \right)$.

For an incompressible inner core, the expression for the spring constant with any $\alpha \ne 1/3$ is:

$$\hat{\kappa}_{\nu,inc} = 4\pi\left(5-3\alpha\right)\frac{1}{r_{out}}\frac{1}{r_{out}'^{\,3\alpha-2}}\frac{1}{\left(r_{out}{}^{3}-r_{in}{}^{3}\right)}\left(\frac{1}{3\alpha-1}r_{out}{}^{3}r_{out}'^{\,3\alpha}-\frac{1}{4-3\alpha}\frac{1}{\alpha}\left(\left(1-\alpha\right)r_{out}'^{\,3\alpha}-r_{out}{}^{3\alpha}\right)r_{in}{}^{3}\right) \tag{5}$$

Equation (27) of McMahon (2023) is Eqn. (5) evaluated for $\nu = 0,\ \alpha = 1$ and shown in Eqn. (6). Another case

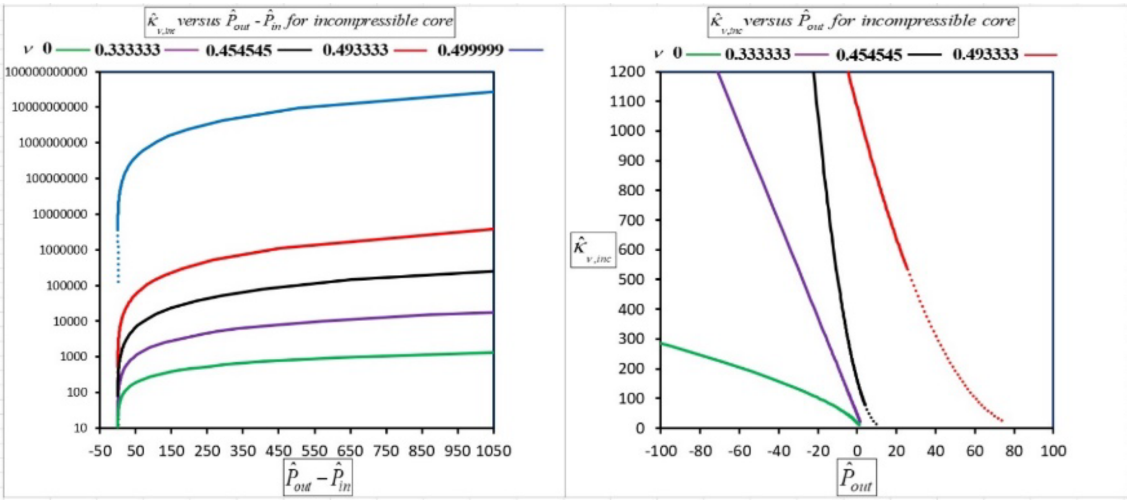


*Figure 3 – Poisson ratio, pressure difference and outer pressure dependence of the spring constant for a spherical elastic shell with incompressible inner core.*

of interest is $\nu = 1/3,\ \alpha = 2/3$ similar to many solids also shown in Eqn. (6). Requiring special analysis is idealised rubber with the maximum possible Poisson ratio $\nu = 1/2,\ \alpha = 1/3$. Owing to the constant $1-2\nu$ appearing in the elasticity equations, finite results require an infinitesimal offset of $1-2\nu$ from zero then taking the limit to zero at the end of deriving an expression for $\hat{\kappa}_{1/2,inc}$. Infinitesimal but nonzero offsets from $\nu$ slightly below 1/2 in Eqn. (5) are a good approximation to $\hat{\kappa}_{1/2,inc}$.

$$\begin{aligned}\hat{\kappa}_{0,inc} &= 4\pi \frac{r_{out}^{\,2}}{r'_{out}} \frac{r'^{\,3}_{out} + 2r_{in}^{\,3}}{r_{out}^{\,3} - r_{in}^{\,3}} \\ \hat{\kappa}_{1/3,inc} &= 3\pi \frac{1}{r_{out}} \frac{1}{\left(r_{out}^{\,3} - r_{in}^{\,3}\right)} \left(4r_{out}^{\,3} r'^{\,2}_{out} - r_{in}^{\,3} r'^{\,2}_{out} + 3r_{out}^{\,2} r_{in}^{\,3}\right)\end{aligned} \tag{6}$$

