N:M Coherence for determining the causality of a transfer function with a non-linear frequency relationship

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ABSTRACT
The standard coherence statistic provides a numerical value indicative of the causality between an input signal and an output signal, for a linear system, where there is a 1:1 relationship in the frequency of the input signal that causes the output response. However, in many systems the characteristics of the system has a frequency multiplication effect, such as a fan with blades, where the blade-pass-frequency is a multiple of the shaft rotation frequency. The use of the standard coherence for this type of system would result in a coherence value of 0. The n:m coherence algorithm addresses the non-linear frequency relationship and provides an estimate of the coherence.

1 INTRODUCTION
The coherence function is commonly used for evaluating the linear dependence between input and output signals. The term 'linear dependence' means that the excitation at one frequency will cause a response in the system at the same frequency, termed iso-frequency, which is the case for many engineering systems. Vibration specialists utilise the coherence statistic when conducting modal analysis, which can involve striking an instrumented hammer on a structure and using accelerometers to measure the vibration response. The coherence statistic is used to quantify the quality of the measured frequency response function, and results with poor coherence will be discarded. The coherence metric can also be used to determine the coherent-output-power (COP), and non-coherent-output-power (NCOP), which is used to identify or separate uncorrelated noise from a desired signal (Bendat and Piersol, p. 185, p. 194). Coherence is also used to characterise or identify the transmission paths of energy through a system.

However, there are many examples in engineering systems where the excitation at one frequency induces a response at other frequencies. For example, consider a fan with 6 blades that rotates at frequency f_1, such that the blade-pass-frequency (BPF) of the fan is $6f_1$. If a standard coherence measurement were undertaken between the resulting acoustic signal which has a dominate response at the blade pass frequency $(6f_1)$ and the shaft rotational frequency (f_1), the coherence would be 0. Whereas it is appreciated that there is strong causality between the two. Hence, a different algorithm of coherence is required to account for the non-linear interaction.

There are numerous non-linear coherence analysis methods such as:

1. Higher-order coherence functions that includes Bicoherence and Tricoherence.
2. Phase and amplitude coupling techniques that include Phase-Amplitude Coupling (PAC), Cross-Frequency Coupling (CFC), Phase Locking Value (PLV), Nonlinear Phase Synchronization Metrics.
3. Time-frequency based nonlinear coherence that includes Wavelet Coherence Analysis and Hilbert-Huang Transform (HHT) based Coherence, Empirical Mode Decomposition (EMD) with Hilbert Spectral Analysis, Wigner-Ville Distribution, Short-Time Fourier Transform (STFT) with coherence measures.
4. Data-driven and machine learning approaches that includes Empirical Mode Decomposition (EMD) based Coherence, Phase Locking Value (PLV), Support Vector Machines (SVMs) and Deep Learning (CNNs, RNNs, Kernel Canonical Correlation Analysis (KCCA), Nonlinear Principal Component Analysis (NLPCA), Independent Component Analysis (ICA).

5. Nonlinear dynamical system approaches that include Phase Space Reconstruction, Lyapunov Exponents (for coherence in chaotic systems), Recurrence Quantification Analysis (RQA), Synchronization Measures (e.g., phase locking value, mutual synchronization).
6. Generalized coherence frameworks that includes Cross-Spectral Coherence and Nonlinear Transfer Function Estimation.
7. Other specialized methods that include Hyper-coherence function Partial Coherence.

The bicoherence statistic (Poloskei et al., 2018) can be used for analysing second-order nonlinearities, such as $y(t) = x^2(t-\tau)$, and can detect the second-order harmonic $(2f_i)$ and intermodulation coupling $(f_i \pm f_j)$. However, bicoherence cannot detect higher-order intermodulation between multiple input frequencies (such as $f_i \pm f_j \pm f_k, 2f_i \pm 2f_j$) or subharmonic coupling $(4f_i/5)$.

It is beyond the scope of this paper to review the large number of non-linear coherence algorithms with their merits and limitations. The purpose of this paper is to demonstrate the use of one non-linear coherence algorithm, namely n:m coherence, and to provide Matlab code for other investigators to use.

Yang et al. (2015) and Yang et al. (2016) proposed the n:m coherence algorithm, which is a variation of the standard linear coherence algorithm, to measure the coherence in non-linear systems where harmonic or sub-harmonic responses are generated by the input signal. Their research area is in the study of neurological systems, that are highly nonlinear and exhibit various forms of nonlinear interaction between input and output responses. Nonlinear system identification techniques have been used to study neurological system since the 1970s (Marmarelis & Naka, 1972). Yang et al. used the n:m coherence signal processing algorithm, along with other algorithms in the study of neurological systems. The n:m coherence algorithm has also been applied to the study of cryptocurrency trading (Sinha & Yang, 2021) and can also be applied in the analysis of vibration and acoustic systems.

2 ORDINARY COHERENCE AND N:M COHERENCE

2.1 Ordinary Coherence

The squared-modulus of the coherence function, typically referred to as just coherence, is very important in the analysis of linear systems. In addition to quantifying the similarity between signals, it is related to other quantities of interest, such as the signal-to-noise ratio (SNR), coherent output, and noise output spectra.

