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VIBRATION ANALYSIS OF VIBRATORY GYROS

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Abstract

The purpose of this paper is to investigate the vibration of a vibratory gyroscope with imperfections. A linear model due to imperfections of material or manufacturing tolerance of a vibratory gyroscope is established. The operations of vibratory gyroscopes without or with imperfections are described under free vibration. Effects of the imperfections in terms of damping, gyroscopic, stiffness and circulation in the governing equations are analysed with multiple time scale method. Effects of the resulting angular frequency variation only and anisoelasticity are investigated via the variation of the elliptical orbit of a reference point on the element relative to a coordinate system fixed on the gyro.

1. INTRODUCTION

Vibration gyroscope based on the modal vibration pattern of the ring or the hemispherical shell moves when the shell is rotated about its axis and this movement provides a measure of the applied rate of turn. Hemispherical resonator gyroscope excited with electrostatic field is not only expensive and hard to fabricate. We are developing a new type of hemispherical resonator gyroscope which is excited with discrete piezoceramic actuation and sensing elements bonded on the outer surface and close to the rim of the shell. A linear error model due to imperfections of material or manufacturing tolerance of the shell in terms of damping, gyroscopic, stiffness and circulatory was established [1]. The behaviour of the resulting angular frequency varying only and anisoelasticity are investigated via the variation of the elliptical orbit of a reference point on the element relative to a coordinate system fixed on the gyro.

2. IDEAL VIBRATORY GYROSCOPE

Governing equations for a vibratory gyroscope can be written in a general form as

$$\ddot{x}+\omega^2 x = 2G\Omega\dot{y}, \qquad \ddot{y}+\omega^2 y = -2G\Omega\dot{x}. \tag{1}$$

where x and y represent a set of orthogonal general coordinates of a point on the longitudinal axis of the vibrating member measured in a plane fixed in the gyro and normal to the axis, ω is the radial frequency of the vibratory gyroscope, $\Omega(<<\omega)$ is the rotating speed of the gyro, and G is the sensing coefficient. Equation (3) is normalized with

$$\underline{t} \equiv \omega t,\ \frac{d^2}{\omega^2 dt^2} = \frac{d^2}{d\underline{t}^2},\ \underline{\Omega} = \frac{\Omega}{\omega} = \varepsilon\Omega^*,\ x(t) = X(\underline{t}),\ y(t) = Y(\underline{t}) \tag{2}$$

to obtain

$$\ddot{X} + X = 2\varepsilon G\Omega^*\dot{Y}, \qquad \ddot{Y} + Y = -2\varepsilon G\Omega^*\dot{X}, \tag{3}$$

where ε is a small parameter in perturbation method and ω=1.

2.1 Elliptical orbit

For non-rotating case; ω=0, the general solution of Eq. (3) is given by

$$\begin{aligned} X(\underline{t}) &= A\cos\varphi\cos(\underline{t}+\vartheta) - B\sin\varphi\sin(\underline{t}+\vartheta),\\ Y(\underline{t}) &= A\sin\varphi\cos(\underline{t}+\vartheta) + B\cos\varphi\sin(\underline{t}+\vartheta). \end{aligned} \tag{4}$$

The motion of the point (X,Y) is an ellipse in the X-Y plane, as shown in Figure 2. Parameters *A*, *B*, φ, ϑ are constants that depend on the initial conditions. Parameters *A*, *B*, φ define the shape and orientation of the ellipse, and ϑ the orbital angle.

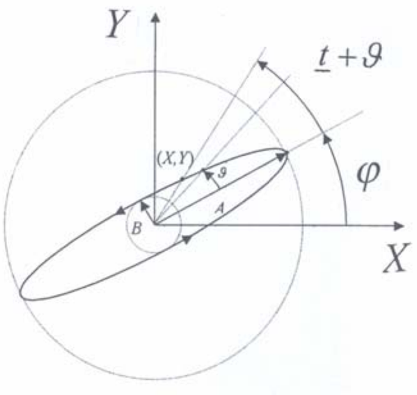


Figure 2 Elliptical orbit

The normalized general energy E_{NG} and angular momentum are A_{NG} given by

$$E_{NG} = \frac{1}{2}(X^2 + Y^2 + \dot{X}^2 + \dot{Y}^2), \qquad A_{NG} = X\dot{Y} - \dot{X}Y. \tag{5}$$

and

$$E_{NG} = \frac{1}{2}(A^2 + B^2), \qquad A_{NG} = AB.$$

The motion of the point (X,Y) is a straight line(*B*=*0*) when the angular momentum is zero.

