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GEOMETRIC SHAPE IDENTIFICATION FOR MULTI-MODE TUNING OF PERCUSSION INSTRUMENT BARS

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Abstract

The lower registers of the marimba, vibraphone and xylophone consist of percussion bars that have the lower three bending modes tuned. Tuning is achieved by removing material from the underside of the beam. Accurate prediction of the geometry of this undercut would be necessary for automated tuning. This paper models the beam free vibrations and accurately predicts the non-unique shape of the undercut that results in the simultaneous tuning of the three frequencies.

Accurate natural bending frequencies are modelled using receptance sub-structuring and Timoshenko beam receptances. Allowance for the frequency dependency of the elastic modulus of wood is made. Search algorithms are implemented to locate the geometric shape of the undercut curve that satisfies the multi-mode frequency requirements. The sensitivity of the frequencies to dimensional variations is reported.

Manufacture of aluminium and wooden bars show that the predictions are very accurate and suitable for the basis of automated manufacture.

1. INTRODUCTION

The marimba, xylophone and vibraphone are each examples of idiophones where the primary vibrating element is a free-free bar excited by being struck with a mallet. As prismatic bars do not have transverse bending frequencies that form a harmonic series, these instruments utilise bars that are non-prismatic, being shaped in such a way as to bring the first one or two overtones into the desired frequency relationship with the fundamental. The usual approach is to remove an arch of material from the underside of the bar, symmetrically about the centre of its long dimension. The goal for the marimba, especially in the lower registers, is to tune the first overtone to the double octave of the fundamental ($f_1 = 4 f_0$) and the second overtone to somewhere between the minor and major third above the triple octave ($f_1 \approx 9.8 f_0$) [1]. The primary difference in the xylophone is that the first overtone is tuned to a twelfth above the fundamental ($f_1 = 3 f_0$) [2]. The marimba and xylophone bars are traditionally made of dense wood (although some synthetic bars are used) while the vibraphone utilises metal. Each form makes use of tubular resonators to amplify the sound. Those on the vibraphone have rotating paddles at their upper ends to periodically open and close the resonator, producing an amplitude vibrato effect.

When building one of these instruments, it would be advantageous to be able to accurately predict the shape of the undercut that would produce the desired pitch and overtone frequencies. This would be especially so for volume manufacture where the avoidance of labour intensive manual tuning can reduce costs. This paper offers a prediction method that has resulted in bars very close to their tuning goals without any subsequent fine tuning. Given the non-homogeneous nature of wood, that cannot be accurately characterised, clearly some fine tuning of the wooden bars will always be required. However, being able to reliably pre cut bars to within a known multi-mode frequency tolerance will go some way to aiding rapid manufacture. Automated application of known tuning maps [3] could take care of the fine tuning.

Several methods of modelling the non-prismatic beam have been published [1,4-12]. It is clear that the Timoshenko beam model is required [8, 12], as it is more accurate than the Euler-Bernoulli model, especially for the higher vibration modes – whether using mathematical or finite element models. The mathematical models generally approximate the smoothly curved undercut by dividing the bar into a number of prismatic sections thus forming a 'staircase' approximation. The boundary conditions at the ends of each section are matched to its neighbour and solved along with the support conditions (usually free-free, closely approximating the condition in the real instrument). Various solution techniques have been used.

A feature of wood is its frequency-dependent elastic properties. Of particular importance in the present study is the change in Young's modulus (E) as the frequency varies. It is usually higher than the static value but plateaus above about 150 Hz [12]. For accurate natural frequency prediction, this property should be taken into account.

This paper presents a model for marimba bar tuning with three novel features. Firstly, the non-prismatic beam is modelled using receptance sub structuring techniques [13] that account for both coordinates – bending and shear. While not widely used, the Timoshenko beam receptances are available [14] and are used in this work. Secondly, the geometrical shape of the undercut is not prescribed but rather a mathematical curve with three adjustable parameters. The same goal is achieved by Henrique and Antunes [15] and Petrolito and Legge [16] with the use of finite element modelling and numerical optimisation techniques. In this paper, a three dimensional Newton-Raphson solution technique solves the three variable parameters such that the lowest three required natural bending frequencies are achieved. Thirdly, the receptance model easily allows the Young's modulus to be frequency dependent.

2. THE MODEL

The tip receptance of a prismatic beam is the steady state amplitude response at one end to a sinusoidal input at that (direct) or the other (cross) end. The input may be either a transverse force or a moment and the response may be a displacement or a rotation. Hence there are eight direct receptances and eight cross receptances, half of which are duplicated due to symmetry. These can be calculated in closed form, including for the Timoshenko formulation. [13,14]. Prismatic sections (perhaps with different cross sectional dimensions) may then be 'assembled' in the following fashion.

