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NUMERICAL SIMULATION OF NONLINEAR SOUND WAVE PROPAGATION USING CIP METHOD

Takao Tsuchiya[1]*, Kan Okubo[2]†, Nobunao Takeuchi[2]‡

[1]Department of Information Systems Design
Doshisha University
1-3 Miyakodani, Tatara, Kyotanabe-shi, Kyoto 610-0321, Japan

[2]Department of Electrical Information Systems
Akita Prefectural University
84-4 Ebinokuchi, Tsuchiya, Yurihonjyo-shi, Akita 015-0055, Japan

ttsuchiy@mail.doshisha.ac.jp
kanne@akita-pu.ac.jp
ntake@akita-pu.ac.jp

Abstract

In this paper, numerical simulation of nonlinear sound wave propagation in time domain is performed by Constrained Interpolation Profile (CIP) method. CIP is a novel numerical scheme which was recently proposed by Yabe. It is one kind of method of characteristics (MOC) and is a high accuracy numerical scheme in which numerical dispersion errors are hardly caused. To achieve high accuracy, not only the acoustic field values on the grid point but also their spatial derivatives are used in the scheme. It is suitable for analysis of nonlinear wave propagation including weak shock formation because the rapid pressure change such as shock front easily causes numerical dispersion error in the conventional numerical scheme. Some numerical demonstrations are made for the one-dimensional nonlinear sound propagation in air. The results are compared with the conventional FDTD method and the analytical solutions.

1. INTRODUCTION

Time domain numerical analysis of acoustic field has been familiar as a result of recent progress of computing environments. Although many numerical schemes have been proposed for time domain analysis, the finite difference time domain (FDTD) method [1] is the most popular scheme in acoustics. In FDTD method, continuity equation and equation of motion are transformed into central-difference equations, then the equations are solved in a leapfrog manner. However it is known that the scheme easily causes the error due to numerical dispersion. This means that the scheme is not so suitable for analysis of nonlinear wave propagation including shock formation because the rapid pressure change such as shock front causes numerical dispersion error.

In this paper, numerical simulation of nonlinear sound wave propagation in time domain is performed by Constrained Interpolation Profile (CIP) method. CIP is a novel numerical scheme which was recently proposed by Yabe[2]. It is one kind of method of characteristics (MOC) and is a high accuracy numerical scheme in which numerical dispersion errors are hardly caused. To achieve high accuracy, not only the acoustic field values on the grid point but also their spatial derivatives are used in the scheme. Some numerical demonstrations are made for the one-dimensional nonlinear sound propagation in air under the weak shock assumption. The results are compared with the conventional FDTD method and the analytical solutions.

2. GOVERNING EQUATIONS

The governing equations for the nonlinear acoustic field with the velocity dispersion under the assumption of weak shock are given as follows [3]

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\rho_0 c_0^2 \nabla \cdot \boldsymbol{u} + \frac{\delta_1}{\rho_0}\nabla^2 p + \frac{\beta}{\rho_0 c_0^2}\frac{\partial p^2}{\partial t} && (1) \\ \frac{\partial \boldsymbol{u}}{\partial t} &= -\frac{1}{\rho_0}\nabla p + \frac{\delta_2}{\rho_0}\nabla^2 \boldsymbol{u} && (2) \end{aligned}$$

where p is pressure, $\boldsymbol{u}$ is particle velocity vector, ρ_0 is ambient medium density, κ is bulks modulus, $c_0 = \sqrt{\kappa/\rho_0}$ is sound velocity of small amplitude. Equation (1) is the continuity equation and (2) is the equation of motion. δ_1 and δ_2 are respectively given as

$$\delta_1 = \chi\left(\frac{1}{c_v} - \frac{1}{c_p}\right), \quad \delta_2 = \zeta + \frac{4}{3}\eta \tag{3}$$

where χ is thermal conductivity, ζ is bulk viscosity, η is shear viscosity, and c_v and c_p are specific heat at constant volume and one at constant pressure, respectively.