The magnitude-squared coherence at frequency f between an input signal $x(t)$ and the output signal $y(t)$ that vary with time t is given by (Bendat and Piersol, 2011, p. 135, Eq (5.90))

$$\gamma^2(f) = \frac{\left|S_{xy}(f)\right|^2}{S_{xx}(f) S_{yy}(f)} \tag{1}$$

where $S_{xy}(f)$ is the cross-spectral density between $x(t)$ and $y(t)$, $S_{xx}(f)$ and $S_{yy}(f)$ are the auto-power spectral densities of $x(t)$ and $y(t)$, respectively. The expressions can be calculated in terms of the Fourier transform of the windowed input $X_i(f)$ and output $Y_i(f)$ signal segments, where their time series have been divided into M segments as (Miranda de Sá, 2006)

$$\gamma^2(f) = \frac{\left|\frac{1}{M}\sum_{i=1}^{M} X_i(f)\, Y_i^*(f)\right|^2}{\left|\frac{1}{M}\sum_{i=1}^{M} X_i(f)\right| \left|\frac{1}{M}\sum_{i=1}^{M} Y_i(f)\right|} \tag{2}$$

The magnitude-squared coherence value is between 0 and 1, and is indicative of how well the input signal corresponds to the output signal at each frequency f. This metric is adequate when there is a 1:1 relationship between the excitation signal causing an influence in the response signal.

However, if the response signal that is generated is at a frequency multiple of the frequency of the input signal, then the coherence value will be zero, and an alternative algorithm for evaluating the coherence between the input and output signals would be beneficial.

2.2 N:M Coherence

Yang et al. (2015) and Yang et al. (2016) proposed a variation of the coherence function in Eq. (2), that accounts for a frequency ratio between the input and output signal. The n:m coherence between an input signal at frequency f_{in} and output signal at frequency f_{out} is (Yang et al., 2016, Eq (3); Sinha & Yang, 2021, Eq. (14))

$$\text{nmc}(f_{\text{in}}, f_{\text{out}}) = \frac{\left|S_{xy}^{nm}(f_{\text{in}}, f_{\text{out}})\right|^2}{S_{xx}^{n}(f_{\text{in}})\, S_{yy}^{m}(f_{\text{out}})} \quad , \tag{3}$$

where the frequency ratio of the output and input signals is $f_{\text{out}}\colon f_{\text{in}} = n\colon m$ where n and m are positive integers, the n:m cross-spectrum given by the average of the n'th and m'th order Fourier transforms of the input and output signals and is given by

$$S_{xy}^{nm}(f_{\text{in}}, f_{\text{out}}) = \langle X^n(f_{\text{in}}) \ \left(Y^m(f_{\text{out}})\right)^* \rangle = \frac{1}{M}\sum_{i=1}^{M} X_i^n(f_{\text{in}})\, (Y_i^m(f_{\text{out}}))^* \, , \tag{4}$$

where the * superscript is the complex conjugate operator, and the average n'th and m'th order auto-power spectral densities are given by

$$S_{xx}^{n}(f_{\text{in}}) = \frac{1}{M}\sum_{i=1}^{M} X_i^n(f_{\text{in}})\, (X_i^n(f_{\text{in}}))^* \ ; \ S_{yy}^{m}(f_{\text{out}}) = \frac{1}{M}\sum_{i=1}^{M} Y_i^m(f_{\text{out}})\, (Y_i^m(f_{\text{out}}))^* \, . \tag{5,6}$$

The expressions can be calculated in terms of the Fourier transform of the windowed input $X_i(f)$ and output $Y_i(f)$ signal segments, where their time series have been divided into M segments. The n:m coherence is a magnitude-squared coherence value between 0 – 1, and is indicative of how well the input signal corresponds to the output signal. Thus with the appropriate selection of n & m, the n:m coherence algorithm can quantify harmonic and subharmonic coupling.

The following section provides examples demonstrating the application of the n:m coherence using idealised sine wave signals, and also using measured exhaust noise. The Matlab code to generate the results is available from the Mathworks FileExchange website (Howard, 2025).

3 EXAMPLE APPLICATIONS OF N:M COHERENCE

3.1 Example 1: single sine wave input, single sine wave output

The first example demonstrates the evaluation of n:m coherence for an input signal comprising a single sine wave at 40 Hz, and an output signal that comprises a single sine wave at 120 Hz, both with added noise, and hence $f_{\text{out}}\colon f_{\text{in}} = f_{\text{y}}\colon f_{\text{x}} = 120\colon 40 = 3\colon 1 = n\colon m$. Figure 1 (a) shows the auto-power spectral densities of the input and output signals. Figure 1 (c) shows the n:m coherence map, where n=3, and m=1, and shows there is high coherence at (fx, fy) = (40, 120) Hz, as expected. Figure 1 (d) shows the same results as Figure 1 (c), only that the results have been masked to only show the colour map for coherence values greater than 0.7, and dotted lines are included to highlight the location of the high coherence. Figure 1 (b) shows the evaluation of the ordinary magnitude-squared coherence (blue curve) calculated using Eq. (1) and shows there is no coherence between the input and output signals, which is to be expected as the input excitation and output response frequencies are different. The figure also shows (orange curve) a 'vertical slice' through the map in Figure 1 (c) at fx=40 Hz, and shows that there is high coherence at fy=120 Hz. Hence, the n:m coherence statistic has correctly identified the causality between the input and output signals.

