Easy-to-use closed-form equations for modal cut-on frequencies of a surface duct with an exponential sound speed profile

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ABSTRACT

An acoustic surface duct with the sound speed tending exponentially towards an asymptotic value with increasing depth is considered. It is demonstrated that the exponential sound speed profile (SSP) is a limiting case of the SSP in the transitional layer described by Brekhovskikh. It is shown that the SSP caused by wind-induced bubbles near the surface fits well to the exponential profile. The theory by Brekhovskikh describing wave propagation in the transitional layer is applied to the layer with exponential SSP and an equation for an acoustic wave propagating through this layer is derived. This equation is used to obtain a solution for an acoustic wave trapped in the surface duct with the exponential SSP. Conditions of existence of this solution are obtained and utilised to derive easy-to-use equations for the modal cut-on frequencies of such a duct. The duct modal cut-on frequencies so obtained are compared with the frequencies obtained by means of a well-known formula. It is shown that the values of these two sets of cut-on frequencies are very close in the case of the surface layer due to gas bubbles. The slight difference between these values and its decrease with mode number is explained by the wave-based nature of the solution obtained in this work.

1. Introduction

Formation of an acoustic duct near the ocean surface occurs when the gradient of the sound speed below the surface becomes positive with respect to depth. It is well-known that a sound speed profile (SSP) with positive gradient leads to refraction of acoustic waves towards the surface, which affects acoustic propagation in the duct and sonar detection ranges in at least two ways. First, it increases the angle at which the acoustic energy approaches the surface thus increasing the rate of reflection loss if the surface is rough due to gravitational waves caused by wind. This effect was considered, for example, by Zinoviev et al (2012) and Jones et al. (2014). Also, due to multiple reflections from the surface, the acoustic energy can propagate in the duct to distances much longer than the ones achieved in an environment with uniform sound speed.

An important characteristic of a surface duct is a set of cut-on frequencies of its normal modes. The propagation range at any particular frequency depends on the number of modes excited at this frequency as well as on their vertical pressure profile. The value of the cut-on frequency of the first mode is the most significant, as at frequencies below this value no mode can propagate in the duct and, therefore, propagation to large ranges is not possible (Jones et al., 2015).

The duct cut-on frequencies can be calculated by an existing formula (Kerr, 1951). However, this formula has some limitations. First of all, it is derived on the basis of the geometrical (ray) acoustics, so that it can be less accurate at low frequencies. In addition, it is inconvenient to use in practice, as an integral in the formula can be found analytically only for some specific sound speed profiles.

The main purpose of this paper is to derive closed-form equations for cut-on frequencies of a duct with the sound speed tending exponentially towards its asymptotic value with increasing depth. As shown in the paper, this profile can fit well to the SSP of a surface layer containing wind-induced gas bubbles. Also, if linearized with respect to the vertical coordinate, the exponential profile can approximate the SSP in the isothermal surface layer as well as in the deep water duct. Therefore, a solution for a sound wave obtained for the exponential SSP can be applied to these two cases. However, consideration of these cases is outside the scope of this work.

This paper has the following structure. In Section 2, the exponential SSP is derived from the SSP of the transitional layer described by Brekhovskikh (1960). The exponential SSP and the sound speed profile of a bubbly layer are combined in Section 3. A solution for the vertical pressure profile of an acoustic wave propagating through a layer with the exponential SSP is derived in Section 4. In Section 5, a condition for the existence of a wave trapped in the duct is obtained, and Section 6 is devoted to derivation of equations for the cut-on frequencies of the duct and their comparison with the results obtained by the existing formula.

2. Exponential SSP as a limiting case of the transitional layer SSP

Consider an acoustic medium which is a fluid half-space with the sound speed c_0 = 1500 m/s far below the surface and with the equilibrium fluid density ρ_0 = 1000 kg/m^3. The boundary of the half-space is considered to be pressure-release and smooth. Assume that, in the fluid immediately below the surface, the sound speed profile coincides with the SSP of the transitional layer described by Brekhovskikh (1960) and is determined by the following equation (Zinoviev et al., 2012):

$$c_{tr}(z) = c_0\sqrt{\frac{1-N}{1-N\frac{e^{m(z+z_0)}}{1+e^{m(z+z_0)}}}},\ \ z \geq 0. \tag{1}$$

In this analysis, the water surface is considered to be located at $z = 0$. The parameters $m^{-1} > 0$ and $N > 0$ characterise the depth of the layer and the difference between c_0 and the sound speed at the surface, $c_s = c_{tr}(0)$, respectively. The vertical coordinate of the surface is denoted z_0.