Now we consider the one-dimensional case. Using the linear relation $\partial p/\partial t = -\rho_0 c_0^2 \partial u/\partial x$, equations (1) and (2) are rewritten for the one-dimensional field as

$$\begin{aligned} \frac{\partial p}{\partial t} + c_0\frac{\partial (Zu)}{\partial x} &= \frac{\delta_1}{\rho_0}\frac{\partial^2 p}{\partial x^2} - 2\beta p\frac{\partial u}{\partial x} && (4) \\ \frac{\partial (Zu)}{\partial t} + c_0\frac{\partial p}{\partial x} &= \delta_2 c_0 \frac{\partial^2 u}{\partial x^2} && (5) \end{aligned}$$

where $Z = \sqrt{\rho_0 \kappa}$ is characteristic impedance. By addition and subtraction of two equations, we obtain

$$\begin{aligned} \frac{\partial f_+}{\partial t} + c_0\frac{\partial f_+}{\partial x} &= \frac{\delta_1}{2\rho_0}\frac{\partial^2}{\partial x^2}(f_+ + f_-) + \frac{\delta_2 c_0}{2Z}\frac{\partial^2}{\partial x^2}(f_+ - f_-) - \frac{\beta}{2Z}(f_+ + f_-)\frac{\partial}{\partial x}(f_+ - f_-) && (6) \\ \frac{\partial f_-}{\partial t} - c_0\frac{\partial f_-}{\partial x} &= \frac{\delta_1}{2\rho_0}\frac{\partial^2}{\partial x^2}(f_+ + f_-) - \frac{\delta_2 c_0}{2Z}\frac{\partial^2}{\partial x^2}(f_+ - f_-) - \frac{\beta}{2Z}(f_+ + f_-)\frac{\partial}{\partial x}(f_+ - f_-) && (7) \end{aligned}$$

where $f_\pm$ are defined as $f_\pm = p \pm Zu$. When the medium dispersion and nonlinearity can be ignored, equations (8) and (9) can be denoted as

$$\frac{\partial f_+}{\partial t} + c_0 \frac{\partial f_+}{\partial x} = 0 \tag{8}$$

$$\frac{\partial f_-}{\partial t} - c_0 \frac{\partial f_-}{\partial x} = 0 \tag{9}$$

These equations are advection equations and they show that the waves $f_\pm$ propagate along $\pm x$ direction with speed c_0.

When one can obtain $f_\pm$, the sound pressure and the particle velocity can be calculated respectively as

$$p = \frac{f_+ + f_-}{2}, \quad u = \frac{f_+ - f_-}{2Z} \tag{10}$$

3. CIP METHOD

3.1. Advection phase

First we are to solve advection equations (6) and (7) by CIP method which is called advection phase. The field values $f_\pm$ are defined at the one-dimensional grid points. For the propagating waves $f_\pm$ along $\pm x$ directions, the field values $f_\pm^n(x_i \pm c_0\Delta t)$ are advected to the point x_i after the time step Δt as shown in Figure 1.

$$f_\pm^{n+1}(x_i) = f_\pm^n(x_i \mp c_0\Delta t) \tag{11}$$

Because the field values $f_\pm^n(x_i \mp c_0\Delta t)$ are not defined at grid points, they are interpolated by using the field values at the grid points as follows

$$f_\pm^{n+1}(x_i) \doteqdot F_{i\pm}^n(x_i \mp c_0\Delta t) \tag{12}$$

where $F_{i\pm}(x)$ is the interpolation function defined for the interval $[x_{i\mp1}, x_i]$.