3.2 Example 2: single sine wave input, two sine wave output

The second example is an input signal that comprises a single sine wave at 40 Hz, and the output signal comprises one sine wave at 120 Hz and a second sine wave at 480 Hz. The values of $f_{\text{out}}\colon f_{\text{in}} = f_{\text{y}}\colon f_{\text{x}} = 120\colon 40 = 3\colon 1 = n\colon m$. Figure 2 (a) shows the auto-power spectral densities of the input and output signals. Figure 2 (c) shows the n:m coherence map, where n=3, and m=1, and shows there is high coherence at (fx, fy) = (40, 120) Hz, as expected, and no other high coherence regions. Figure 2 (d) shows the same results as Figure 2 (c), only that the results have been masked to only show the colour map for coherence values greater than 0.7 and dotted lines are included to highlight the location of the high coherence. Figure 2 (b) shows the evaluation of the ordinary magnitude-squared coherence (blue curve) calculated using Eq. (1) and shows there is no coherence between the input and output signals, which is to be expected as the input excitation and output response frequencies are different. The figure also shows (orange curve) a 'vertical slice' through the map in Figure 2 (c) at fx=40 Hz, and shows that there is high coherence at fy=120 Hz, and not at 480 Hz. Hence, the n:m coherence statistic has correctly identified the causality between the input and output signals.