Consider two sections (B and C) of uniform prismatic beams that are to be joined into a single section A. (Figure 1)

Determination of the tip receptances of A $\begin{bmatrix} \alpha_{dd} & \alpha_{ds} \\ \alpha_{sd} & \alpha_{ss} \end{bmatrix}$ would then enable a further section (D) to be joined in the same fashion as C was joined to B. Repeating this process enables a complete staircase approximation of a non-prismatic beam to be constructed. The resonant frequencies will be those for which any $\alpha_{ij} \to \infty$.

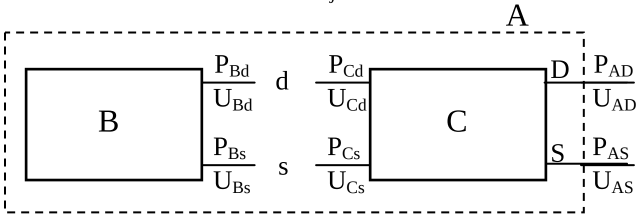

Figure 1. Joining beam sections B and C to give tip receptances of A

At the joint, four conditions must be fulfilled:

The internal shear forces on B and C must sum to zero: i.e. $V_B + V_C = 0$ (1)

The internal moments on B and C must sum to zero: i.e. $M_B + M_C = 0$ (2)

The deflections must be the same: i.e. $d_B = d_C$ (3)

The rotation (slopes) must be the same: i.e. $s_B = s_C$ (4)

Introducing the nomenclature that all loads (forces and moments) are *P* and all displacements (deflection and slope) are *U*, then equations (1-4) become:

$$\begin{bmatrix} P_{Bd} \\ P_{Bs} \end{bmatrix} + \begin{bmatrix} P_{Cd} \\ P_{Cs} \end{bmatrix} = 0 ; \qquad \begin{bmatrix} U_{Bd} \\ U_{Bs} \end{bmatrix} = \begin{bmatrix} U_{Cd} \\ U_{Cs} \end{bmatrix} \tag{5}$$

Use is made of the conventional symbols for receptances. α_{ij} are the receptances of system A, β_{ij} of B, γ_{ij} of C, etc. The first subscript is the response location (lower or upper case) and type (*d* is transverse load and deflection, *s* is applied moment and slope) while the second subscript is the location and type of the excitation. Then from the definition of a receptance and assuming linearity:

$$\begin{bmatrix} U_{Cd} \\ U_{Cs} \end{bmatrix} = \begin{bmatrix} \gamma_{dd} & \gamma_{ds} \\ \gamma_{sd} & \gamma_{ss} \end{bmatrix} \begin{bmatrix} P_{Cd} \\ P_{Cs} \end{bmatrix} + \begin{bmatrix} \gamma_{dD} & \gamma_{dS} \\ \gamma_{sD} & \gamma_{sS} \end{bmatrix} \begin{bmatrix} P_{CD} \\ P_{CS} \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} U_{Bd} \\ U_{Bs} \end{bmatrix} = \begin{bmatrix} \beta_{dd} & \beta_{ds} \\ \beta_{sd} & \beta_{ss} \end{bmatrix} \begin{bmatrix} P_{Bd} \\ P_{Bs} \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} U_{AD} \\ U_{AS} \end{bmatrix} = \begin{bmatrix} \gamma_{DD} & \gamma_{DS} \\ \gamma_{SD} & \gamma_{SS} \end{bmatrix} \begin{bmatrix} P_{AD} \\ P_{AS} \end{bmatrix} + \begin{bmatrix} \gamma_{Dd} & \gamma_{Ds} \\ \gamma_{Sd} & \gamma_{Ss} \end{bmatrix} \begin{bmatrix} P_{Cd} \\ P_{Cs} \end{bmatrix} \tag{8}$$

and by definition

$$\begin{bmatrix} U_{AD} \\ U_{AS} \end{bmatrix} = \begin{bmatrix} \alpha_{DD} & \alpha_{DS} \\ \alpha_{SD} & \alpha_{SS} \end{bmatrix} \begin{bmatrix} P_{AD} \\ P_{AS} \end{bmatrix} \tag{9}$$