In CIP method, the third order polynomial is used for the interpolation function $F_{i\pm}(x)$.

$$F_{i\pm}^n(x) = a_\pm X_i^3 + b_\pm X_i^2 + c_\pm X_i + f_\pm^n(x_i) \tag{13}$$

where $X_i = x - x_i$. Although the third order polynomial is usually defined by four grid points, it is defined by two adjoining grid points in CIP method at which field values and their derivatives

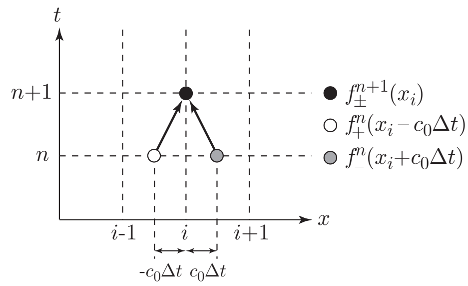

Figure 1. Characteristic curve.

$\partial f_\pm/\partial x \equiv g_\pm$ are defined. This means that the derivative of field value is also advected. The interpolation function for derivative defined for interval $[x_{i\mp1}, x_i]$ is given as

$$G^n_{i\pm}(x) = 3a_\pm X_i^2 + 2b_\pm X_i + c_\pm \tag{14}$$

where

$$\begin{aligned}
a_\pm &= \pm\frac{2\{f^n_\pm(i\mp1) - f^n_\pm(i)\}}{(\Delta x)^3} + \frac{g^n_\pm(i\mp1) + g^n_\pm(i)}{(\Delta x)^2} && (15)\\
b_\pm &= \frac{3\{f^n_\pm(i\mp1) - f^n_\pm(i)\}}{(\Delta x)^2} \pm \frac{g^n_\pm(i\mp1) + 2g^n_\pm(i)}{\Delta x} && (16)\\
c_\pm &= g^n_\pm(i) && (17)
\end{aligned}$$

where $f(i)$ and $g(i)$ are the field value and its derivative at $x = x_i$ and Δx is distance between grid points. Using equations (13) and (14), the field values and their derivatives at the next time step $n+1$ are expressed as

$$\begin{aligned}
f^{n+1}_\pm(x_i) \doteqdot F^n_{i\pm}(x_i+\xi) &= a_\pm\xi^3 + b_\pm\xi^2 + g^n_\pm(i)\xi + f^n_\pm(i) && (18)\\
g^{n+1}_\pm(x_i) \doteqdot G^n_{i\pm}(x_i+\xi) &= 3a_\pm\xi^2 + 2b_\pm\xi + g^n_\pm(i) && (19)
\end{aligned}$$

where $\xi = \mp c_0\Delta t$.

3.2. Non-advection phase

Next we consider the dispersion and nonlinear terms shown in the right side of equations (6) and (7).

$$h_\pm = \frac{\delta_1}{2\rho_0}\frac{\partial^2}{\partial x^2}(f_+ + f_-) \pm \frac{\delta_2 c_0}{2Z}\frac{\partial^2}{\partial x^2}(f_+ - f_-) - \frac{\beta}{2Z}(f_+ + f_-)(g_+ - g_-) \tag{20}$$

These terms are non-advection terms which are calculated after the advection phase by the following equations

$$\begin{aligned}
\frac{\partial f^*_\pm}{\partial t} &= h^*_\pm && (21)\\
\frac{\partial g^*_\pm}{\partial t} &= \frac{\partial h^*_\pm}{\partial x} && (22)
\end{aligned}$$

where superscript * denotes the solution obtained by the advection phase. The time development equations are given as

