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MODAL PROBABILISTIC ANALYSIS : THEORY AND APPLICATION

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Abstract

The randomness of the parameters such as Young's modulus or the loss factor of a mechanical system can have considerable influence upon its response. The present study aims at providing an analytical method to evaluate the impact of this randomness. We focus on the frequency response of a discrete vibrating system with separated modes and random damping. The probabilistic method is based on normal modes with the Finite Element Method. This new method will be called the modal probabilistic analysis. This theory leads to the definition of an envelope of the response. Then these envelopes are used to characterize the law of probability that is governing the damping : this identification method is illustrated using a vibrating system composed of an Euler-Bernoulli beam.

1. INTRODUCTION

This paper presents a parametrical method for the study of the influence of a random damping, upon the response of a multi degrees of freedom vibrating system. In the literature about random mechanics [1], people always try to adapt deterministic approaches to the resolution of systems with uncertain parameters. For example in the non-parametrical approach like in [2, 3, 4, 5], the assumptions upon the deterministic system condition the randomness of the matrices involved in the stochastic problem. That's all the more true for non-intrusive method like Monte-Carlo simulations [6, 7], where the stochastic behavior of the system is evaluated without changing anything in the deterministic one. There exist a lot of other methods together with their derivatives [8]. The approach in the present paper has the same confidence is the deterministic model, but proposes a direct analytical resolution.
The paper is organized as follows. Section 1 presents quickly the probabilistic description used in the following. Section 2 describes the general method based on normal modes. Then in Section 3 the analytical PDF and the expression of the first moments are derived in the case of a single mode. The definition of envelopes is extended to a vibrating system with n degrees of freedom in Section 4. An identification procedure to obtain probabilistic parameters from

simulated experimental frequency response data is proposed in Section 5.

2. MODAL PROBABILISTIC ANALYSIS

2.1. Description of the mechanical problem

Consider a discrete structure consisting of n degrees of freedom, arising for example from a finite element discretization. Using Rayleigh's approach [9] for homogeneous structures, the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ can be written $\mathbf{M} = m\,\mathbf{M}_d$ and $\mathbf{K} = k\,\mathbf{K}_d$, where m and k are scalars and $\mathbf{M}_d$ and $\mathbf{K}_d$ are matrices only depending on the discretization of the structure. Let η be the loss factor and $\omega_p^2 = \dfrac{k}{m}$, it comes :

$$\left(-\omega^2\,\mathbf{M}_d + \omega_p^2(1 + \mathtt{i}\,\eta)\,\mathbf{K}_d \right)\,\mathbf{u} = \frac{\hat{\mathbf{f}}}{m} \tag{1}$$

Assuming that $\mathbf{K}_d$ is a definite positive matrix (this can be performed after imposing boundary conditions), it allows the co-diagonalization of $\mathbf{M}_d$ and $\mathbf{K}_d$. Let $\left(\lambda_j\,,\,\mathbf{p}_j\right)_{j\leq n}$ be the solution of the eigenvalues problem $\mathbf{M}_d\,\mathbf{p}_j = \lambda_j\,\mathbf{K}_d\,\mathbf{p}_j$. With $\mathbf{P} = \mathrm{Vect}(\mathbf{p}_j)$, one has $\mathbf{P}^{-1}\mathbf{M}_d\mathbf{P} = \mathbf{I}_n$ and $\mathbf{p}_j^{\mathrm{T}}\mathbf{K}_d\mathbf{p}_j = \lambda_j \quad \forall j \leq n$. As η and ω_p^2 are scalars, it is possible to write :

$$\begin{pmatrix} \ddots & & \cdots\ 0\ \cdots & & \cdots\ 0 \\ 0 & \cdots & -\omega^2 + \omega_p^2\,(1+\mathtt{i}\,\eta)\,\lambda_j & & \cdots\ 0 \\ 0 & \cdots & \cdots\ 0\ \cdots & & \ddots \end{pmatrix} \mathbf{U} = \mathbf{F}$$

where $\mathbf{U} = \mathbf{P}^{-1}\mathbf{u}$ et $\mathbf{F} = \mathbf{P}^{-1}\dfrac{\hat{\mathbf{f}}}{m}$. $\mathbf{U}$ and $\mathbf{F}$ are the generalized displacements and excitation vectors respectively. Then $\mathbf{U} = \mathbf{H}\,\mathbf{F}$ where $\mathbf{H}$ is the transfer function matrix, which is diagonal of size $n \times n$.