The substitution of $c_s = c_{tr}(0)$ to Eq. (1) leads to the following equation for the parameter N:

$$N = \frac{c_0^2 - c_s^2}{c_0^2 - c_s^2\frac{e^{mz_0}}{1+e^{mz_0}}}, \tag{2}$$

which, if substituted into Eq. (1), transforms the latter equation as follows:

$$c_{tr}(z) = c_0\sqrt{\frac{c_s^2 + c_s^2 e^{mz+mz_0}}{c_0^2 + c_0^2 e^{mz_0} - c_s^2 e^{mz_0} + c_s^2 e^{mz+mz_0}}}. \tag{3}$$

Assume that the parameters m and z_0 are such that $e^{-mz_0} \ll 1$. Then, Eq. (3) reduces to the following equation for the exponential sound speed profile c_{exp}, where the sound speed exponentially tends to c_0 with increasing z:

$$c_{exp}(z) = \frac{c_0 c_s}{\sqrt{(c_0^2 - c_s^2)e^{-mz} + c_s^2}}\ . \tag{4}$$

If the difference $\Delta_c = c_0 - c_s$ is small in comparison with c_0, the exponential sound speed profile can be further simplified:

$$c_{exp}(z) = c_0\left(1 - \frac{\Delta_c}{c_0}e^{-mz}\right), \Delta_c/c_0 \ll 1. \tag{5}$$

Eq. (5) allows one to clarify the meaning of the parameter m. The sound speed gradient at the surface, g_s, can be expressed as follows:

$$g_s = c'_{exp}(0) = \Delta_c m. \tag{6}$$

As a result, for the SSP determined by Eq. (5), m can be represented as a ratio between the sound speed gradient at the surface and the difference between the sound speed values below the layer and at the surface:

$$m = \frac{g_s}{\Delta_c}. \tag{7}$$

The above argument demonstrates that Eq. (4) for the exponential SSP is a limiting case of Eq. (1) for the SSP of the transitional layer with $e^{-mz_0} \to 0$. An additional assumption that $\Delta_c/c_0 \ll 1$ leads to a simplified exponential SSP described by Eq. (5). Therefore, sound propagation within a layer with the exponential SSP can be described by equations derived for the transitional layer SSP applying the same assumptions.

3. Fitting the exponential SSP to the SSP for a layer due to gas bubbles

There is at least one type of a surface duct for which an existing model predicts a sound speed profile fitting very well to the exponential one described by Eq. (5). Ainslie (2005) derived equations determining the SSP in a layer below the ocean surface where, due to wind action, gas bubbles accumulate thus affecting the compressibility of water and, therefore, changing the sound speed.

Figure 1 shows the SSP of the layer due to bubbles for the wind speed, $w = 12.5$ m/s. The wind speed is measured at the height of 19.5 m above the water surface. The parameter m for the corresponding exponential SSP, which is also shown in Figure 1, is found by the "best fit" method.

It can be clearly seen that the two sound speed profiles are very close. In fact, the discrepancy between the two curves shown in Figure 1 is much smaller than the one achieved by Zinoviev et al. (2014) between the SSP of the layer due to bubbles and the transitional layer SSP described by Eq. (1).