$$
\begin{aligned}
f_{\pm}^{n+1}(i) &= f_{\pm}^{*}(i) + h_{\pm}^{*}(i)\Delta t \\
&= f_{\pm}^{*}(i) + \frac{\delta_1 \Delta t}{2\rho_0} \frac{g_{+}^{*}(i+1) - g_{+}^{*}(i-1) + g_{-}^{*}(i+1) - g_{-}^{*}(i-1)}{2\Delta x} \\
&\quad \pm \frac{\delta_2 c_0 \Delta t}{2Z} \frac{g_{+}^{*}(i+1) - g_{+}^{*}(i-1) - g_{-}^{*}(i+1) - g_{-}^{*}(i-1)}{2\Delta x} \\
&\quad - \frac{\beta \Delta t}{2Z} (f_{+}^{*}(i) + f_{-}^{*}(i))(g_{+}^{*}(i) - g_{-}^{*}(i)) \qquad (23) \\
g_{\pm}^{n+1}(i) &= g_{\pm}^{*}(i) + \frac{f_{\pm}^{n+1}(i+1) - f_{\pm}^{*}(i+1)}{2\Delta x} - \frac{f_{\pm}^{n+1}(i-1) - f_{\pm}^{*}(i-1)}{2\Delta x} \qquad (24)
\end{aligned}
$$

3.3. CIP-CSL4 method

A more accurate CIP scheme has been proposed as CIP-Conservative Semi-Lagrangian of the 4th order (CIP-CSL4) method[4]. In CIP-CSL4 method, the fourth order polynomial is used for the interpolation function $F_{i\pm}(x)$ and its derivative $G_{i\pm}(x)$ is the third order, which are respectively given as

$$
\begin{aligned}
F_{i\pm}^{n}(x) &= a_{4\pm} X_i^4 + b_{4\pm} X_i^3 + c_{4\pm} X_i^2 + d_{4\pm} X_i + f_{\pm}^{n}(i) \qquad (25) \\
G_{i\pm}^{n}(x) &= 4a_{4\pm} X_i^3 + 3b_{4\pm} X_i^2 + 2c_{4\pm} X_i + d_{4\pm} \qquad (26)
\end{aligned}
$$

To define the fourth order polynomial, the integrated value between two grid points is introduced as follows

$$
\begin{aligned}
s_{\pm}^{n}(i \mp \frac{1}{2}) &= \int_{x_{i\mp1}}^{x_i} F_{i\pm}^{n}(x)dx = \pm \frac{a_{4\pm}}{5}(\Delta x)^5 - \frac{b_{4\pm}}{4}(\Delta x)^4 \\
&\quad \pm \frac{c_{4\pm}}{3}(\Delta x)^3 - \frac{d_{4\pm}}{2}(\Delta x)^2 \pm f_{\pm}^{n}(i)\Delta x \qquad (27)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{4\pm} &= -5\left[6\{f_{\pm}^{n}(i \mp 1) + f_{\pm}^{n}(i)\}\Delta x \pm \{g_{\pm}^{n}(i \pm 1) - g_{\pm}^{n}(i)\}(\Delta x)^2 \right. \\
&\quad \left. \mp 12 s_{\pm}^{n}(i \mp 1/2)\right] / \{2(\Delta x)^5\} \qquad (28) \\
b_{4\pm} &= \mp 2\left[14\{f_{\pm}^{n}(i \mp 1) + 16 f_{\pm}^{n}(i)\}\Delta x \right. \\
&\quad \left. \pm \{2g_{\pm}^{n}(i \mp 1) - 3g_{\pm}^{n}(i)\}(\Delta x)^2 \mp 30 s_{\pm}^{n}(i \mp 1/2)\right] / (\Delta x)^4 \qquad (29) \\
c_{4\pm} &= -3\left[8\{f_{\pm}^{n}(i \mp 1) + 12 f_{\pm}^{n}(i)\}\Delta x \right. \\
&\quad \left. \pm \{g_{\pm}^{n}(i \mp 1) - 3g_{\pm}^{n}(i)\}(\Delta x)^2 \mp 20 s_{\pm}^{n}(i \mp 1/2)\right] / \{2(\Delta x)^3\} \qquad (30) \\
d_{4\pm} &= g_{\pm}^{n}(i) \qquad (31)
\end{aligned}
$$