Now consider that η is a random variable characterized by an absolute continuous law with PDF (Probability Density Funstion) f_η. The aim of what follows is to study the effect of the randomness of η upon transfer function $\mathbf{H}$.

2.2. Probabilistic description

The fundamental assumption is that the modes are well separated. The system behaves like the superposition of n elementary systems. This assumption is already made in the deterministic resolution with normal modes and it is also assumed in the probabilistic problem. The resolution of the probabilistic problem already exists for a single degree-of-freedom oscillator with random damping [10], this is why a direct extension is possible thanks to the separation assumption. As $\mathbf{H}$ is a complex quantity, it has been decided to study respectively the real part Re($\mathbf{H}$) and the imaginary part c, which are considered as functions of η. In the present case, the calculation considers ω_p^2 as a constant. So there is a single uncertain parameter, which is the hysteretic damping η.

In the present case, damping is considered as a random variable whose probability law is uniform. This choice allows one to control easily the value of the parameter so that the physical sense is preserved. Furthermore, the separation of eigenfrequencies of normal modes will be

preserved for damped ones.
It is possible to use another law of probability, but on one side the analytical aspect is lost very quickly and on an other side the choice is always arbitrary because of the lack of knowledge about the PDF of parameters. The identification method which is proposed here relies on the hypothesis of a uniform law because analytical results can then be used. The 100%-envelopes are the basis of the identification method.

2.3. PDF and mean value

It is possible to calculate analytically the PDF of Re(**H**) and Im(**H**) on each mode j, denoted by $f_{\mathrm{Re}(\mathbf{H}_{jj})}$ and $f_{\mathrm{Im}(\mathbf{H}_{jj})}$ directly by supposing that the PDF of η (denoted f_η) is a uniform law (with mean value $\bar{\eta}$ and standard deviation σ_η). This allows one for calculating all moments of Re(**H**) and Im(**H**), especially the first one. The expressions of the mean values for the j^{th} mode are given thereafter.
The analytical expression for $M_1^{\mathrm{Re},j}$ is :

$$M_1^{\mathrm{Re},j} = \frac{\mathrm{sign}(A)}{2\sqrt{3}\omega_p^2\lambda_j\sigma_\eta}\left(\arcsin\left(\frac{-A^2 + \lambda_j^2\omega_p^4Y_-^2}{A^2 + \lambda_j^2\omega_p^4Y_-^2}\right) - \arcsin\left(\frac{-A^2 + \lambda_j^2\omega_p^4Y_+^2}{A^2 + \lambda_j^2\omega_p^4Y_+^2}\right)\right)$$

The analytical expression for $M_1^{\mathrm{Im},j}$ is :

$$M_1^{\mathrm{im},j} = \frac{1}{4\sqrt{3}\omega_p^2\lambda_j\sigma_\eta}\ln\left(\frac{-A^2 + \lambda_j^2\omega_p^4Y_+^2}{-A^2 + \lambda_j^2\omega_p^4Y_-^2}\right)$$

with $A = -\omega^2 + \omega_p^2\lambda_j$ and $Y_- = \bar{\eta} - \sigma_\eta\sqrt{3}$ and $Y_+ = \bar{\eta} + \sigma_\eta\sqrt{3}$.
Then by using the assumption of well separated modes (fixed to N), it is possible to derive the mean value of the whole transfer function :

$$M_1^{\mathrm{Re}} = \sum_{j=1}^{N} M_1^{\mathrm{Re},j} \quad \text{and} \quad M_1^{\mathrm{Im}} = \sum_{j=1}^{N} M_1^{\mathrm{Im},j}$$