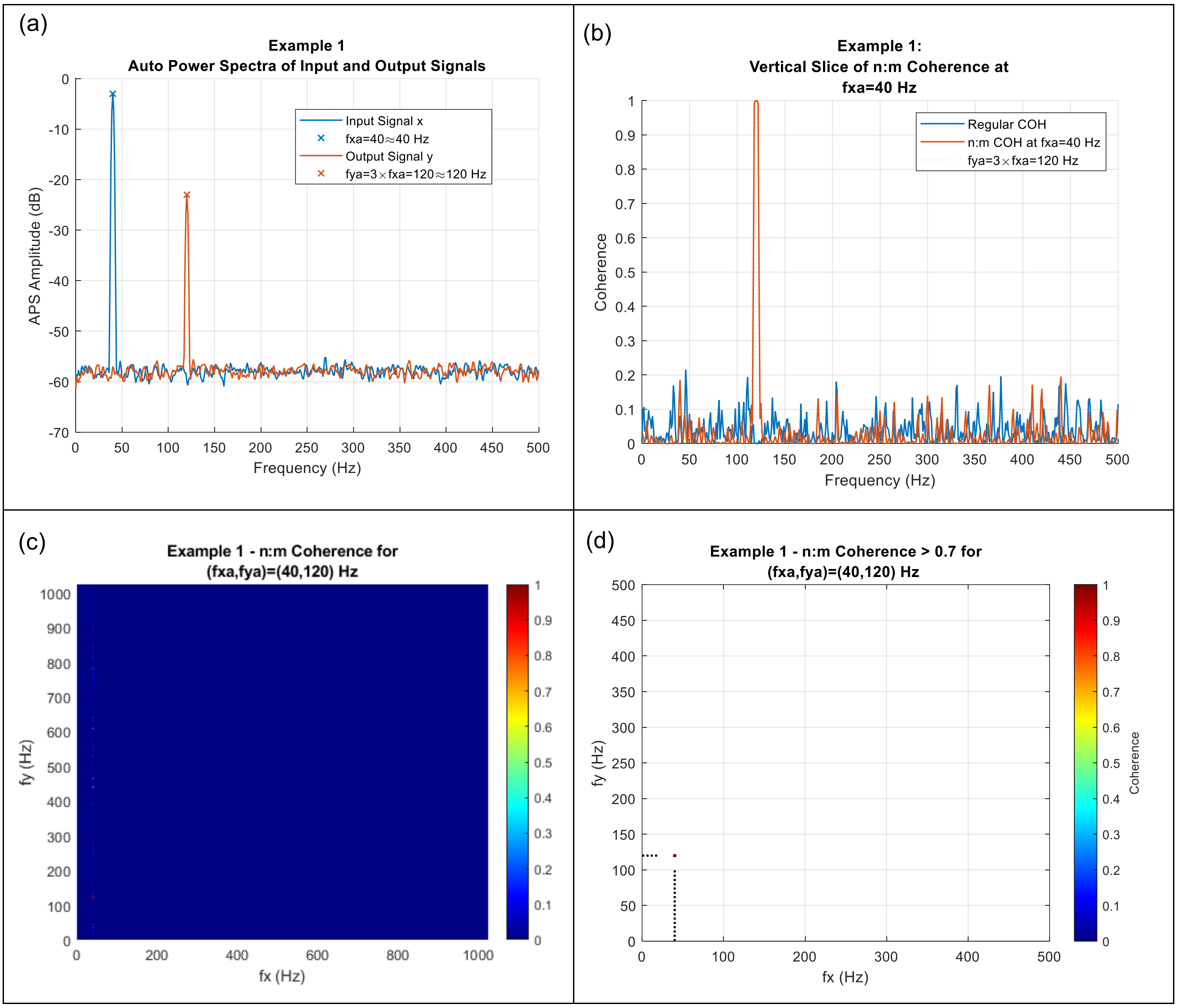

Figure 1: Example 1 of the evaluation of n:m coherence for an input signal of a single sine wave at 40 Hz and an output signal that is harmonically related with a single sine wave at 3x40=120 Hz.

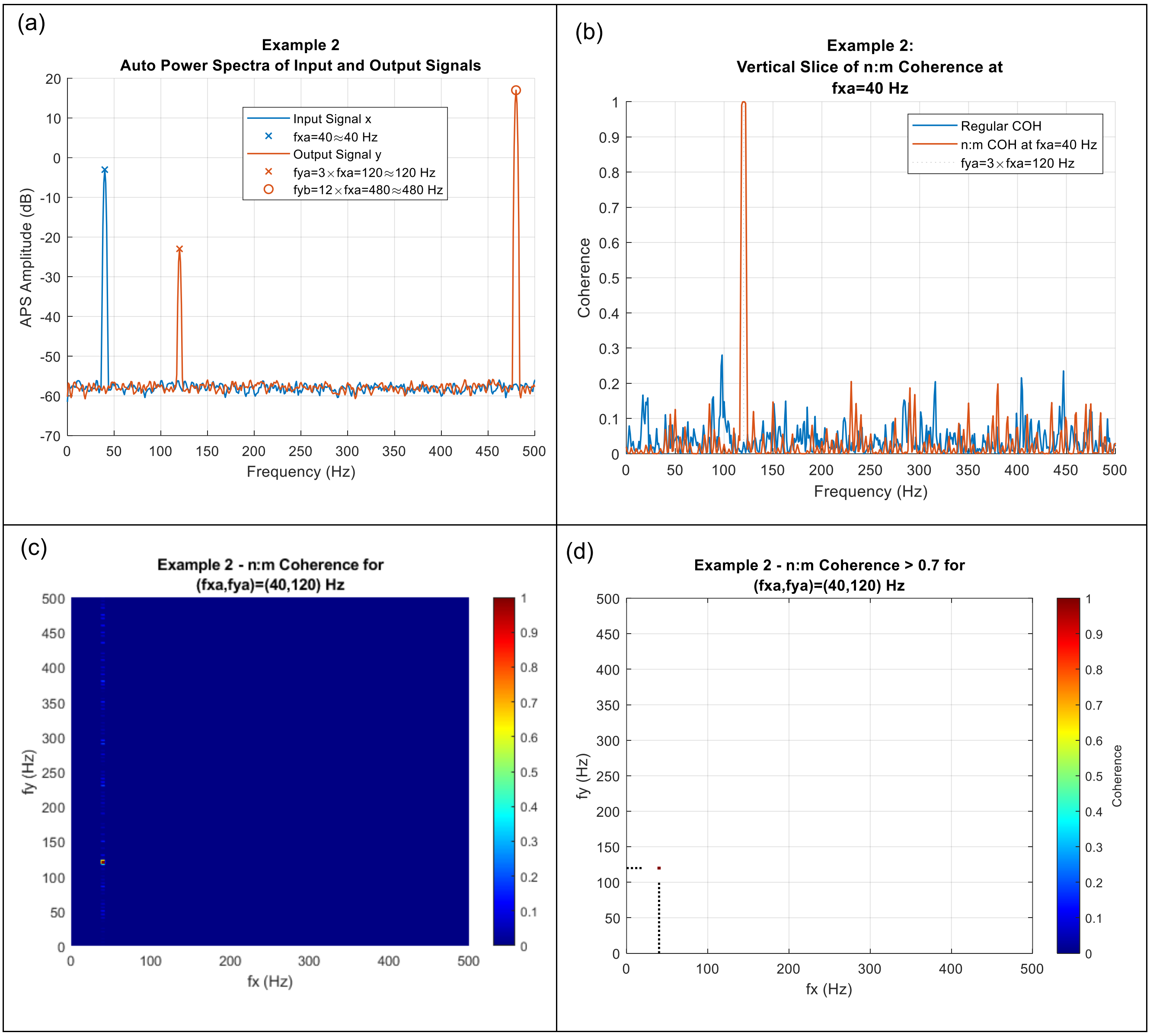

Figure 2: Example 2 of the evaluation of n:m coherence for an input signal of a single sine wave at 40 Hz and an output signal that is harmonically related with a sine wave at 3x40=120 Hz and an additional sine wave at 480 Hz.