## 3.3 Analytical solutions for spring constants for an elastic shell with vacuum core

Applying the inner radius boundary condition $\hat{P}_{in}=0$ the *E*-normalised system spring constant for any $\alpha \neq 1/3$ is:

$$\begin{aligned}\hat{\kappa}_{\nu,vac} &= 4\pi \frac{5-3\alpha}{3\alpha-1} \frac{1}{(4-3\alpha)} \frac{r'^2_{out}}{r_{out}} \left[ BC\left(1 - \frac{C}{A}\right) - \frac{1}{\alpha}(\alpha-1)(3\alpha-1)\right] \\ A &= \frac{1}{3\alpha}\left[\left(\frac{r_{out}}{r_{in}}\right)^{3\alpha}\left(\frac{r'_{in}}{r'_{out}}\right)^{3\alpha} + (3\alpha-1)\frac{r_{out}^{\,3}}{r_{in}^{\,3}}\left(1 - 3\frac{(1-\alpha)}{(4-3\alpha)}\left(\frac{r'_{in}}{r_{in}}\right)\right)\right] \\ B &= (4-3\alpha)\left(\frac{r'_{in}}{r_{in}}\right)^{-1} - \frac{1}{\alpha}(1-\alpha)(3\alpha-1),\; C = \left(\frac{r_{out}}{r_{in}}\right)^{3\alpha}\left(\frac{r'_{in}}{r'_{out}}\right)^{3\alpha}\end{aligned} \tag{7}$$

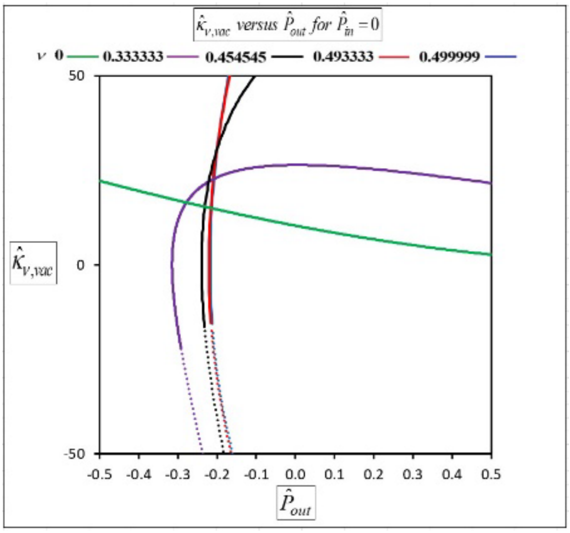


*Figure 4 – Poisson ratio and outer pressure dependence of the spring constant for an elastic shell with a vacuum core.*

These “soft” inner core cases produce a negative spring constant close to the transition to an unphysical inner layer thickness $\gamma_{r,in} \leq 0$.

Two cases that reduce to simpler formulae, $\nu = 0,\ \alpha=1$ and $\nu = 1/2,\ \alpha = 1/3$, give

$$\begin{aligned}\hat{\kappa}_{0,vac} &= 8\pi\eta^3 \frac{r_{out}^{\,2}}{r'_{out}} \frac{r'^{\,3}_{out} - r'^{\,3}_{in}}{2r'^{\,3}_{out} + r'^{\,3}_{in}} \\ \hat{\kappa}_{1/2,vac} &= \frac{32\pi}{3} \frac{r'^2_{out} r_{out}^{\,2}}{r_{in}^{\,3} r_{out}^{\,3}} \left(r_{out}^{\,3} - r_{in}^{\,3}\right)\left(\frac{3}{2}\frac{r_{in} r_{out}}{r_{in}' r'_{out}} - 1\right)\end{aligned} \tag{8}$$