2.4. Envelopes definition

The following definition is given for $\mathrm{Re}\,(\mathbf{H}_{jj})$ but it is easy to extend this definition by a similar way to $\mathrm{Im}\,(\mathbf{H}_{jj})$.
Let's define $s_1^{\mathrm{Re},\lambda_j}$ and $s_2^{\mathrm{Re},\lambda_j}$ by $\int_{s_1^{\mathrm{Re},\lambda_j}}^{s_2^{\mathrm{Re},\lambda_j}} f_{\mathrm{Re}(\mathbf{H}_{jj})}(s)\,ds = 1$. Outside $[s_1^{\mathrm{Re},\lambda_j}, s_2^{\mathrm{Re},\lambda_j}]$, $f_{\mathrm{Re}(\mathbf{H}_{jj})}$ is zero. Consider α_1 and α_2 two probabilities and the set $\{\beta_1^{\mathrm{Re},\lambda_j},\, \beta_2^{\mathrm{Re},\lambda_j}\}$ satisfying :

$$\left\{\begin{array}{rcl} \displaystyle\int_{s_1^{\mathrm{Re},\lambda_j}}^{s_1^{\mathrm{Re},\lambda_j}+\beta_1^{\mathrm{Re},\lambda_j}} f_{\mathrm{Re}(\mathbf{H}_{jj})}(s)\,ds & = & \alpha_1 \\ \displaystyle\int_{s_2^{\mathrm{Re},\lambda_j}-\beta_2^{\mathrm{Re},\lambda_j}}^{s_2^{\mathrm{Re},\lambda_j}} f_{\mathrm{Re}(\mathbf{H}_{jj})}(s)\,ds & = & \alpha_2 \\ \alpha_1 + \alpha_2 & = & \alpha \end{array}\right.$$

involving positive values $\beta_1^{\mathrm{Re},\lambda_j}$ and $\beta_2^{\mathrm{Re},\lambda_j}$. The $(1-\alpha)$%-envelope for Re $(\mathbf{H}_{jj})$ is denoted by $\Upsilon^{\alpha}_{\mathrm{Re},\lambda_j}$ and is defined as the complementary of the union of intervals :

$$\Upsilon^{\alpha}_{\mathrm{Re},\lambda_j} = \overline{\bigcup_{\omega} \left[s_1^{\mathrm{Re},\lambda_j} + \beta_1^{\mathrm{Re},\lambda_j},\ s_2^{\mathrm{Re},\lambda_j} - \beta_2^{\mathrm{Re},\lambda_j} \right]}$$

3. APPLICATION TO A EULER-BERNOULLI CANTIVELER BEAM

3.1. Description

The Euler-Bernoulli approximation assumes that the length of each beam section is much greater than the height of each section and the shear modulus and rotary inertia effects are ignored. The governing equation of the flexural vibration of a uniform Euler-Bernoulli cantilever beam of length L vibrating sinusoidally at circular frequency ω and excited by a force F is given by :

$$-\,\omega^2\,\hat{v}(x,\omega) + \frac{E(1+\mathtt{i}\eta)I}{\rho S}\,\frac{\partial^4 \hat{v}(x,\omega)}{\partial x^4} = \frac{\hat{F}}{\rho S} \tag{2}$$