The field values, their derivatives and integrated value at the next time step $n+1$ are expressed respectively as

$$
\begin{aligned}
F^n_{i\pm}(x_i+\xi) &= a_{4\pm}\xi^4 + b_{4\pm}\xi^3 + c_{4\pm}\xi^2 + g^n_\pm(i)\xi + f^n_\pm(i) && (32)\\
G^n_{i\pm}(x_i+\xi) &= 4a_{4\pm}\xi^3 + 3b_{4\pm}\xi^2 + 2c_{4\pm}\xi + g^n_\pm(i) && (33)\\
s^{n+1}_\pm(i\mp 1/2) &= s^n_\pm(i\mp 1/2) + \Delta s^n_\pm(i\mp 1) - \Delta s^n_\pm(i) && (34)
\end{aligned}
$$

where

$$\Delta s^n_\pm(i) = \int_{x_{i+\xi}}^{x_i} F^n_{i\pm}(x)dx = -\frac{a_{4\pm}}{5}\xi^5 - \frac{b_{4\pm}}{4}\xi^4 - \frac{c_{4\pm}}{3}\xi^3 - \frac{g^n_\pm(i)}{2}\xi^2 - f^n_\pm(i)\xi \tag{35}$$

The non-advection term for the integrated value is given as

$$\frac{\partial s}{\partial t} = \int h^*_\pm dx \tag{36}$$

The time development equation for the integration is given as

$$
\begin{aligned}
s^{n+1}_\pm(i) &= s^*_\pm(i) + \Delta t\int_{x_i\mp 1}^{x_i} h^*_\pm dx \\
&= s^*_\pm(i) + \frac{\delta_1\Delta t}{2\rho}\{g^*_+(i) + g^*_-(i) - g^*_+(i\mp 1) - g^*_-(i\mp 1)\} \\
&\quad + \frac{\delta_2 c_0\Delta t}{2Z}\{g^*_+(i) - g^*_-(i) - g^*_+(i\mp 1) + g^*_-(i\mp 1)\} && (37)
\end{aligned}
$$

4. NUMERICAL EXPERIMENTS

To verify the validity of the present schemes, some numerical examinations are made for the nonlinear sound propagation in air. An acoustic pipe of 34m in length is considered for the one-dimensional model. Number of grid points is 20000 and the grid spacing is $\Delta x = 1.7$mm. The time step Δt is chosen to be 2.5μS where CFL number is 0.5. The sound speed c_0 is 340m/s, the medium density ρ_0 is 1.2kg/m^3 and the nonlinearity parameter β is 1.2. A single-shot pulse of sinusoidal whose amplitude and frequency are 400Pa and 5kHz is given around $x = 1$m as the initial pressure at $t = 0$.

The linear wave propagation is first demonstrated. Figure 2 shows the pressure distribution as the wave propagates until the time $2t_s$ where

$$t_s = \frac{\rho_0 c_0^2}{\beta\omega p_0} \tag{38}$$

is the shock formation time. In the figure, the thin red line indicates the CIP solution, the bold line indicates the CIP-CSL4 solution and the dashed line indicates FDTD solutions whose calculation conditions are same with CIP method. Both CIP and CIP-CSL4 solutions well agree with the exact solution, although the numerical dispersion error appears in the FDTD solution.

Figure 3 shows the numerical results of nonlinear wave propagation. In the figure, the broken line indicates the analytical solution calculated by the Cole-Hopf transformation of Burgers'