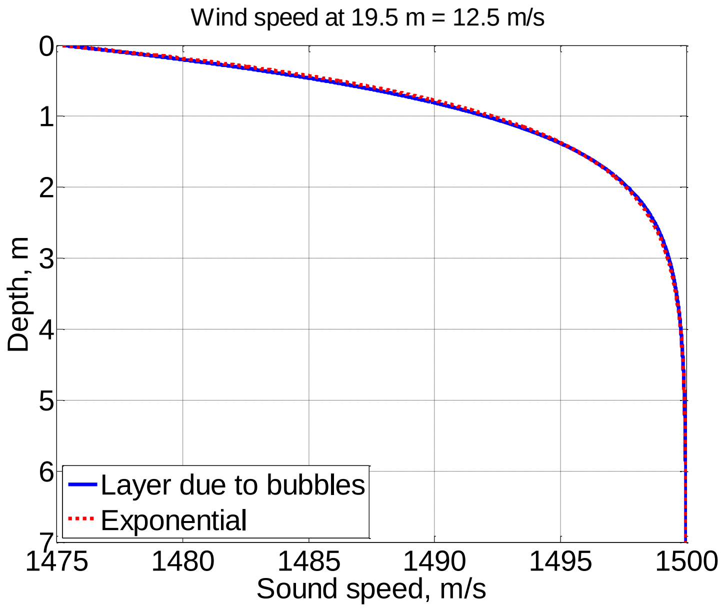

Figure 1: SSP of the surface layer due to gas bubbles compared with the "best fit" exponential SSP. Parameters used in the calculations: $\Delta_c = 24.7942$ m/s, m=1.17 m^{-1}.

4. Solution for the pressure profile of an acoustic wave in the layer with the exponential SSP

4.1 Derivation of the pressure profile

Brekhovskikh (1960) considered propagation of a plane acoustic wave through an infinite medium with a horizontal transitional layer where the sound speed *increases* along the propagation path of the wave. He showed that the vertical profile of the acoustic pressure, $p(z)$, in such a wave can be represented via hypergeometric series. Based on Brekhovskikh's results, Zinoviev et al. (2012) formulated a solution for a plane wave in a fluid half-space with sound speed *decreasing* towards the surface as determined by Eq. (1). According to this solution, for a plane wave of the frequency, f, propagating through the layer after having approached its lower boundary with grazing angle θ_0, the vertical dependence of acoustic pressure $p(z)$ can be determined as follows:

$$p(z) = \sum_{n=0}^{\infty} p_n(z), \tag{8}$$

$$p_0(z) = \left(e^{-m(z+z_0)} + 1\right) e^{-ik_0(z+z_0)\sin\theta_0}, \tag{9}$$

$$p_n(z) = -p_{n-1}(z)\left(1 + \frac{\mu^2}{n(n+2ik_0\sin\theta_0/m)}\right) e^{-m(z+z_0)}, \tag{10}$$

$$\mu^2 = \frac{Nk_0^2}{m^2(1-N)}, \tag{11}$$

$$k_0 = \frac{2\pi f}{c_0}. \tag{12}$$

It can be easily shown that, if $e^{-mz_0} \to 0$, Eq. (2) can be simplified as follows:

$$N = 1 - e^{-mz_0} \frac{c_s^2}{c_0^2 - c_s^2}. \tag{13}$$

Utilising Eq. (13) and considering that $e^{-mz_0} \to 0$ it is possible to obtain the following equations for the vertical profile of the pressure field of an acoustic wave of the unit amplitude propagating through the exponential layer determined by Eq. (4):

$$p(\zeta) = \sum_{n=0}^{\infty} p_n(\zeta), \tag{14}$$

$$p_0(\zeta) = e^{-i\delta\zeta}, \tag{15}$$

$$p_n(\zeta) = -p_{n-1}(\zeta) \frac{\phi^2 e^{-\zeta}}{n(n+2i\delta)}, \tag{16}$$

where

$$\zeta = mz, \tag{17}$$

$$\phi = \frac{2\pi f}{mc_0 c_s} \sqrt{c_0^2 - c_s^2}, \tag{18}$$

and

$$\delta = \frac{2\pi f}{mc_0} \sin\theta_0 = \frac{k_0}{m} \sin\theta_0, \tag{19}$$

are the non-dimensional depth, frequency, and vertical wavenumber respectively. Denoting $\kappa \equiv i\delta$, one can write the solution for $p(\zeta)$ in a shorter way using the product symbol:

$$p(\zeta) = e^{-\kappa\zeta} \left(1 + \sum_{n=1}^{\infty} \frac{\phi^{2n} e^{-n\zeta} (-1)^n}{n!} \prod_{l=1}^{n} \frac{1}{l+2\kappa}\right). \tag{20}$$

With the first four terms shown explicitly, Eq. (20) takes the following form:

$$p(\zeta) = e^{-\kappa\zeta} \left(1 - \frac{\phi^2 e^{-\zeta}}{1+2\kappa} + \frac{\phi^4 e^{-2\zeta}}{2(1+2\kappa)(2+2\kappa)} - \frac{\phi^6 e^{-3\zeta}}{6(1+2\kappa)(2+2\kappa)(3+2\kappa)} + \cdots\right). \tag{21}$$