### 3.4 Radial vibrational mass derivations for an elastic shell

Omitting the added mass of the outer fluid, the RVM of an elastic shell $M_{\nu,core}$ of any $\nu$ and core material is defined from the vibrational KE by

$$K_{r'_{out}} = \frac{1}{2} M_{\nu,core} \left( \frac{\partial \delta_{r'_{out}}}{\partial t} \right)^2 \tag{9}$$

where $\delta_{r'_{out}}$ is a deviation from the average outer radius $r'_{out}$. Evaluation of $K_{r'_{out}}$ is by the integration over $r'_{in} \le r' \le r'_{out}$ of all infinitesimal thickness layer KE from speed $\partial \delta_{r'} / \partial t$. The radial speed $\partial \delta_{r'} / \partial t$ for each layer is proportional to $\partial \delta_{r'_{out}} / \partial t$ and a function of $r'$ derived from the elasticity solutions. Hence apply the relation $\partial \delta_{r'} / \partial t = \left( \partial r' / \partial r'_{out} \right) \partial \delta_{r'_{out}} / \partial t$. Also use the analytical relation to $\psi$ of all radii to evaluate $\partial r' / \partial r'_{out} = \left( \partial r' / \partial \psi \right) / \left( \partial r'_{out} / \partial \psi \right)$. The shell KE and RVM are given by

$$K_{r'_{out}} = \frac{1}{2} \left( 4\pi \int_{r'_{in}}^{r'_{out}} \rho' r'^2 \left( \frac{\partial r'}{\partial r'_{out}} \right)^2 dr' \right) \left( \frac{\partial \delta_{r'_{out}}}{\partial t} \right)^2, \quad M_{\nu,core} = 4\pi \int_{r'_{in}}^{r'_{out}} \rho' r'^2 \left( \frac{\partial r'}{\partial r'_{out}} \right)^2 dr' \tag{10}$$

The displaced density $\rho'$ is derived from the strains and is generally a function of $r'$.

### 3.5 Radial vibrational mass for an elastic spherical shell with an incompressible core

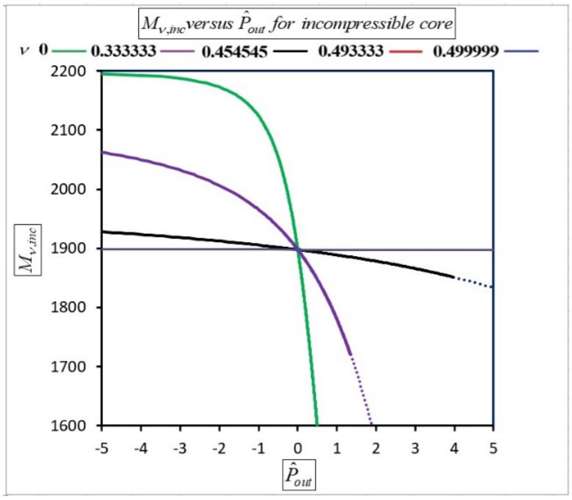


*Figure 5 – Plots of shell RVM vs pressure for an incompressible inner core. The plots show the RVM decreasing due to the outer radius decreasing with increasing pressure, although very little for ν~1/2. Unphysical domains are shown dotted.*

The RVM has the form $M_{\nu,inc} = M_\nu I_\nu$ where

$$M_\nu = 4\pi\rho \frac{{r'_{out}}^{6\alpha-2}}{{r_{out}}^{6\alpha-6}} \frac{1}{\left( {r_{out}}^3 - {r_{in}}^3 \right)^2} \tag{11}$$

and $I_\nu$ is a sum of integrals over the elastic shell radii. The simplest formula for $M_{\nu,inc}$ is for $\nu = 0$ giving

$$M_{0,inc} = \frac{3}{5} M \frac{r'^3_{out}\left(r'^3_{out} + 3r'^2_{out} r_{in} + 6r'_{out} r_{in}{}^2 + 5r_{in}{}^3\right)}{\left(r'^2_{out} + r'_{out} r_{in} + r_{in}{}^2\right)^3}, \quad M = (4\pi/3)\rho\left(r_{out}{}^3 - r_{in}{}^3\right) \tag{12}$$

Numerical results are plotted in Fig. 5. The largest RVM are for the lowest outer pressure, particularly negative pressures where their suction enlarges the shell radii and radial speeds.