Rewriting equations (5) to (9) in an abbreviated form where bold font denotes a matrix:

$$\mathbf{P_B} + \mathbf{P_C} = \mathbf{0}\,; \qquad \mathbf{U_B} + \mathbf{U_C} = \mathbf{0} \tag{5a, 5b}$$

$$\mathbf{U_C} = \boldsymbol{\gamma}_1\mathbf{P_C} + \boldsymbol{\gamma}_2\mathbf{P_A}\,; \qquad \mathbf{U_B} = \boldsymbol{\beta}\mathbf{P_B} \tag{6a, 7a}$$

$$\mathbf{U_A} = \boldsymbol{\gamma}_3\mathbf{P_A} + \boldsymbol{\gamma}_4\mathbf{P_C}\,; \qquad \mathbf{U_A} = \boldsymbol{\alpha}\mathbf{P_A} \tag{8a, 9a}$$

We wish to solve for $\boldsymbol{\alpha}$ in terms of the known tip receptances of B and C.

From (5a) to (7a):
$$\mathbf{P_C} = -(\boldsymbol{\gamma}_1 + \boldsymbol{\beta})^{-1}\boldsymbol{\gamma}_2\mathbf{P_A} \tag{10}$$

Substituting (10) into (8a):
$$\mathbf{U_A} = \boldsymbol{\gamma}_3\mathbf{P_A} - \boldsymbol{\gamma}_4(\boldsymbol{\gamma}_1 + \boldsymbol{\beta})^{-1}\boldsymbol{\gamma}_2\mathbf{P_A}$$

Then from (9a):
$$\boldsymbol{\alpha} = \frac{\mathbf{U_A}}{\mathbf{P_A}} = \boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_4(\boldsymbol{\gamma}_1 + \boldsymbol{\beta})^{-1}\boldsymbol{\gamma}_2 \tag{11}$$

Finally, due to reciprocity resulting from the assumed linearity, $\gamma_{Dd} = \gamma_{dD}$ etc., and therefore $\boldsymbol{\gamma}_4 = \boldsymbol{\gamma}_2^{\mathbf{T}}$, (11) becomes:

$$\boldsymbol{\alpha} = \boldsymbol{\gamma}_3 - \boldsymbol{\gamma}_2^T(\boldsymbol{\gamma}_1 + \boldsymbol{\beta})^{-1}\boldsymbol{\gamma}_2 \tag{12}$$

The tip receptances for an Euler-Bernoulli beam are available in [13] and those for a Timoshenko beam in [14]. Note that these are exact results in both cases and hence $\boldsymbol{\alpha}$ from equation (12) is exact. The only source of accuracy limiting is computer round off error. Natural frequencies occur whenever any element of $\boldsymbol{\alpha}$ becomes infinite (its denominator goes to zero). Due to the zero damping assumption, a sign change occurs as the resonance is traversed. Therefore a resonant frequency can be easily located as the zeros of $1/\alpha_{ij}$ of the assembled non-prismatic bar. Use of software such as Matlab, where the default variable type is a matrix, makes coding the above method straight forward.

The model was validated against exact natural frequency solutions for prismatic beams and also those offered for tapered cantilevers by Wang [17]. It was found that results correct to five significant figures were obtained for heavily tapered beams when the staircase approximation was given 300 or more steps. This accuracy is within one cent (one hundredth of a semitone) even for the fundamental frequency of an A2 bar (110 Hz).

3. APPLICATION

As the goal is to alter the shape of the undercut in order to simultaneously tune the lower three bending natural frequencies, the mathematical description of the undercut geometry requires three adjustable parameters. For example, each of the following is possible where the dimensions are: L, the total length of the bar, x, the dimension from the centre of the bar along the length, d is the depth at any point and d_0 is the depth at the centre of the span. The width of the beam is constant.

Cubic:
$$d\left(\frac{x}{L}\right) = a_1\left|\frac{x}{L}\right|^3 + a_2\left(\frac{x}{L}\right)^2 + d_0 \tag{13}$$

Exponential 1: $$d\left(\frac{x}{L}\right) = d_0\left(1 + e^{a_1\frac{x}{L}} - e^{a_2\frac{x}{L}}\right) = d_0\left(1 + \sum_{i=1}^{\infty}\frac{a_1^i - a_2^i}{i!}\left(\frac{x}{L}\right)^i\right) \quad (14)$$

Exponential 2: $$d\left(\frac{x}{L}\right) = \frac{d_0}{2}\left(e^{a_1\frac{x}{L}} + e^{-a_2\frac{x}{L}}\right) = \frac{d_0}{2}\left(2 + \sum_{i=1}^{\infty}\frac{a_1^i + (-1)^i a_2^i}{i!}\left(\frac{x}{L}\right)^i\right) \quad (15)$$

Flat bottom with parabolic transitions (d_1 is the uncut depth of the beam):

$$\begin{aligned} d\left(\frac{x}{L}\right) &= d_0; \quad \frac{x}{L} \le a_1 \\ d\left(\frac{x}{L}\right) &= a_2\left(\frac{x}{L} - a_1\right)^2 + d_0; \quad \frac{x}{L} > a_1, d < d_1 \end{aligned} \quad (16)$$

The cubic and the flat bottomed parabola have the advantage of zero slope at the centre of the bar thus eliminating any cusp. The exponentials do not have such a constraint.