Utilising Equations (8.331) and (8.402) of Gradshteyn & Ryzhik (2000), one can conclude that Eq. (20) can be written via the Bessel function, $J_\alpha(x)$, of the order $\alpha = 2\kappa$:

$$p(\zeta) = 2\kappa\Gamma(2\kappa)\phi^{-2\kappa} J_{2\kappa}\left(2\phi e^{-\zeta/2}\right). \tag{22}$$

In Eq. (22), $\Gamma(2\kappa)$ is the gamma function. If the normalising multipliers are omitted in Eq. (22), the shape of the vertical sound speed profile can be described by the following equation:

$$p(\zeta) \sim J_{2\kappa}\left(2\phi e^{-\zeta/2}\right). \tag{23}$$

Eqs. (17) to (23) describe the vertical acoustic pressure profile in the exponential layer and represent one of the main results of this paper. The solution determined by these equations is derived without any assumptions about environmental and acoustic parameters and, therefore, can be considered to be exact for any layer where the SSP is determined by Eq. (4).

4.2 Convergence of the series

Since the pressure field within the exponential layer is determined by the Bessel function (Eq. (22)), the corresponding series in Eqs. (20) and (21) converges. However, the number of terms required for the series to converge can be important for carrying out calculations in some specific cases. It can be shown that the series converges quickly in a wide range of parameters. For example, consider the convergence of the series at the surface ($\zeta = 0$) where it converges slower than at other depths as seen from Eq. (16). If $\phi > 1$, it is possible that the absolute value of terms increases with n for the first several terms of the series, so that $|p_n(0)| > |p_{n-1}(0)|$. At the same time, it is clear from Eq. (16) that

$$|p_n(0)|\big|_{n\to\infty} \to \frac{\phi^{2n}}{(n!)^2}, \tag{25}$$

and, therefore, the series converges quickly when the term number n becomes greater than some value depending on ϕ.

The convergence of the series is demonstrated in Figure 2 for the frequency $f = 6$ kHz at the water surface ($\zeta = 0$). The Figure shows the absolute value, $|p_N(0)|$, of the term of the order $n = N$ as well as that of the partial sum, $|\sum_{n=0}^{N} p_n(0)|$ of N+1 first terms in dependence on N. The calculations are carried out for the exponential SSP shown in Figure 1. The value of the grazing angle, θ_0, used in the calculations is 3°. This value is a typical value considered in models predicting the loss of acoustic energy due to reflection from the rough sea surface (Jones et al., 2010).

It is clearly seen that, although the absolute value of the series terms grows with N for the first few terms, starting with the 3rd term the absolute value quickly decreases and any term with $N \geq 6$ does not affect significantly the sum of the series.

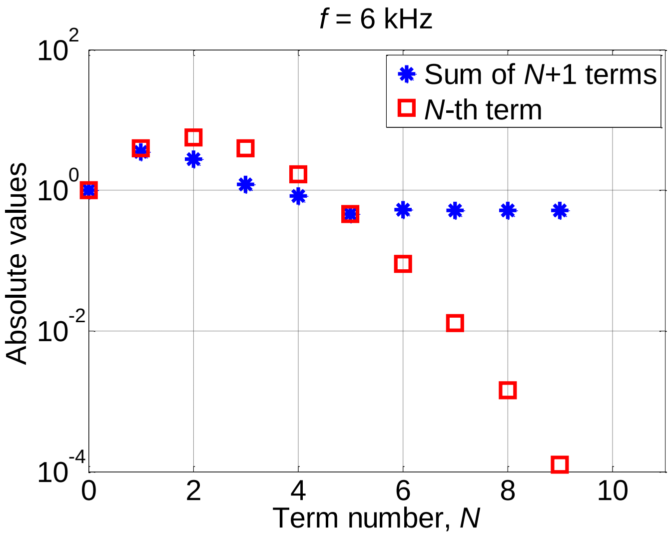

Figure 2: The absolute value of the partial sum of N terms of the series and that of the N-th term vs N. The parameters used in the calculations are: $\Delta_c = 24.7942$ m/s, m=1.17 m^{-1}, $\theta_0 = 3°$.