3.3 Example 3: two sine wave input, two sine wave output

The third example is similar to the second example, only the input signal comprises two sine waves at 40 Hz and 160 Hz, and the output signal comprises two sine waves that are harmonically related to the input signal at 3x40=120 Hz and 3x160=480 Hz. The values of $f_{\mathrm{out}}\colon f_{\mathrm{in}} = f_{\mathrm{y}}\colon f_{\mathrm{x}} = 120\colon 40 = 3\colon 1 = n\colon m$. Figure 2 (a) shows the auto-power spectral densities of the input and output signals. Figure 3 (c) shows the n:m coherence map, where n=3, and m=1, and shows there is high coherence at (fxa, fya) = (40, 120) Hz and (fxb, fyb) = (160, 480) Hz, as expected. Figure 3 (d) shows the same results as Figure 3 (c), only that the results have been masked to only show the colour map for coherence values greater than 0.7, and dotted lines are included to highlight the location of the high coherence. Figure 3 (b) shows the evaluation of the ordinary magnitude-squared coherence (blue curve) calculated using Eq. (1) and shows there is no coherence between the input and output signals, which is to be expected as the input excitation and output response frequencies are different. The figure also shows a 'vertical slice' through the map in Figure 3 (c) at fxa=40 Hz (orange curve) and shows that there is high coherence at fya=120 Hz, and a vertical slice at fxb=160 Hz (yellow curve) and shows that there is high coherence at fyb= 480 Hz. Hence, the n:m coherence statistic has correctly identified the causality between the input and output signals.

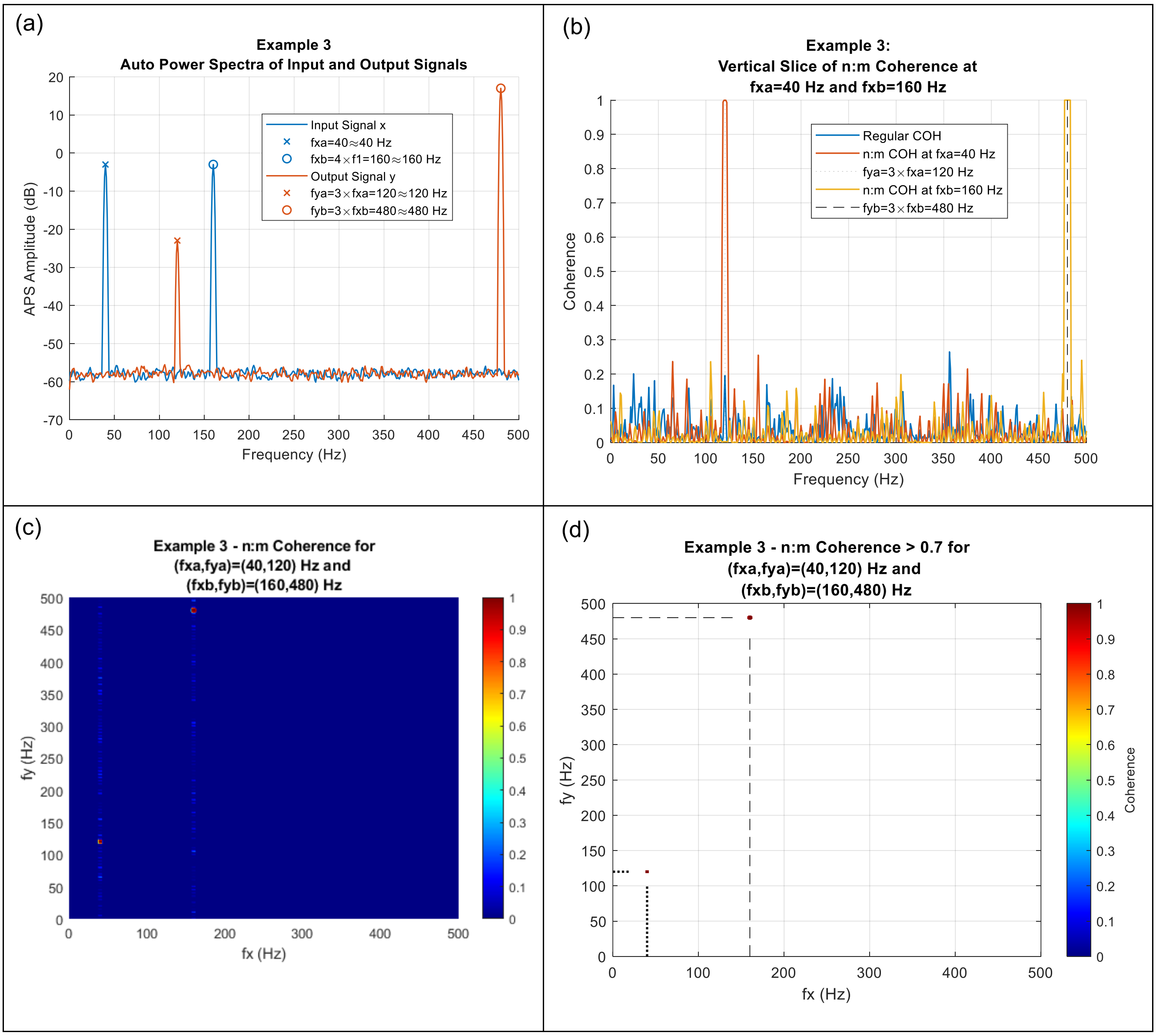

Figure 3: Example 3 of the evaluation of n:m coherence for an input signal with two sine waves and an output signal that is harmonically related with two sine waves.

