Inverse problems in musical acoustics

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ABSTRACT

Inverse problems in musical acoustics have a long history. Essentially they involve deducing something about the structure of the instrument simply by examining its sound. Can you tell the metal from which a flute is made from its sound? Can you hear the shape of a drum? Can you hear the shape of a wooden beam? What makes a Stradivarius violin sound special? This paper will examine some of these problems to find those which are, at least to some degree, answerable. Among the questions considered in addition to those above are hearing the diameter of a metal rod, discovering the diameter of an organ pipe, and deducing the shape of a bent rod. While many of these problems are only of 'academic' interest, since direct measurements can usually be made, others such as that concerning the violin can often be approached only in an inverse manner. The answers to such inverse problems – even partial answers – can be of practical importance in helping to understand and improve the sound and ease of performance of musical instruments.

INTRODUCTION

The design of most musical instruments has evolved over long periods of time with contributions from many makers. A few exceptions are the silver flute of Theobald Boehm [1], the saxophone of Adolphe Sax [1], the new violin octet of Carleen Hutchins [2], and the guitar quartet of Graham Caldersmith [3]. With the 'evolved' instruments in particular it is often found that some are easier to play or sound much better than others, and so we are faced with the inverse problem 'Why is this so?' Perhaps the most famous from a musical viewpoint is the 'secret of Stradivarius'. Why are instruments by this famous maker so greatly desired? We return to this and related questions later, but we should start with simpler and more formal problems.

Perhaps the most famous inverse problem is that posed by Mark Kac [4] in 1966: 'Can you hear the shape of a drum?' Here a drum is simply defined to be a plane membrane under uniform tension with its edges rigidly clamped, and the information upon which the inverse problem relies is the set of frequencies of all the vibrational modes of the membrane. If the answer to this question turns out to be 'No', then this can be demonstrated by finding two different drum shapes that have the same mode spectrum. This was accomplished by Gordon, Webb and Wolpert [5] in 1992, while Gottlieb and McManus [6] demonstrated it for a large class of shapes based on triangular elements. So the answer to Kac's question is 'No'. Of course one could determine the drum shape by using a more sophisticated measurement system, for example a small microphone that could be scanned across the near-field, but this is almost like 'looking at' the drum, which is not allowed!

In this paper we examine some of the inverse problems related to musical instruments that can indeed be solved, though often only partially, and see how such deductions from the sound properties of excellent instruments can be used to aid the design of modern versions or even new instruments.

STRINGS, RODS AND BARS

The simplest instruments to examine are those that are impulsively excited like Kac's drum, but recognising that the number of physical parameters describing the structure must be reduced. The simplest such problem can be stated as `Can you hear the diameter of a taut string?' and this is indeed a musically significant question for instruments such as guitars and pianos. If we assume that the ends of the string are simply pinned and that the string is under tension and ideally thin, then its mode frequencies will be in exact harmonic (or integer) relation. When the effect of a finite diameter is included then the bending stiffness of the string becomes important. For a string of length L and radius a, the frequency f_n of the nth mode is given by

$$f_n = n f_1 \left[\frac{1 + B\,n^2}{1 + B} \right]^{1/2} \tag{1}$$

where f_1 depends on the string tension and

$$B \approx \frac{\pi^2 E\,a^2}{16 f_1^2 \rho L^2} \tag{2}$$

with E being the Young's modulus and ρ the density of the string material [1]. The stiffness of the string thus stretches the mode frequencies and, assuming that the length of the string is known, the string diameter $2a$ can be deduced from the degree of stretching of the mode frequencies f_n provided the material of the string, and thus the elastic modulus E, is known. While this may seem a purely technical problem, it has important applications in musical instrument acoustics, where the tuning of a piano is stretched by about a fifth of a semitone over its full compass to match the string inharmonicity [7].

A rather different but related approach can be used to determine the cross-section of a bar with no applied tension. Here the transverse vibration of the bar is essentially by bending at low frequencies where the wavelength is large compared to the bar thickness, and by shear distortion at higher frequencies where this is no longer true. For bending vibration of a bar with free ends [1] the mode frequency varies approximately as

$$f_n \approx \frac{\pi r}{8L^2}\left(\frac{E}{\rho}\right)^{1/2}(2n-1)^2 \qquad (3)$$

where L is the bar length, r is the radius of gyration of the bar cross section about the neutral plane, and E the Young's modulus of the bar material. At higher frequencies where shear distortion applies, the mode frequencies are

$$f_n = \frac{n}{2L}\left(\frac{G}{\rho}\right)^{1/2} \qquad (4)$$

where $G \approx E$ is the shear modulus of the bar material. The transition between these two series, as shown in Fig. 1, gives the ratio r/L between the radius of gyration, which is proportional to the bar thickness, and the bar length. Extrapolation of the curves of best fit for high and low frequencies gives

$$\frac{r}{L} = \frac{4n^*}{\pi(2n^*-1)^2} \approx \frac{1}{\pi n^*} \qquad (5)$$

where n^* is the extrapolated intercept position. The exact shape of the bar cross section is, however, not revealed.