### 3.6 Radial vibrational mass for an elastic spherical shell with a vacuum core

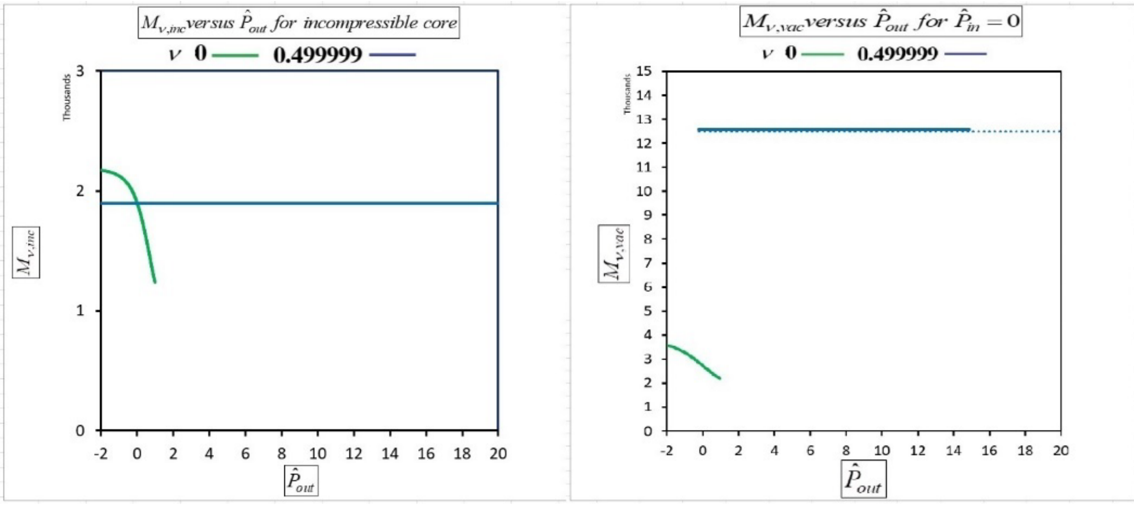


*Figure 6 –The RVM of an elastic shell with an incompressible core (left) and a vacuum core (right).*

The less constrained motion for an inner vacuum and $\nu > 0$ gives rise to the larger RVM seen in Fig. 6. Equation (13) gives analytic formulae for the RVM.

$$M_{0,vac} = \frac{3}{5} M \frac{r'_{out}{}^3}{\left(r'_{out}{}^2 + r'_{out} r'_{in} + r'_{in}{}^2\right)\left(2r'_{out}{}^3 + r'_{in}{}^3\right)^2}\left(5r'_{in}{}^5 + 14r'_{out} r'_{in}{}^4 + 14r'_{out}{}^2 r'_{in}{}^3 + 4r'_{out}{}^3 r'_{in}{}^2 + 4r'_{out}{}^4 r'_{in} + 4r'_{out}{}^5\right)$$

$$M_{1/2,vac} = 3M \frac{r_{out}{}^3}{r_{in}\left(r_{out}{}^2 + r_{out} r_{in} + r_{in}{}^2\right)} \tag{13}$$