For practical purposes, it is necessary to know the number of terms required to achieve a given accuracy. The following criterion of the series convergence is considered here. The series $p(\zeta) = \sum_{n=0}^{\infty} p_n(\zeta)$ is deemed to converge at $n = N$ with the accuracy $\varepsilon \ll 1$ if the absolute value of its term of the order N+1 is less than the absolute value of the sum of all previous terms multiplied by ε:

$$|p_{N+1}(\zeta)| < \varepsilon|\sum_{n=0}^{N} p_n(\zeta)|, \qquad 0 < \varepsilon \ll 1. \tag{26}$$

Figure 3 shows the number of terms, $N+1$, required for the series to converge for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-4}$. The Figure demonstrates that the series converges quickly. Although the required number of terms grows approximately linearly with frequency, the series needs no more than 10 terms to achieve the accuracy $\varepsilon = 10^{-4}$ if the frequency is not greater than 6 kHz. In addition, improving the accuracy by two orders of magnitude is achieved by increasing the number of terms by only 1 or 2. Consequently, the summation of the series can be stopped after first several terms are taken into account.

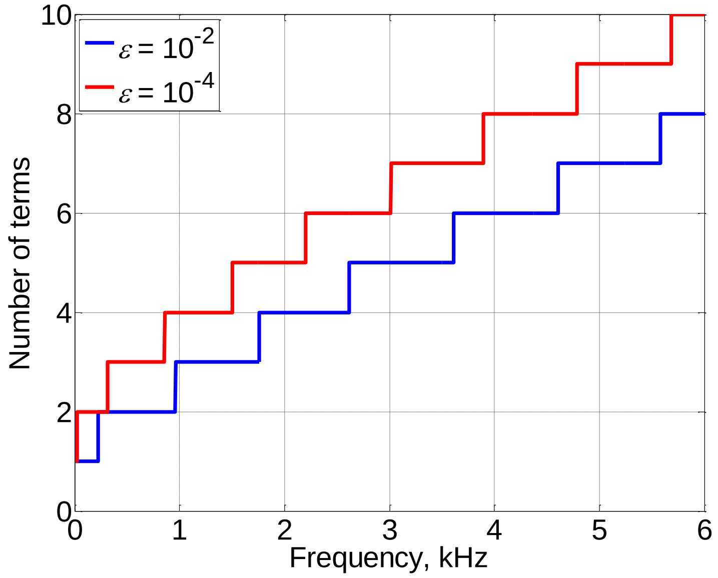

Figure 3: Number of terms required to achieve the convergence of the series determined by Eqs. (17) to (20). The parameters used in the calculations are: Δ_c= 24.7942 m/s, m=1.17 m^{-1}, $\theta_0 = 3°$

It can be concluded from Figure 3 that, for the environment under consideration, only one term in the brackets in Eq. (21) is required to achieve the accuracy of $\varepsilon = 10^{-2}$ at frequencies below approximately 200 Hz. As this term describes the incident plane wave, the influence of the sound speed gradient on sound propagation in the layer can be neglected at these frequencies. Analogously, the first two, three and four terms in the brackets in Eq. (21) appropriately describe the wave propagation in the layer at frequencies below 1 kHz, 1.7 kHz and 2.6 kHz respectively. More terms can be easily added for consideration of wave propagation at higher frequencies.

5. Condition of existence of a wave trapped in the duct with the exponential sound speed profile

The layer with the exponential SSP constitutes a surface duct, where propagation of a trapped wave is possible. The pressure field of a wave trapped in the duct must satisfy the following two conditions:

$$p(0) = 0, \tag{27}$$

$$p(\zeta \to \infty) = 0. \tag{28}$$

The first of the above conditions is due to the pressure-release surface. The solution for $p(z)$ described by Eq. (20) can satisfy this condition only if the expression in brackets is zero, which can be achieved only if $\mathrm{Re}(\delta) = 0$. The second condition (Eq. (28)) is the condition of the wave being trapped in the duct as it prevents the wave from propagating in the vertical direction. The solution for $p(z)$ described by Eq. (20) can satisfy this condition only if the exponential term tends to zero as $\zeta \to \infty$, which occurs when $\mathrm{Im}(\delta) < 0$ and, therefore, when the following condition for the variable κ is satisfied:

$$\kappa = i\delta > 0. \tag{29}$$

The substitution of Eq. (29) to Eq. (22) and, subsequently, to Eq. (27), leads to the following condition of existence of the trapped wave:

$$J_{2\kappa}(2\phi) = 0. \tag{30}$$

This condition determines relationships between the frequency and the vertical wavenumber in a wave trapped in the duct. It does not depend explicitly on the parameters of both the layer and the wave and, therefore, is valid for any layer that can be described by the exponential layer SSP determined by Eq. (4).