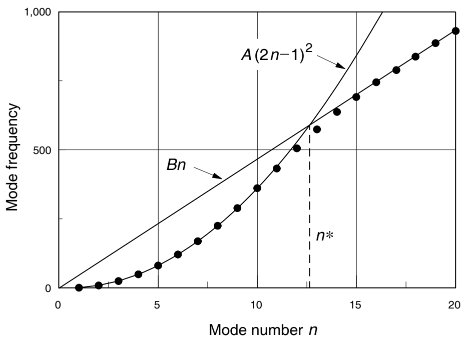

Figure 1. Transition between bending and shear distortion of a bar as revealed by a plot of mode frequencies (dot points).

A common form of musical percussion instrument is that using wooden bars with rectangular cross section, the thickness of which is not uniform but instead reduced near the centre of the bar to give nearly harmonic ratios for the first few modes. Common instruments of this class are the marimba and the xylophone. The direct design problem is to decide the shape of the bar that will give approximately the desired tuning and this can be done either by trial and error or by computation [8], while the inverse problem is to determine the bar thickness gradation from its measured spectrum. Since the unperturbed mode shapes of the bar vibration are known, it is possible to use perturbation theory to calculate the effect of a thickness change on the mode frequencies, second-order perturbation theory generally being necessary because the changes in thickness are usually large enough to significantly perturb the mode shapes. Assuming that the material properties, length, and average thickness of the bar are known, then the measured frequency shifts of N modes can be used to deduce the amplitudes of N orthogonal thickness perturbations and thus the approximate shape of the bar.

Musical percussion instruments also sometimes use rods bent into particular shapes, the common one being the triangle, which has three equal sides and one open vertex. Less well known is the pentangle [9] in which the rod is bent into five sections with reflection symmetry, the aim being the tuning of the first few in-plane modes to nearly harmonic ratios. Even a rod with a single sharp bend presents a complex inverse problem, since nonlinear interactions between transverse modes at the bend generate harmonics of the normal modes [10], but these can be discounted by using only gentle excitations. Because such a singly bent rod is not 'musical', however, we shall not consider it here.

Leaving aside the complication of the bar thickness discussed before and also the overall length, there are two parameters a_2/a_1 and a_3/a_1 specifying the relative lengths of the rod segments of a pentangle and two angles θ and ϕ, as shown in Fig. 2. This suggests that the frequency ratios of the first five modes might be enough to determine the shape to which the rod is bent. Fig. 3 shows a set of sections through such a 4-dimensional parameter space for the particular ratios a_2/a_1 = 2 and a_3/a_1 = 1. Taking Mode 2 as reference point, three solutions S_I, S_{II} and S_{III} were identified and are shown by the black symbols. In the inverse problem the mode ratios would be known and, provided the length ratios were known, two of the contour plots could be superimposed to find the angles θ and ϕ. If the length ratios were not known, then this procedure would have to be carried out progressively in the 4-dimensional space involving these ratios. Points would become lines in 3 dimensions and then planes in 4 dimensions, and the intersections would define the geometry [9].

Whether this result is unique or whether more than one shape could have the same frequency ratios for these modes, as was shown to be the case for the drum, has not yet been determined. The resemblance of a bent bar to the boundary of the indistinguishable drums studied by Gottlieb and McManus [6] suggests, however, that there may be two shapes with many bending points that have identical mode frequencies, so that the inverse problem is in general not completely solvable, though it may be for a small number of bends.

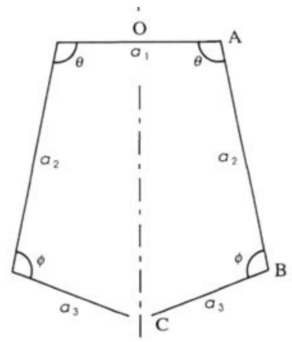

Figure 2. Parameters defining the shape of a simple pentangle, a realistic version actually having curved bends and angles very different from those shown [9].

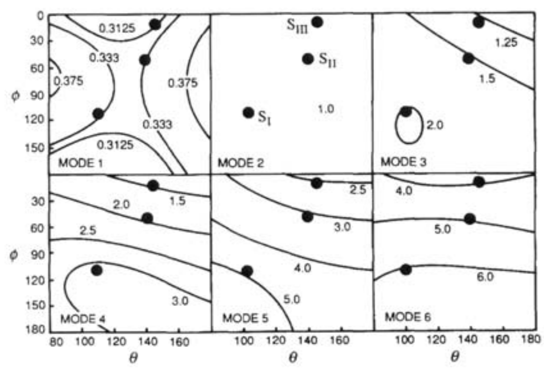

Figure 3. Contours for the frequencies of the first five modes of a pentangle with a_2/a_1 = 2 and a_3/a_1 = 1. S_I, S_{II} and S_{III} are close to possible practical design solutions [9].