To estimate the sensitivity of the three transverse vibration frequencies to each of the three variable parameters, d_0 (central thickness of beam), a_1 and a_2, each parameter can, in turn, be slightly altered and the modelled frequencies predicted. Hence the Jacobian matrix [18] can be estimated:

$$J = \left[\frac{\partial f_i}{\partial a_j}\right]; \quad i, j = 0,1,2. \quad (17)$$

These rates of change, or sensitivities, can then be used in a three dimensional Newton-Raphson method to drive the undercut shape to the objective of $f_0 : \frac{f_1}{f_0} : \frac{f_2}{f_0}$. It is known, and confirmed here, that an undercut geometry that achieves this objective is not unique as more than one of equations (13) to (16) may yield a conforming geometry. Furthermore, it may be that no solution exists.

4. EXAMPLE PREDICTIONS

As an example, Figure 2 shows the undercut geometry predicted using the cubic curve, equation (13), for various target second overtone ratios $\frac{f_2}{f_0}$. In this case the fundamental is A2 (110 Hz) with bar length 450 mm. Figure 3 shows alternative geometries that result in the same frequency ratios and fundamental, choosing $\frac{f_2}{f_0} = 9.8$ as an example. The sensitivity of the fundamental frequency to the depth of the undercut (d_0 in equation (13)) is high, *viz.* 50 μm/Hz for the examples above. This implies that manufacturing must be performed to tight tolerances.

5. MANUFACTURING AND FREQUENCY MEASUREMENTS

Initially three bars were machined in 6061 Aluminium alloy rectangular bar so that the non homogeneity and anisotropy of wood were avoided. The elastic properties were inferred by measuring the vibration frequencies of a prismatic bar and adjusting E and G in the Timoshenko receptance model until they agreed.

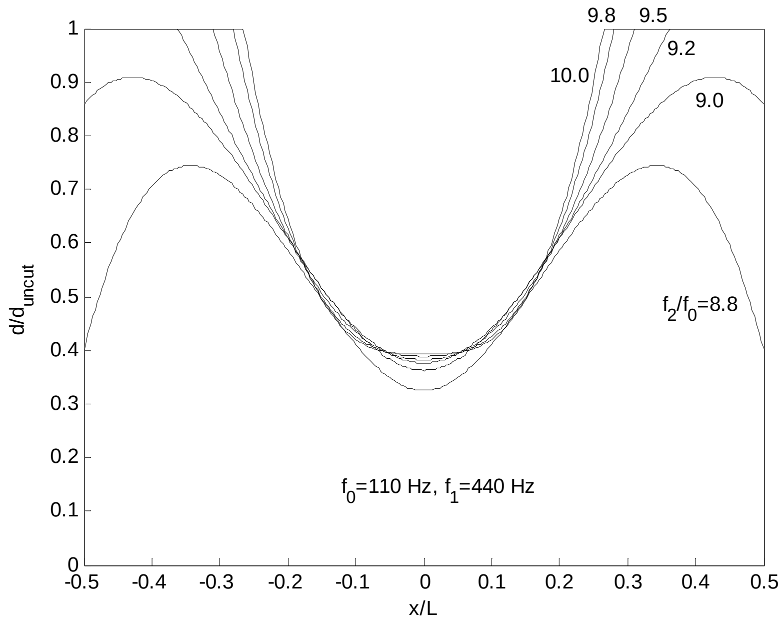

Figure 2. Predicted undercut geometries using cubic equation (13) and various $\frac{f_2}{f_0}$ ratios.