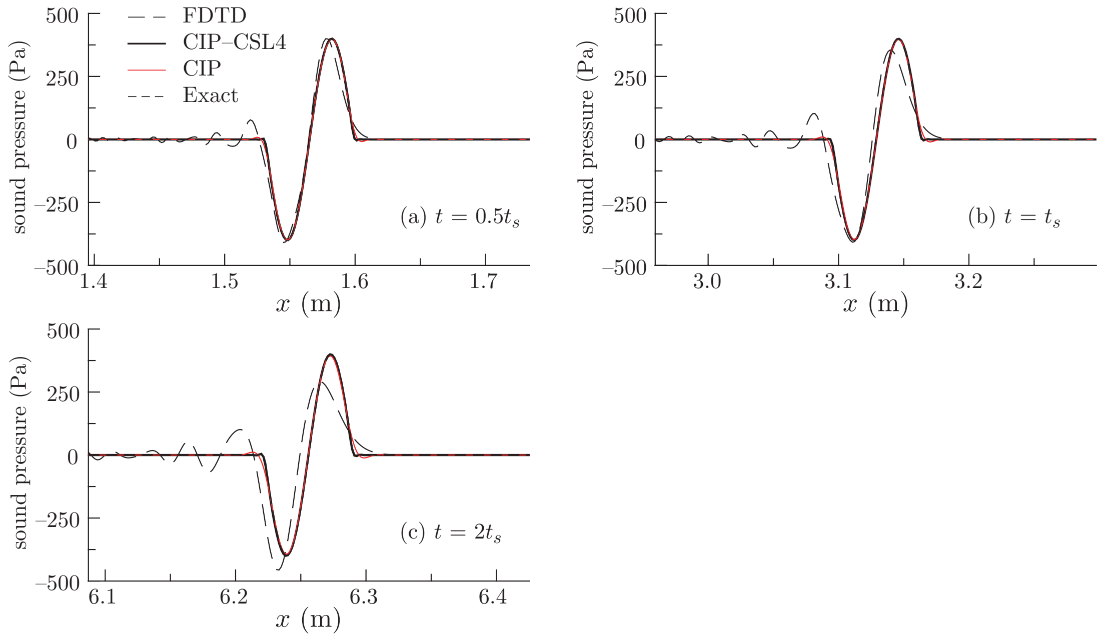

Figure 2. Sound pressure distribution of linear wave.

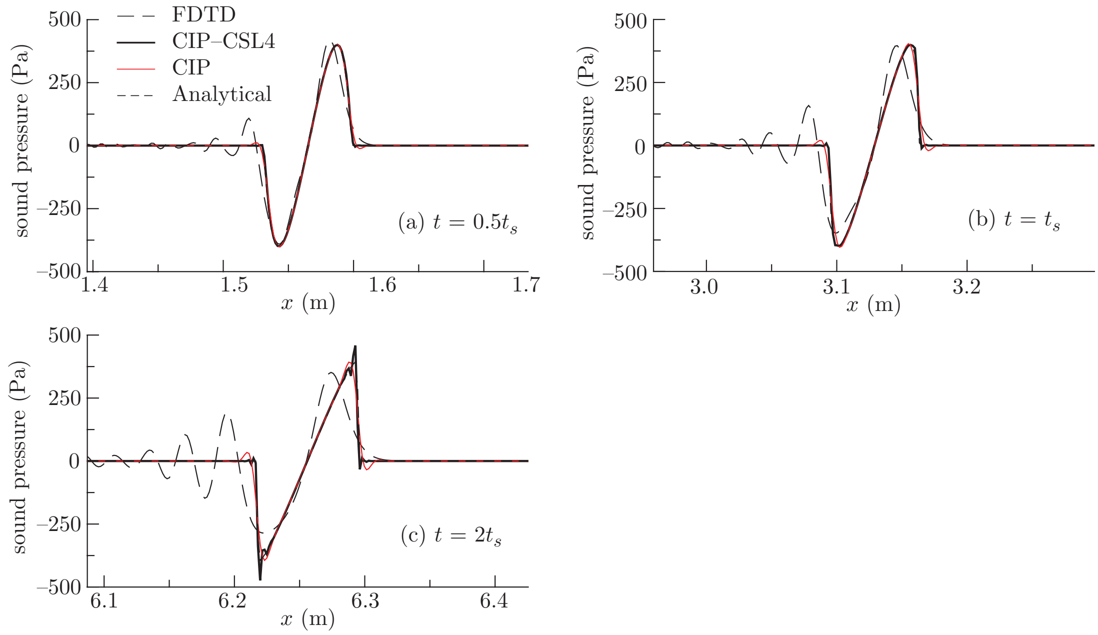

Figure 3. Sound pressure distribution of nonlinear wave.