BELLS, GONGS AND CYMBALS

Bells and gongs are much more complex to analyse because they are three-dimensional objects, though, with the exception of Chinese two-tone bells, they generally have axial symmetry. If this axial symmetry is not perfect, then this can be deduced from the sound, because the two pairs of a normally degenerate mode will have slightly different frequencies, giving rise to the beating sound called `warble'. The inverse problem of practical interest again involves the tuning of the vibrational modes, since a harmonic relationship, including a minor-third ratio 6:5, is the normal objective. This tuning is carried out by identifying the part of the bell profile where the walls are thicker than desired and removing material from the inside wall using a large lathe. In this case the inverse problem has generally been codified into a set of rules relating the excess wall thickness to the mistuning of the modes, so that a practical approach can be used.

Gongs are more difficult because they have many varied shapes and are often excited by the striker to the extent that the oscillation amplitude is larger than the wall thickness and comparable to some of the geometric dimensions. This leads to acoustically dominant nonlinear effects that contribute much of the character of sound, particularly for cymbals [11]. One easily solvable inverse problem, however, relates to the geometry of gongs such as the small Chinese Opera gong, in which the vibrating section is a shallow spherical shell stiffened by a conical and then cylindrical surrounding section. When such a gong is struck vigorously, the sound frequency is initially below the small-excitation value and rises towards this value as the vibration decays. For extreme excitation, however, the frequency rises again, the fractional extent by which the minimum frequency lies below the small-amplitude frequency by an amount depending upon the ratio of shell thickness d to dome height x_0 as shown in Fig. 4 [12]. The ratio x_0/d can therefore be deduced from the frequency minimum and so from the sound, thus solving an interesting inverse problem.

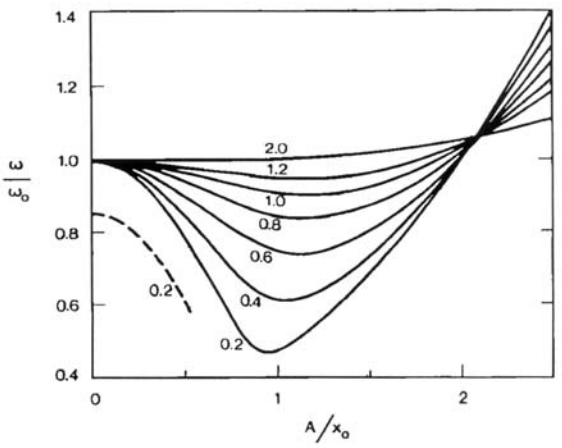

Figure 4. Calculated vibration frequency ω of a shallow domed shell as a function of the ratio of amplitude A to dome height x_0 with the ratio of shell thickness d to dome height x_0 as a parameter. The dashed curve is for an everted dome. [12]

BOWED-STRING INSTRUMENTS

Musical instruments with bowed strings have a very long history and represent the most central and coherent instrument class of modern orchestral music. Their design has changed very little during the past three centuries, and some of the most respected instruments in the world were made early in this period by the Italian makers Stradivari, Amati, Guaneri and their colleagues working in Cremona. The inverse problem in this case asks why these particular instruments are so excellent in sound and playability. Cynics might say it is just a tradition, since many of the instruments have been modified and some modern violins are judged better in double-blind tests, but certainly many of the violins from this period still sound really fine.

The inverse problem asks what it is that is special about these instruments, and various answers have been proposed. Modern makes have for more than a century reproduced the general shape and structure of these famous violins, and more recent versions have shaped the thickness of the top plate so that the mode shapes and frequencies closely match those of some of the classic violins [13]. The results do indeed give a sound quality close to those of the 'great masters', but there are still detectable differences.

Perhaps the most notorious proposal is that the secret of Stradivari resides in the varnish with which he finished his instruments. Certainly varnish does influence the vibration of the violin body because it penetrates the wood to some extent and changes the elastic properties of the surface layer, but analysis shows there is nothing very special about the Stradivari varnish. It is, of course, necessary to compare such iconic instruments with modern instruments which have been built so that the mode frequencies of the violin body match those of Stradivari as closely as possible. This is of supreme importance, for the distribution of mode frequencies and the shapes of those mode patterns have a great influence on the radiated sound.