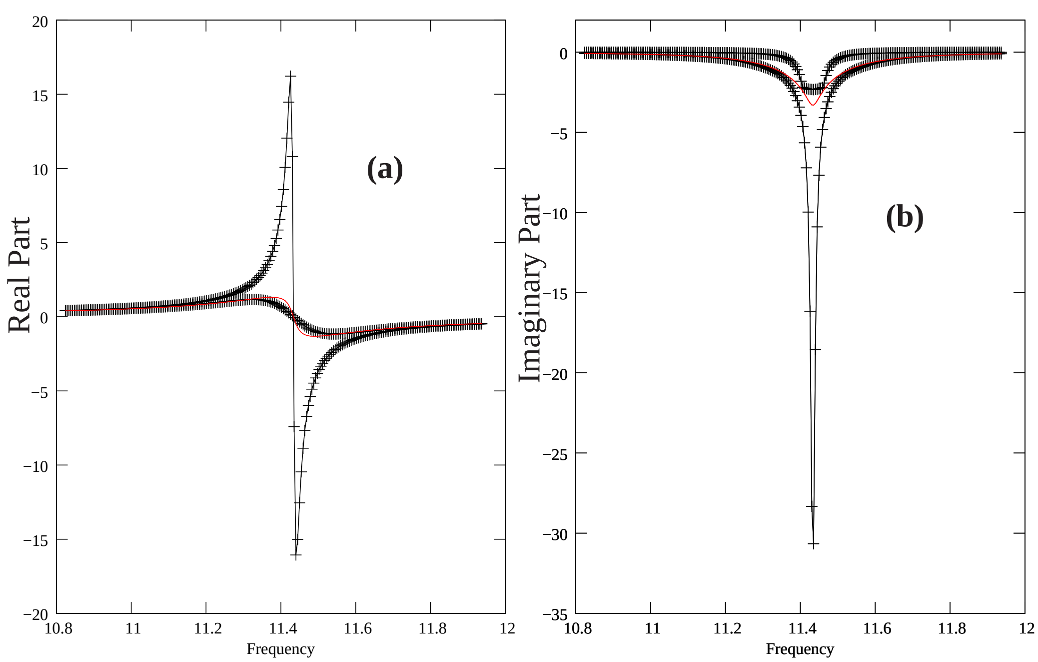

Figure 1. Mean value (-) and envelope (+) of Re $(\mathbf{H})$ (a) and Im $(\mathbf{H})$ (b) for the response of system composed by a clamped-free beam arround the first mode. $h = 5mm$, $\bar{\eta} = 0.01$, $\sigma_\eta = 50\%\,\bar{\eta}$, $\rho = 2700 kg/m^3$, $E = 70MPa$.

Posing $\omega_p^2 = \dfrac{EI}{\rho S}$ and applying the Finite Element Method to the beam, equation (2) returns to equation (1). So results in Section 2 can be applied. The eigenvalues problem is solved with the very efficient QZ-algorithm [11].

In order to illustrate the results on the cantilever beam, the mean value together with 100%-envelopes around the first mode are diplayed on figure 1. Two observations can be made. The first one, is that the mean value lies into the envelope, but it is not centered. This comes from the fact that the PDF is not centered (nor reduced). What is very important to notice is that this analytical approach shows that at a resonance frequency with $\sigma_\eta = 50\%\bar{\eta}$, the response can be 5 times bigger that the mean value. This shows clearly the impact of the randomness on the response and the need to identify correctly the parameters of dispersion (here σ_η for an uniform law). The second remark is that the envelope of $\mathrm{Im}\left(\mathbf{H}\right)$ isn't a "smooth" function of ω. This comes from the fact that it is not injective regarding η. This is an important point that can be noticed on the figure and which has guided one of the identification method that is proposed next.

4. IDENTIFICATION

4.1. Presentation of the technique

Assuming that the parameter of interest is a specific damping η governed by a uniform law, the issue is to identify σ_η and $\bar{\eta}$ from experimental data, which are $s_1^{\mathrm{Re},\lambda_j}$, $s_2^{\mathrm{Re},\lambda_j}$, $s_1^{\mathrm{Im},\lambda_j}$ and $s_2^{\mathrm{Im},\lambda_j}$, which are the 100%-envelopes of $\mathrm{Re}\left(\mathbf{H}\right)$ and $\mathrm{Im}\left(\mathbf{H}\right)$.
Letting $\eta_+ = \bar{\eta} + \sqrt{3}\sigma_\eta$ and $\eta_- = \bar{\eta} - \sqrt{3}\sigma_\eta$, equations of 100%-envelope for $\mathrm{Re}\left(\mathbf{H}\right)$ and $\mathrm{Im}\left(\mathbf{H}\right)$ lead to the following system :