equation[5]. The results calculated by CIP method and CIP-CSL4 method well agree with the analytical solution until $t = t_s$. As the nonlinear distortion becomes strong, the waveforms calculated by FDTD method however collapse because of the numerical dispersion. Although the shock formation can not be confirmed in the FDTD solutions, the shock fronts are clearly calculated in the CIP and CIP-CSL4 solutions. As the wave propagates, the attenuation becomes little larger in the CIP solution because of the numerical dissipation. Although the numerical dissipation hardly appears in the CIP-CSL4 solution, the overshoots appear in the shock fronts at $t = 2t_s$.

Figure 4 shows the numerical results of nonlinear wave propagation calculated by the

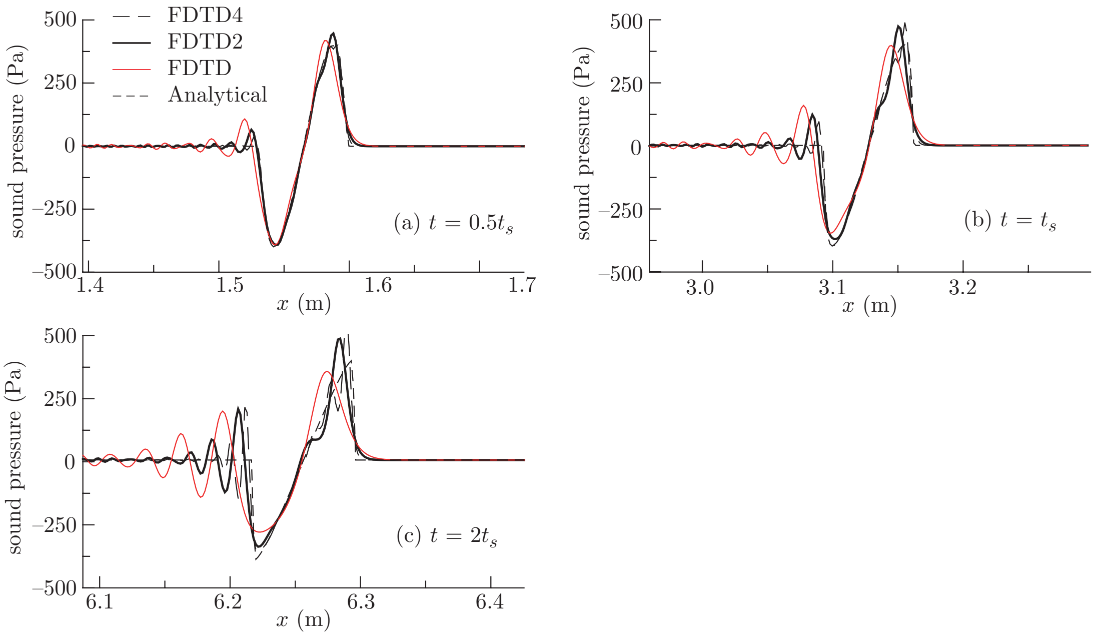

Figure 4. Sound pressure distribution of nonlinear wave calculated by FDTD method.

FDTD method using the finer mesh size. In the figure, the bold line indicates the FDTD solution whose grid points are twice as much as Figure 3 and the dashed line indicates 4 times as much. It is found that the numerical dispersion doesn't disappear even if the grid points increases.

5. CONCLUSIONS

We have examined the numerical simulation of nonlinear sound wave propagation in time domain by using CIP method and CIP-CSL4 method under the assumption of weak shock. Some numerical demonstrations are made for the one-dimensional nonlinear sound propagation in air. It is verified that the shock front is clearly calculated using CIP method or CIP-CSL4 method, because the numerical dispersion is hardly caused in these methods.

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