3.4 Example 4: n:m = 1:1 coherence reduces to ordinary coherence

The equation for n:m coherence in Eq. (3) suggests that when n=m=1, the equation reduces to the equation for the ordinary magnitude-squared coherence shown in Eq. (2). This fourth example shows the evaluation of n:m coherence for an input signal of a single sine wave at 40 Hz and an output signal with the same frequency, 40 Hz, and hence n=m=1. Figure 4 (b) shows the n:m coherence statistic (orange dashed curve) evaluates to the same result as the ordinary magnitude-squared coherence (blue curve). Note that in previous examples a 'vertical slice' was taken through Figure 1 (c), Figure 2 (c), and Figure 3 (c) graphs, whereas in this example, as fx=fy, Figure 4 (b) shows a diagonal slice taken through Figure 4 (c).

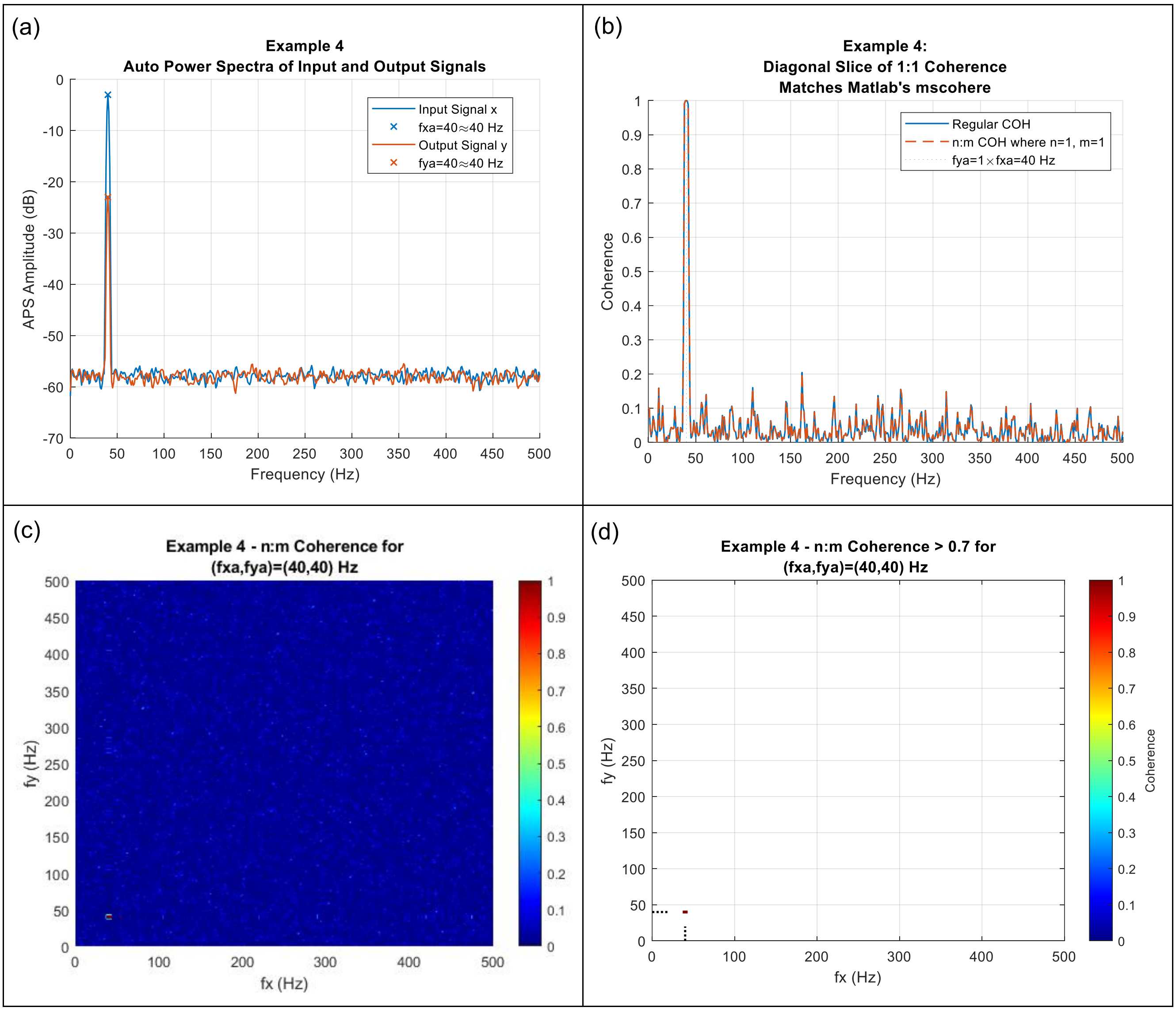

Figure 4: Example 4 of the evaluation of n:m coherence for an input signal of a single sine wave at 40 Hz and an output signal single sine wave at 40 Hz, and shows that for n=m=1 the n:m coherence statistic evaluates to the same result as the ordinary magnitude-squared coherence.