$$(4.1)\quad \begin{cases} \Upsilon^0_{\mathrm{Re},\lambda_j}(\eta_+,\omega) & = & s_1^{\mathrm{Re},\lambda_j} \text{ or } s_2^{\mathrm{Re},\lambda_j} \\ \Upsilon^0_{\mathrm{Re},\lambda_j}(\eta_-,\omega) & = & s_2^{\mathrm{Re},\lambda_j} \text{ or } s_1^{\mathrm{Re},\lambda_j} \\ \Upsilon^0_{\mathrm{Im},\lambda_j}(\eta_+,\omega) & = & s_1^{\mathrm{Im},\lambda_j} \text{ or } s_2^{\mathrm{Im},\lambda_j} \\ \Upsilon^0_{\mathrm{Im},\lambda_j}(\eta_-,\omega) & = & s_2^{\mathrm{Im},\lambda_j} \text{ or } s_1^{\mathrm{Im},\lambda_j} \end{cases}$$

There are 6 domains of frequency to distinguish, so there are actually 6 differents system (4.1) to solve for a fixed ω. These domains are determined by the resonance frequency ω_r of the j^{th} mode and the inflexions of the 100%-envelope of $\mathrm{Re}\left(\mathbf{H}\right)$, so there are 5 frequencies to evaluate from the experimental values, which are noted gradually ω_1, ω_2, ω_r, ω_3 and ω_4 (with I_1, I_2, I_3, I_4, I_5 and I_6 the corresponding intervals). The situation is explained on figure 2.

4.2. Deterministic parameters identification

Before the identification of $\bar{\eta}$ and σ_η, it is natural to find the deterministic parameters, that is the modal coefficient B_j and the resonance frequency ω_r. It is important to notice that the system has no solution for $\omega \in I_2$ and I_5. The resonance frequency ω_r can be determined by usual techniques on $(s_1^{\mathrm{Re},\lambda_j}, s_1^{\mathrm{Im},\lambda_j})$ or $(s_2^{\mathrm{Re},\lambda_j}, s_2^{\mathrm{Im},\lambda_j})$ like the bandwidth method or by observing the change of phase. Note that the deterministic model is based here on hysteretic damping, so damping doesn't change eigenfrequencies. In the present case only hysterical damping is supposed to be random, so ω_r isn't random. That's why deterministic methods can be applied in order to find the resonance frequency. It is interesting to use also the envelopes to determine the modal contribution B_j with system (4.1), but in I_2 and I_5 there is no solution for the system.