To determine one of the parameters ϕ and κ via the other one for the trapped wave, it is necessary to find the roots of Eq. (30). Figure 4 shows the value of the function $J_{2\kappa}(2\phi)$. It is clear that the light blue diagonal streaks correspond to combinations of the parameters where the function changes its sign and, therefore, assumes the value of zero thus defining the relationships between the frequency and the vertical wavenumber of the wave trapped in the duct.

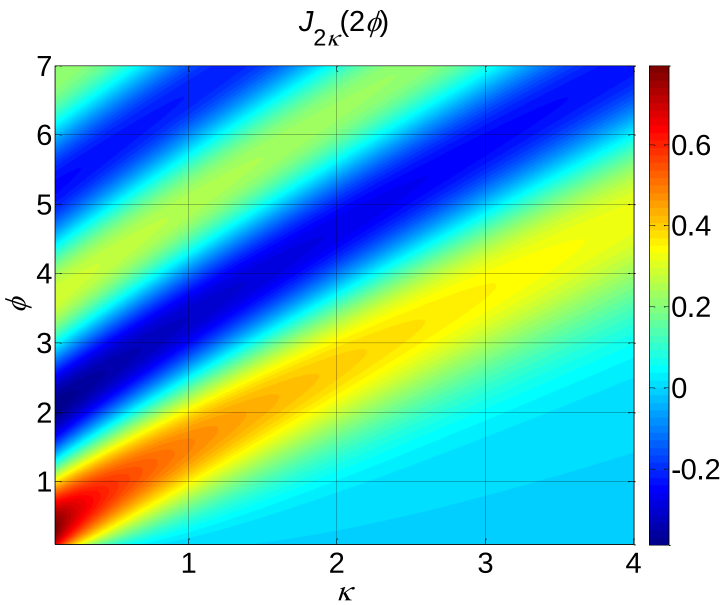

Figure 4: The value of the function $J_{2\kappa}(2\phi)$ for $\kappa > 0$.

6. Cut-on frequencies of the duct with the exponential sound speed profile

6.1 Non-dimensional cut-on frequencies

Figure 4 shows that, for every value of the vertical wavenumber κ, there is a set of frequencies, $\phi_j(\kappa)$, $j = 1,2,3,\ldots$, for which $J_{2\kappa}(2\phi) = 0$. It can be assumed that each of these frequencies corresponds to a normal mode of the duct. It is clear that the mode of the order j exists only if the frequency ϕ exceeds the value $\phi_j(0)$, which is the cut-on frequency of this mode.

The substitution of $\kappa = 0$ to Eq. (30) leads to the following equation with respect to ϕ:

$$J_0(2\phi) = 0. \tag{31}$$

The roots of Eq. (31) determine the modal cut-on frequencies, $\phi_j \equiv \phi_j(0)$, of the duct under consideration. Since the roots of the Bessel function $J_0(x)$ are known, the corresponding cut-on frequencies can be easily calculated. The cut-on frequencies in the non-dimensional form ϕ_j are constant numbers and do not depend on any environmental or acoustical parameters. The dimensional cut-on frequencies, f_j, can be calculated by the following formula derived from Eq. (18):

$$f_j = \frac{\phi_j m c_0 c_s}{2\pi\sqrt{c_0^2 - c_s^2}}. \quad (32)$$

Using an asymptotic expression for the Bessel function at large arguments (Gradshteyn & Ryzhik, 2000, Eq. 8.451.1), it is possible to suggest the following approximate equation for the non-dimensional cut-on frequencies shown in Table 1:

$$\phi_j \approx 1.2 + (j-1)\frac{\pi}{2} \approx 1.2 + 1.57(j-1). \quad (33)$$

Table 1 compares the values of the lowest eight non-dimensional cut-on frequencies calculated using the exact Eq. (31) and the approximate Eq. (33). The two sets of cut-on frequencies are denoted $\phi_j^{(1)}$ and $\phi_j^{(2)}$ respectively. It is clear from Table 1 that the cut-on frequencies $f_j^{(2)}$ calculated with the use of the approximate Eq. (33) are within 0.4% discrepancy as compared with their more precise values $f_j^{(1)}$ and that the discrepancy decreases with increasing mode number j if $j > 2$.