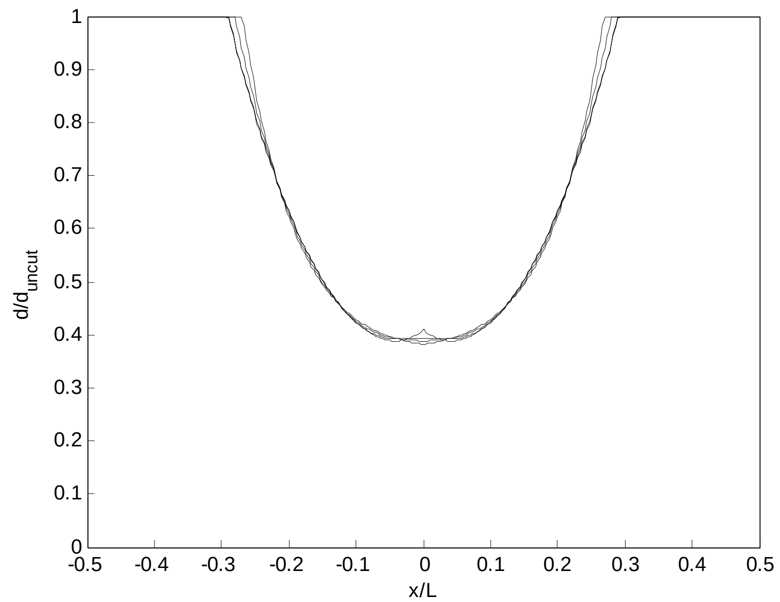

Figure 3. Predicted undercut geometries using equations (13-16), $f_0 : \frac{f_1}{f_0} : \frac{f_2}{f_0}$ =110:4:9.8.

The resulting coordinates for the predicted undercuts were then calculated and used to control a CNC machining centre operating with a long end mill. As mentioned previously, considerable care was required in setting up the job and avoiding distortion so that tolerances of about 0.01 mm could be achieved.

Frequency measurements were made by supporting a bar approximately at its nodes on polyurethane foam and exciting with a percussion mallet. A small (1 gram) accelerometer was attached to the bar with bee's wax close to, but not at, a node in order to reduce mass loading effects. A HP35665 spectrum analyser allowed 0.125 Hz resolution when using its zoom FFT mode. Table 1 shows the frequencies obtained and compares them to the target frequencies and those measured on a Van Sice concert marimba. These frequencies are those obtained directly from the machined bar. No further adjustments were made.

The as-manufactured mid section dimensions of the C3 and B4 bars were measured and found to be in error by 0.014 and -0.030 mm respectively. Using the predicted fundamental frequency sensitivity to this dimension, the fundamental frequencies can be adjusted. This resulted in f_0 = 131.65 Hz (target 130.8 Hz) and 493.4 Hz (target 493.4 Hz). Clearly the modelling technique was confirmed to be very accurate.

Table 1. Frequency targets and measurements – aluminium (51 x 19 mm).

Pitch and Mode	Target		Experimental bar			Measured Van Sice	
	Hz	f_i / f_0	Length (mm)	Hz	f_i / f_0	Hz	f_i / f_0
C3			449.7				
f_0	130.8	1		132.3	1	131.4	1
f_1	523.2	4		528.8	4.00	524.8	3.99
f_2	1281.8	9.8		1295.8	9.80	1313.8	10.00
A3			389.6				
f_0	220	1		222.3	1	220.1	1
f_1	880	4		885.3	3.98	879.9	4.00
f_2	2156	9.8		2164.3	9.74	2190.0	9.95
B4			320.0				
f_0	493.9	1		491.1	1	499.3	1
f_1	1975.6	4		1976.0	4.02	1996.0	4.00
f_2	4840.2	9.8		4766.5	9.71	4152.0	8.32

A similar procedure was then undertaken with two wooden bars (Jarrah – *Eucalyptus Marginata*)

Table 2. Frequency targets and measurements - wood.

Pitch and Mode	Target		Experimental bar			
	Hz	f_i / f_0	Length (mm)	Depth d_1 (mm)	Hz	f_i / f_0
C3			450.5	30.2		
f_0	130.8	1			130.8	1
f_1	523.2	4			529.5	4.05
f_2	1281.8	9.8			1298.0	9.93
A3			392.0	19.7		
f_0	220	1			220.8	1
f_1	880	4			852.0	3.86
f_2	2156	9.8			2096.0	9.49

6. CONCLUSIONS

The combination of receptance sub-structuring methods and use of the Timoshenko beam receptances has provided accurate predictions of the undercut shapes required to meet the objective transverse vibration frequency ratios of marimba bars. When corrected for manufacturing error, the fundamental frequencies were very accurate. The receptance method also allows frequency dependent elastic properties to be easily incorporated into the solution. Experimental tests have shown that the technique could be used to provide initial shaping of marimba and xylophone bars in wood prior to manual fine tuning. Automated initial and final tuning may be possible.

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