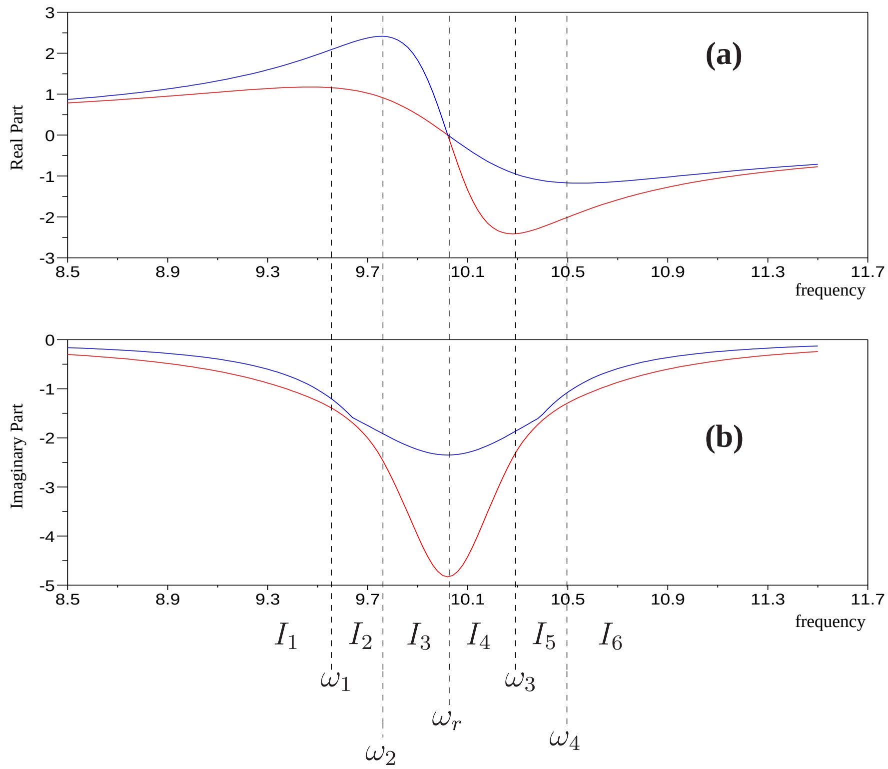

Figure 2. Frequency interval used to distinguish on $\mathrm{Re}\left(\mathbf{H}\right)$ (a) and $\mathrm{Im}\left(\mathbf{H}\right)$ (b) in the global identification method.

$$\left\{\begin{array}{lllr} B_j & = & A\dfrac{\left(s_2^{\mathrm{Im},\lambda_j}\right)^2+\left(s_2^{\mathrm{Re},\lambda_j}\right)^2}{s_2^{\mathrm{Re},\lambda_j}} & \quad \omega\in I_1,\, I_4 \\ B_j & = & A\dfrac{\left(s_1^{\mathrm{Im},\lambda_j}\right)^2+\left(s_2^{\mathrm{Re},\lambda_j}\right)^2}{s_2^{\mathrm{Re},\lambda_j}} & \quad \omega\in I_3,\, I_6 \end{array}\right.$$

with $A = -\omega^2 + \omega_r^2$.

4.3. Identification methods

In what follows, two identification methods are presented. They are based on the assumption of separated modes, this is why a isolated mode is selected. The other important assumption is the uniform law governing the loss factor η. It is in fact the identification of the support of the distribution.

4.3.1. The specific method

This method is based on the evaluation of ω_2 and $s_2^{\mathrm{Im},\lambda_j}(\omega_r) = t_{2,r}$. It is called a specific method because only theses two values together with ω_r and B_j allow for the calculation of $\bar{\eta}$ and σ_η.

Here are the formulas coming from the evaluation at ω_r of the upper envelope of $\mathrm{Im}\,(\mathbf{H})$.

$$\begin{aligned}\bar{\eta} &= \frac{t_{2,\,r}\,(\omega_r^2 - \omega_2^2)\ -\ B_j}{2\,t_{2,\,r}\,\omega_r^2} \\ \sigma_\eta &= -\frac{\sqrt{3}}{6}\,\frac{t_{2,\,r}\,(\omega_r^2 - \omega_2^2)\ +\ B_j}{t_{2,\,r}\,\omega_r^2}\end{aligned}$$

4.3.2. The global method

The issue here is to understand how the 100%-envelopes of $\mathrm{Im}\,(\mathbf{H})\,(\omega)$ behave regarding η_- and η_+. Over intervals I_1 and I_6, the upper envelope of $\mathrm{Im}\,(\mathbf{H})$ corresponds to η_+ and the lower one to η_-. Over intervals I_3 and I_4, it is the contrary : the upper envelope of $\mathrm{Im}\,(\mathbf{H})$ corresponds to η_- and the lower one to η_+. Over intervals I_2 and I_5, the difficulty is that the lower envelope of $\mathrm{Im}\,(\mathbf{H})$ doesn't correspond to η_+ neither to η_-, but always to the value $-1/2|A|$. This is why (4.1) has no solution over I_2 and I_5.
So the system (4.1) is solved separately over intervals I_1 and I_6 then over I_3 and I_4. In order to simplify notations, $x_1 = s_1^{\mathrm{Re},\lambda_j}$, $t_1 = s_1^{\mathrm{Im},\lambda_j}$ and $x_2 = s_2^{\mathrm{Re},\lambda_j}$, $t_2 = s_2^{\mathrm{Im},\lambda_j}$. Posing :