### 3.7 Added mass from an outer incompressible fluid

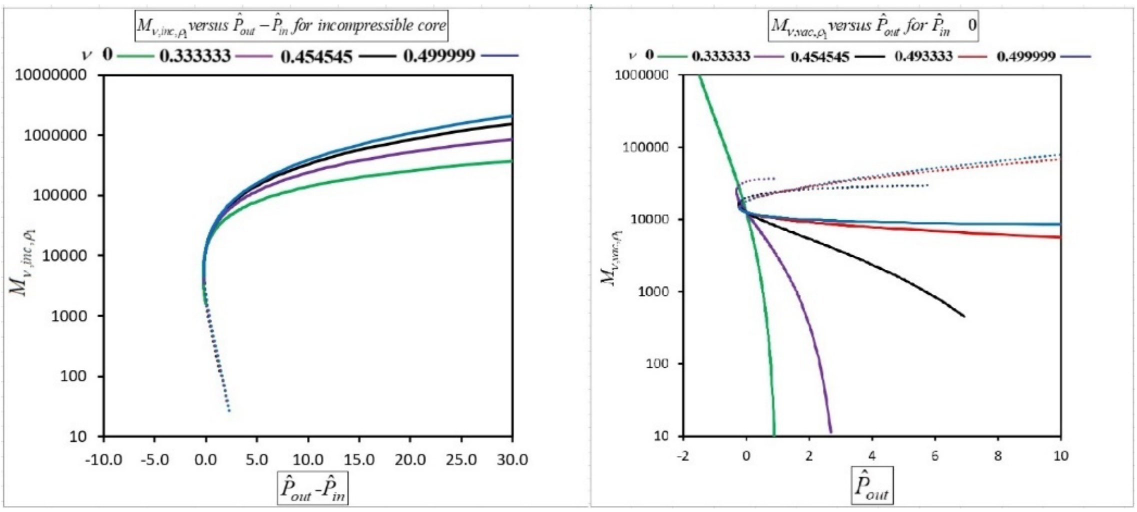


*Figure 7 –Compares added mass vs pressure difference for incompressible core (left) and vacuum core (right). Both plots correspond to more compression hence lower added mass with higher outer pressure but for an incompressible core the pressure difference decreases with higher pressures, opposite to what happens for a vacuum core.*

The contribution of the outer fluid to the system RVM is denoted by the subscripts $\nu,\ core,\ \rho_n$ where $n$=1,2 indicate the two fluid densities listed in Table 1. Hence $M_{\nu,vac,\rho_n} = 4\pi r'^3_{out}\rho_n$ and $M_{\nu,inc,\rho_n} = 4\pi r'^3_{out}\rho_n$ are different added masses because the outer radius $r'_{out}$ is different for vacuum and incompressible cores. The total system RVM by summing the elastic shell RVM and fluid added mass is denoted $M_{\nu,vac,sys\ n} = M_{\nu,vac} + M_{\nu,vac,\rho_n}$, $n = 1,2$ and $M_{\nu,inc,sys\ n} = M_{\nu,inc} + M_{\nu,inc,\rho_n}$, $n = 1,2$ for vacuum and incompressible cores respectively. Figure 7 shows the significant differences of added mass for “hard” and “soft” cores in the case of nonzero $\rho_1$. Both plots are added mass versus the difference of outer and inner pressures $\hat{P}_{out} - \hat{P}_{in}$. The added mass increases with the pressure difference for an incompressible core since the outer radius also increases as shown in Fig. 1.

### 3.8 Poisson ratio effect on the pressure dependence of breathing mode resonant frequency

From the well-known frequency formula for a 1D harmonic oscillator, the resonant frequencies for an incompressible and vacuum core are:

$$f_{\nu,inc,sys\ n} = \frac{1}{2\pi}\sqrt{E\frac{\hat{\kappa}_{\nu,inc}}{M_{\nu,inc,sys\ n}}}, \qquad f_{\nu,vac,sys\ n} = \frac{1}{2\pi}\sqrt{E\frac{\hat{\kappa}_{\nu,vac}}{M_{\nu,vac,sys\ n}}},\ n = 1,2 \tag{14}$$

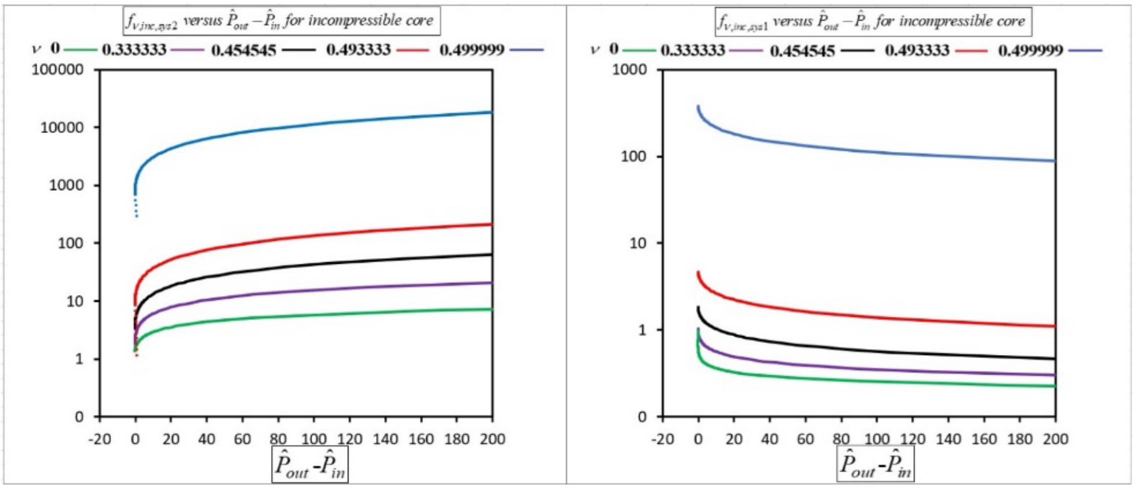


*Figure 8 – Incompressible core comparison of resonant frequencies from several Poisson ratios and outer fluid density zero (left) and nonzero (right). Low and high Poisson ratios show similar pressure difference dependence.*

Figure 8 demonstrates the effect of the Poisson ratio and outer fluid added mass on the resonant frequency for an incompressible core. For zero added mass, the spring constant (Fig. 3) increases the frequency (left) and increasing added mass decreases the frequency (right) with increased pressure difference.