3.5 N:M coherence of exhaust noise

The n:m coherence analysis method was applied to the measurement of in-duct exhaust noise from an MTU 8V1600 DS440, 14-litre, twin turbo-charged, diesel generator that has a crankshaft that rotates at 1500 rpm = 25 Hz, and was loaded by a 200 kW resistive load bank. A water-cooled PCB 106B microphone was installed in the exhaust from the diesel generator, and a tachometer sensor was attached to the crankshaft. The tachometer sensor data was post-processed to create a synchronised sine wave signal at half the rotational speed of the crankshaft, at 12.5 Hz. The auto-power spectral densities of the tachometer signal (blue curve – input signal) and the acoustic pressure in the exhaust (orange curve – output signal), are shown in Figure 5. The acoustic pressure of the exhaust noise has a harmonic series with a frequency spacing of 12.5 Hz, which is half the rotational frequency of the crankshaft (1500 rpm / 60 / 2 = 12.5 Hz), and is typical exhaust noise spectra. It is common that the dominant tonal exhaust noise from 4-stroke engines is at a frequency of

$$f = \frac{\text{(crankshaft RPM)}}{60} \times \frac{\text{(number of cylinders)}}{2} = \frac{1500}{60} \times \frac{8}{2} = 100\text{ Hz.} \tag{7}$$

Whereas it can be seen in Figure 5 (orange curve) that the dominant tone is at 37.5 Hz. This is a result of the configuration of the exhaust manifolds in the MTU diesel generator, the exhaust path lengths from each of the cylinders, and the timings of their firing that result in a different tone (other than 100 Hz) being dominant.

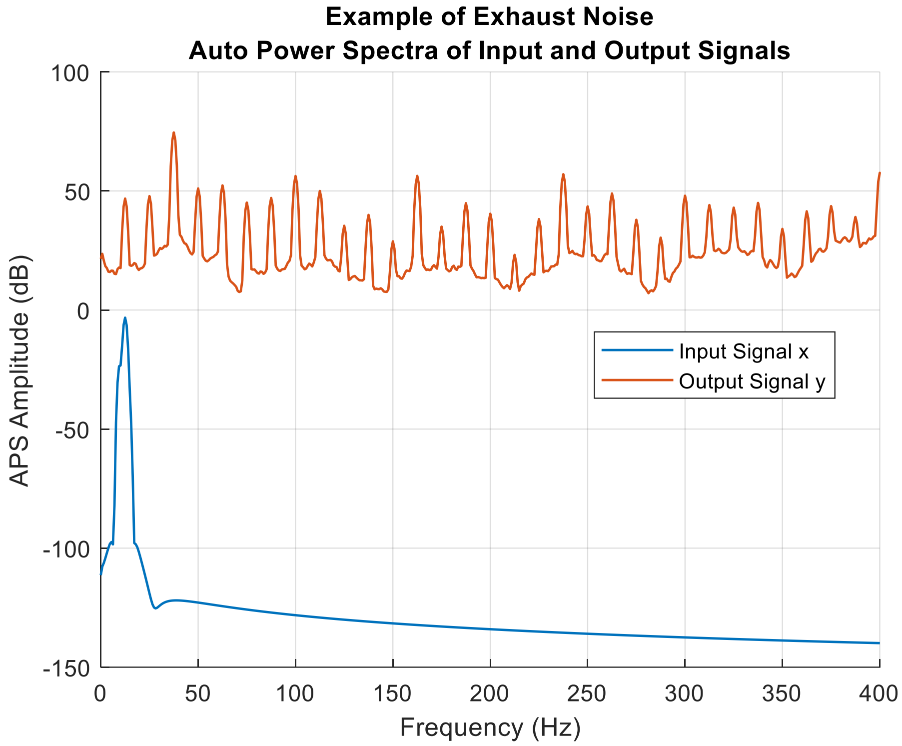

Figure 5: Auto-power spectral densities of the tachometer signal (in dB re 1V), and the acoustic pressure in the exhaust duct (in dB re 1 Pa).

Crocker et al. (1979) used coherence analysis of exhaust noise to examine the radiated noise. Lee et al (2003) used the non-linear bicoherence function to analyse the non-linear characteristics exhaust noise.

Here, the n:m coherence between the (input) tachometer signal and the (output) in-duct exhaust noise was calculated for various ratios of n:m: 2:1, 4:1, 6:1, 8:1, 10:1, and the results are shown in Figure 6. There are sets of strong coherences at each of the ratios considered, where all the peaks of coherence are at a harmonic of 12.5 Hz. Strong coherence is exhibited where n=2, 4, 6, 8, 10, at output frequencies of 25, 50, 75, 100, 125 Hz, respectively.