Another influence is the wood itself, for no two pieces of wood are the same, and this can influence mechanical impedance and damping, both of which are important to sound quality. It has been proposed [14] that the Italian wood of the time was special because it grew during a time of reduced temperatures, known as the Maunder Minimum, and indeed this could influence the ring spacing and also the pore microstructure of the wood, with consequent acoustic effects [15]. This explanation seems likely, since the elastic anisotropy of the wood depends upon its ring structure and the vibrational damping, particularly at high frequencies, is influenced by the density and pore structure. These effects can be demonstrated by making a violin from a non-standard wood species – the sound may be quite pleasant but will be 'different'.

Some of these inverse questions can be answered, but generally only in a limited way because of the number of parameters involved. Working backwards from a long-term averaged spectrum, for example, gives information about the internal damping of the wood, while identification of the frequencies of the main resonances tells something about the thickness shaping of the top-plate. The high frequencies can also often reveal whether the strings are gut or metal.

WIND INSTRUMENTS

When we consider wind instruments such as flutes, clarinets and trumpets the primary inverse question to be answered is one defining the shape of the bore. Certainly there are many questions asked about influence of the material from which the instruments is made: Do tin-rich organ pipes sound better than lead-rich pipes? Is a gold flute better than a silver flute? and so on, but the answer is that the material is usually chosen for non-acoustic reasons and, within limits, has little effect on the sound [16]. But the geometry is crucial.

Beginning with the simplest case, can one determine from its sound the ratio of diameter to length for a cylindrical organ pipe? The answer is a moderate 'Yes' and comes from examination of the spectrum of the sound. Nonlinear interactions at the air jet or the reed driving the pipe produce all harmonics of the fundamental and these are locked into exact phase relations [17]. But the relative amplitudes of these harmonics depend upon the Q-factors and tuning of the pipe resonances and these are largely determined by the ratio of pipe diameter to wavelength. As the harmonic frequency increases past the value at which the pipe diameter is about one half of the wavelength, the Q-factor drops sharply and this reduces the amplitudes of higher harmonics, so that a narrow pipe sounds bright compared with a wide pipe, and is also quieter. This effect is taken into account in the scaling of organ pipe ranks [18]. Examination of the relative strengths of the harmonics of an open pipe also gives information about the geometry of the pipe mouth and adjustment of the jet. If the jet strikes the upper lip nearly symmetrically then the even harmonics are weak relative to the odd harmonics [19], while the relation between loudness and harmonic development gives information about the lip cut-up distance and blowing pressure.

Another interesting feature of the sound that provides some inverse solutions is the nature of the initial transient as the pipe begins to sound. Particularly if the excitation commences abruptly, all the normal modes of the pipe are excited at their natural frequencies, and it is only after about ten cycles of the fundamental that they all become locked together in phase and frequency ratio [17]. Observation of this initial transient can give further information about the geometry and excitation of the pipe.

Since a woodwind instrument must be able to play all the notes in its compass, rather than just a single one, there is no such simple inverse problem, though the equivalent is the scaling of the tone-holes relative to the bore [20]. If this ratio is small then higher harmonics are not adequately reflected from the first open hole and are weak in the sound, giving the mellow tone typical of baroque-style instruments. The sound quality, in addition, gives immediate information about the general shape of the instrument bore or air column. This is because, ideally, a cylindrical tube, as in a clarinet, has only odd resonances, while a conical tube, as in an oboe, has all harmonic resonances. In the first register the sound spectrum reflects this differentiation, though not precisely because the air jet or reed generates all harmonics and these are then filtered by the bore resonances, but the difference between cylindrical and conical instruments is clear in the lower register.

THE INSTRUMENT AND THE PLAYER

As a final inverse problem it is interesting to ask whether one can distinguish between different players on an instrument and recognise a particular player. The answer to the first question is clearly yes, since players differ greatly in their expertise and this is reflected in the sound of the instrument, quite apart from any relation to the work being played. In orchestral instruments such as strings and winds the features that distinguish individual players are mainly the attack and the vibrato. The sound of the attack or initial transient is closely related to the ‘gesture’ of the player – an arm movement on the case of string instruments and an internal motion of tongue and breathing muscles for wind players – while vibrato depends on individual neuro-muscular effects.

Solving this inverse problem is rather like the activities of spies and forensic scientists, since there is a great deal of coupling between the brain and the muscles involved in performance, and the result may depend greatly upon the music being played. Despite this, some performers are clearly recognisable to an experienced listener, though the perception of the relevant clues is more a psychological than a physical exercise.

CONCLUSIONS

Inverse problems relating to musical instruments are difficult because there are so many parameters affecting the sound produced. Only in the simplest cases can a precise answer be obtained, but a great deal can be learned from the study involved and the questions asked and partially answered.

Perhaps most important among these answers to inverse problems are those referring to specific design properties of historical instruments that have been found to have excellent playing properties or musical timbre. Even if the complete answer cannot be found, a great deal can be learned from the investigation.

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