For a vacuum core, Fig. 9 shows that $\nu = 0$ and $\nu = 1/2$ cause opposite pressure dependence of the resonant frequency instead of similar shown by Fig. 8.

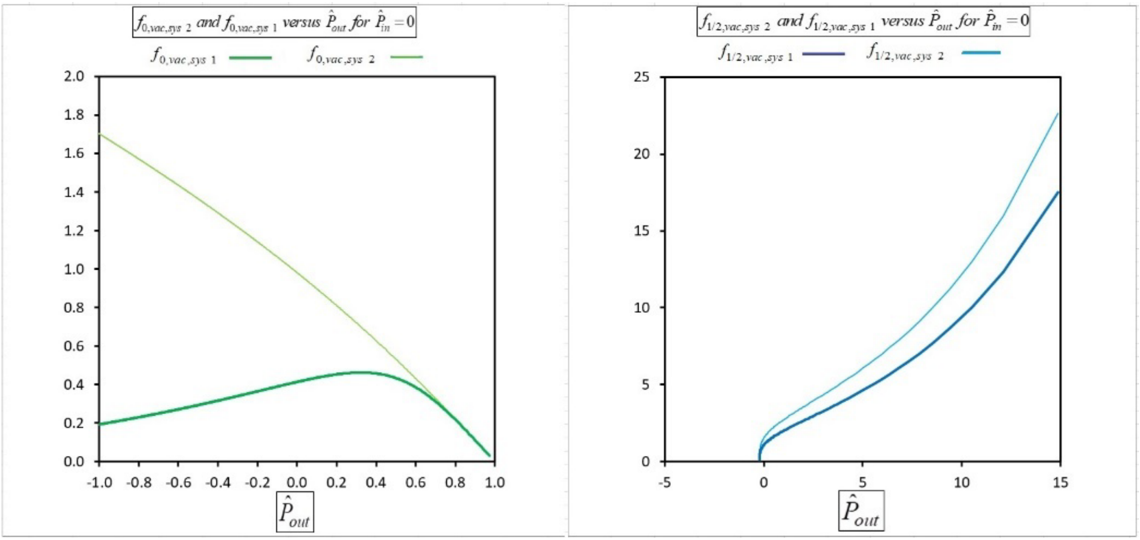


*Figure 9 – Comparison of resonant frequencies for two Poisson ratios and two outer fluid densities. Low and high Poisson ratios produce the opposite pressure dependence in the case of a vacuum core.*

## 4 FINAL REMARKS

It is found that the high strain characteristics of a thick-walled elastic shell with $\nu > 0$ are sensitive to whether the inner volume is a compressible or an incompressible core. The spring constant defined for a "soft" core has a smaller outer pressure range compared to a "hard" core. The softness or hardness of the inner core is "felt" at the outer elastic surface. Both types of inner core have a transition to an unphysical domain defined by zero or negative spherical layer thickness. The "soft" case differs by having a domain where the spring constant is physically realistic but negative, which means that stretching the spring reduces the stored PE. In a dynamic situation expansion of a shell with a negative spring constant would accelerate by energy conservation leading to bursting. The spring constant for the "hard" core case is positive so the unphysical domain would not be reached as a result of the shell's own dynamics. It is likely an inner compressible gas of sufficient pressure would also prevent a negative spring constant.

This paper goes beyond Lame′s solution for a stressed hollow sphere made of Hookean elastic material by investigating unlimited stresses and strains assuming indestructible elastic material. "Bursting" can then be a consequence of the high stain theory for a curved geometry rather than a deviation from Hooke's law.

Further development would allow energy exchanges between the elastic shell and the core material. This may create inner and outer spring constants that interact and cause a more complex vibration spectrum than a single spring harmonic oscillator.

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