$$Q = \sqrt{B_j^2(t_1^2 + t_2^2) - 8t_1^2A^2t_2^2 + 2t_2t_1\sqrt{B_j^4 + 16t_1^2A^4t_2^2 - 4B_j^2A^2(t_1^2 + t_2^2)}}$$

The following formulas give $\bar{\eta}$ and σ_η :

Over I_1 and I_6 : $\left\{\begin{aligned}\bar{\eta} &= -\frac{4B_j(t_1+t_2)+4\,Q}{16\,t_2t_1\omega_r^2} \\ \sigma_\eta &= \bar{\eta}\,\frac{B_j\sqrt{3}(t_2-t_1)}{Q}\end{aligned}\right.$ and over I_3 and I_4 : $\left\{\begin{aligned}\bar{\eta} &= -\frac{4B_j(t_1+t_2)-4\,Q}{16\,t_2t_1\omega_r^2} \\ \sigma_\eta &= \bar{\eta}\,\frac{B_j\sqrt{3}(t_2-t_1)}{Q}\end{aligned}\right.$

Depending on the frequency, there are several calculations to carry out in each interval, except in I_2 and I_5.

4.3.3. Comments about identification methods

In both identification methods, it has been considered that the 100%-envelope is an input datum. A uniform law for hysteretical damping η has been supposed, which enables one to identify the support of the PDF, ie $\bar{\eta}$ and σ_η. Note that the same method can be used with a different law for the damping as soon as the direct analytical problem has been solved.
The specific method is easier to apply, but only relies on three values, whereas the global method provides a set of values that can be compared in order to eliminate the chaotic ones. This point leads to the major problem of these resolutions, which is the determination of the "experimental" 100%-envelope. Statistic tests would allow experimentators to decide when the "experimental" 100%-envelope is known. The question of the number of measurements which are necessary to reach the envelope is quite difficult. It refers to the avalaible information and for vibro-acoustic measurements, the following remarks apply. Usually, whatever measurement at a point of a experimental set is the mean value of a series of measurements. It would be more interesting to keep all the measured values than only the mean value, because the dispersion aspect is lost.

5. CONCLUSION

The purpose of the present work was to illustrate the modal probabilistic analysis. The random flexural response of a cantilever Euler-Bernouilli beam with random hysteretic damping has been calculated using FEM and modal expansion. Firstly, the first moments of the real and imaginary parts of the transfer function together with its envelopes have been calculated in the case of a uniform damping PDF. This approach is very efficient numerically since analytical expressions are available to determine the parameters associated to the response probability law.

Secondly, an identification procedure based on the knowledge of the 100%-enveloppe has been proposed. It allows for the determination of the damping PDF parameters. The greatest difficulty is the experimental determination of the 100%-envelope experimentally and this refers to the problem of the available information.

It must be pointed out that theses formulas are a first step in the identification from experimental data, and can not be exploited directly at this stage. In reality, it is very important to consider that the resonance frequency is random too. In the present model, it means that the eigenvalue λ_j and the stiffness k are also random parameters.

The future prospects of the present work are to proceed with an experimental identification. In order to reach relevant results, it is very important to solve the problem with two parameters, which are damping and the resonance frequency. The resonance frequency is more appropriate than stiffness because this is a physical value than can be directly determined experimentaly.

The extension of the approach to two parameters and n DOF sytems would provide a tool to solve many physical problems with random parameters. The modal probabilistic analysis also needs to deal with non well separated modes.

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