Table 1: Comparison of the non-dimensional cut-on frequencies with their approximate values

Mode number, j	1	2	3	4	5	6	7	8
$\phi_j^{(1)}$, Eq. (31)	1.2024	2.7600	4.3269	5.8958	7.4655	9.0355	10.606	12.176
$\phi_j^{(2)}$, Eq. (33)	1.2000	2.7700	4.3400	5.9100	7.4800	9.0500	10.620	12.190
$f_j^{(1)}/f_j^{(2)}$	1.0020	0.9964	0.9970	0.9976	0.9981	0.9984	0.9987	0.9989

Eqs. (32) and (33) represent the main result of this work. They are obtained from the exact solution of the wave equation in a transitional layer provided by Brekhovskikh (1960) and adapted here for the duct with the exponential SSP. These equations allow quick calculation of the modal cut-on frequencies of any duct with the exponential SSP if the sound speed values at the surface and below the duct, as well as the parameter m, are known.

6.2 Comparison with an existing formula

The modal cut-on frequencies of the duct can also be calculated using an existing formula (Kerr, 1951), which is based on the laws of the geometrical (ray) acoustics:

$$f_j = c_s/\lambda_j, \quad \lambda_j = \frac{2}{j-1/4}\int_0^H \sqrt{N^2(z) - N^2(H)}\,dz. \quad (34)$$

In Eq. (34), H is the duct depth and $N(z) = c_0/c(z)$ is the index of refraction.

Table 2 contains the values of f_j for the first eight modes of the duct under consideration, which SSP is shown in Figure 1. The cut-on frequencies corresponding to the values of ϕ_j calculated via the roots of Eq. (31) and shown in Table 1 are denoted $f_j^{(1)}$, whereas the frequencies calculated by the ray-based Eq. (34) are denoted $f_j^{(3)}$.

Table 2: Comparison of the cut-on frequencies calculated using roots of Eq. (31) and by Eq. (34)

Mode number, j	1	2	3	4	5	6	7	8
$f_j^{(1)}$(Hz), Eq. (31)	1824	4187	6564	8944	11325	13707	16090	18472
$f_j^{(3)}$ (Hz), Eq. (34)	1767	4124	6480	8837	11194	13550	15907	18264
$f_j^{(1)}/f_j^{(3)}$	1.0206	1.0040	1.0017	1.0009	1.0006	1.0004	1.0003	1.0002

It is clear from Table 2 that the cut-on frequencies $f_j^{(1)}$ and $f_j^{(3)}$ calculated respectively by the wave-based theory and by the ray theory are close, but the former ones are somewhat larger. This table also shows that the ratio between the two values tends to unity with increasing frequency. This can be explained by the fact that the ray theory is the limiting case of the wave theory at large frequencies.

7. Conclusions

In this paper, a surface layer with the sound speed profile (SSP) of the transitional layer described by Brekhovskikh (1960) is considered. It is shown that, if the vertical coordinate of the surface with respect to the zero level in the transitional layer is large, the transitional layer SSP reduces to the exponential SSP which implies that the sound speed approaches its equilibrium value with increasing depth according to the exponential law. It is demonstrated that the exponential SSP fits well to the SSP modelled by Ainslie (2005) for the surface layer caused by wind-induced gas bubbles.

An existing wave-based solution in the form of hypergeometric series for a plane acoustic wave propagating through the transitional layer is used to obtain a solution for the exponential layer. An equation in the form of an infinite series describing the vertical pressure profile in the exponential layer is derived. It is demonstrated that the obtained series converges quickly and that it can be represented as a Bessel function.

The obtained equation for the pressure profile is used to derive a condition of existence of a sound wave trapped in the exponential layer, which constitutes an acoustic duct. Using this condition, cut-on frequencies of the normal modes of the duct are obtained, which, in the non-dimensional form, are constant numbers. A simple approximate equation for these cut-on frequencies is suggested. It is shown that this equation provides estimates for the cut-on frequencies with a good accuracy.

The dimensional cut-on frequencies are calculated for the duct, which parameters correspond to the wind-induced bubbly surface layer. The obtained values are compared with the values calculated by means of an existing formula. It is shown that the values of the two sets of cut-on frequencies are close for the duct under consideration and that the ratio between them tends to unity with increasing frequency. It is explained by the fact that the equations obtained here are wave-based, whereas the existing formula is based on the ray acoustics and, therefore, the two solutions should produce closer results at higher frequencies.

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