4 SUMMARY

There are several signal processing algorithms available for determining the coherence between an input and output signal where the system exhibits a non-linear frequency response. One such algorithm has been explored in this paper is the n:m coherence algorithm, which is a variation of the commonly used magnitude-squared coherence metric. Four validation examples were presented that incrementally demonstrated the behaviour of the algorithm using synthetic sine wave signals. Lastly, the n:m coherence metric was applied to the analysis of in-duct exhaust noise measured from a diesel generator, and showed how it can be used to determine the coherence between various frequency components of the noise. The Matlab code for the examples presented in this paper are available for free from the Mathworks web site (Howard, 2025).

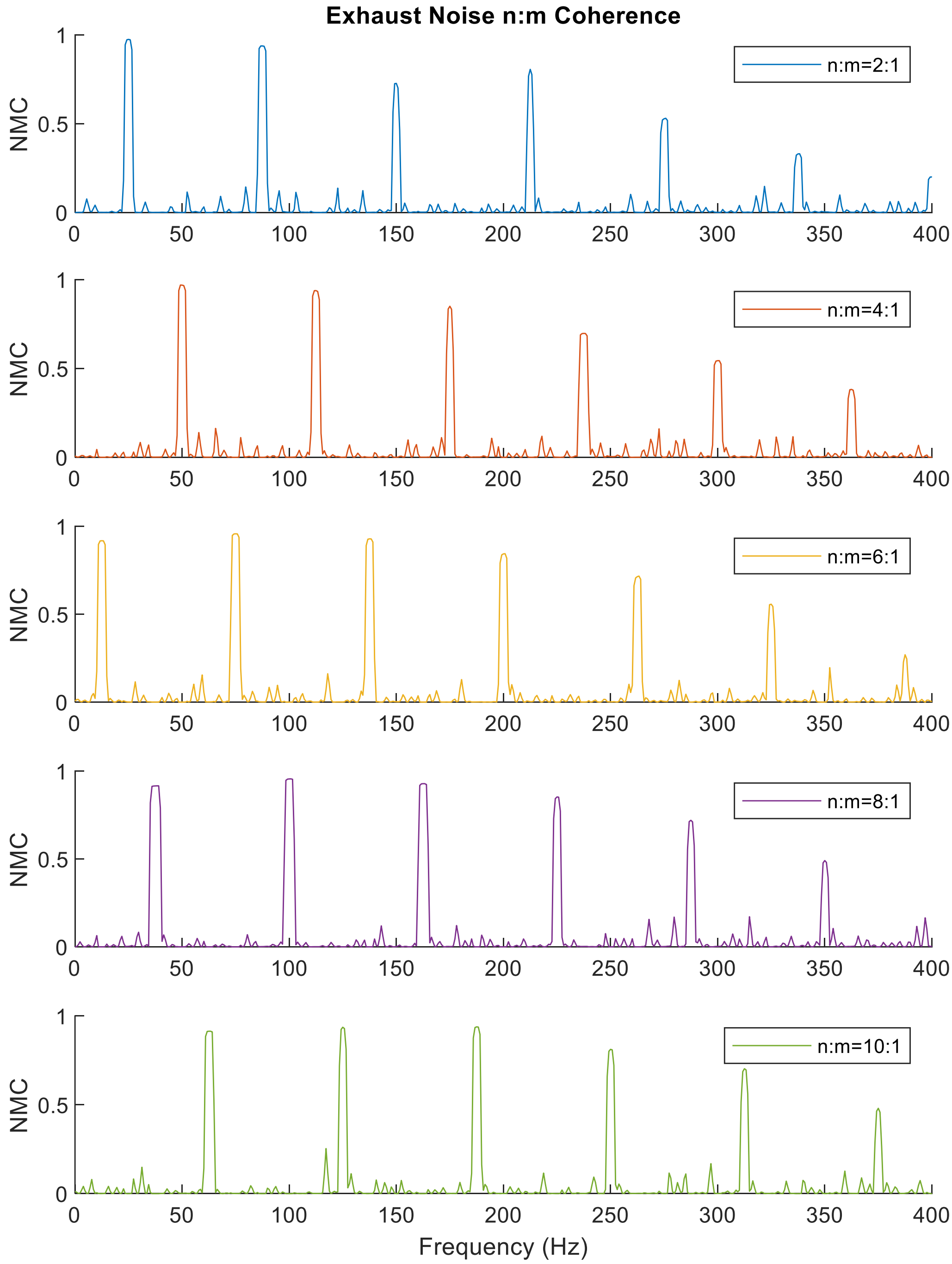

Figure 6: Exhaust noise n:m coherence (NMC) for n:m = 2:1, 4:1, 6:1, 8:1, 10:1 between a tachometer signal and the acoustic pressure measured in the exhaust of a MTU V8 diesel generator.

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