2.2 Angular velocity sensing

For non-zero angular velocity, a perturbation technique based on the method of two time scales is used to solve the governing equation [3]. The vibration solution of an ideal vibratory gyro is

$$X = 2(r_1\cos(\varepsilon G\Omega^*\underline{t}) + r_2\sin(\varepsilon G\Omega^*\underline{t}))\cos\underline{t} + 2(-r_3\cos(\varepsilon G\Omega^*\underline{t}) - r_4\sin(\varepsilon G\Omega^*\underline{t})) + \mathrm{O}(\varepsilon)\sin\underline{t},$$

$$Y = 2(r_2\cos(\varepsilon G\Omega^*\underline{t}) - r_1\sin(\varepsilon G\Omega^*\underline{t}))\cos\underline{t} + 2(-r_4\cos(\varepsilon G\Omega^*\underline{t}) + r_3\sin(\varepsilon G\Omega^*\underline{t})) + \mathrm{O}(\varepsilon)\sin\underline{t}.$$

In form of the elliptical orbit general solution we have (6)

$$X = X_C\cos T_0 + X_S\sin T_0, \qquad Y = Y_C\cos T_0 + Y_S\sin T_0. \tag{7}$$

$$\begin{aligned} X_C &= A\cos\varphi\cos\vartheta - B\sin\varphi\sin\vartheta, \quad X_S = -A\cos\varphi\sin\vartheta - B\sin\varphi\cos\vartheta,\\ Y_C &= A\sin\varphi\cos\vartheta + B\cos\varphi\sin\vartheta, \quad Y_S = -A\sin\varphi\sin\vartheta + B\cos\varphi\cos\vartheta. \end{aligned} \tag{8}$$

The behaviour of the orbit parameters is investigated by differentiating Eq. (8) to obtain

$$A' = X_C{}' \cos\varphi \cos\vartheta - X_S{}' \cos\varphi \sin\vartheta + Y_C{}' \sin\varphi \cos\vartheta - Y_S{}' \sin\varphi \sin\vartheta,$$

$$B' = -X_C{}' \sin\varphi \sin\vartheta - X_S{}' \sin\varphi \cos\vartheta + Y_C{}' \cos\varphi \sin\vartheta + Y_S{}' \cos\varphi \cos\vartheta,$$

$$\begin{aligned}\varphi' = \frac{1}{A^2 - B^2}[X_C{}'(-A\sin\varphi\cos\vartheta + B\cos\varphi\sin\vartheta_0) + X_S{}'(A\sin\varphi\sin\vartheta + B\cos\varphi\cos\vartheta) \\ + Y_C{}'(A\cos\varphi\cos\vartheta + B\sin\varphi\sin\vartheta) + Y_S{}'(-A\cos\varphi\sin\vartheta + B\sin\varphi\cos\vartheta)],\end{aligned} \tag{9}$$

$$\begin{aligned}\vartheta' = -\frac{1}{A^2 - B^2}[X_C{}'(A\cos\varphi\sin\vartheta - B\sin\varphi\cos\vartheta) + X_S{}'(A\cos\varphi\cos\vartheta + B\sin\varphi\sin\vartheta) \\ + Y_C{}'(A\sin\varphi\sin\vartheta + B\cos\varphi\cos\vartheta) + Y_S{}'(A\sin\varphi\cos\vartheta - B\cos\varphi\sin\vartheta)].\end{aligned}$$

The solution of the rotating gyro represent in elliptical orbit form, we have the amplitude terms

$$X_C = 2a_1 = 2(r_1\cos(G\Omega^*T_1) + r_2\sin(G\Omega^*T_1)), \quad X_S = -2b_1 = 2(-r_3\cos(G\Omega^*T_1) - r_4\sin(G\Omega^*T_1)),$$

$$Y_C = 2a_2 = 2(r_2\cos(G\Omega^*T_1) - r_1\sin(G\Omega^*T_1)), \quad Y_S = -2b_2 = 2(-r_4\cos(G\Omega^*T_1) + r_3\sin(G\Omega^*T_1)). \tag{10}$$

Equation (10) is differentiated with respect to T_1, we obtain

$$X_C{}' = G\Omega^* Y_C, \quad X_S{}' = G\Omega^* Y_S, \quad Y_C{}' = -G\Omega^* X_C, \quad Y_S{}' = -G\Omega^* X_S. \tag{11}$$

The change rate of the orbit is obtained as

$$A' = 0, \qquad B' = 0, \qquad \varphi' = -G\Omega^*, \qquad \vartheta' = 0. \tag{12}$$

The change rate of the φ is proportional to the rotating speed of the gyro and G is the sensing coefficient. The orbit of the point (X,Y) is no longer an ellipse but has a roseate appearance as shown in Figure 2.

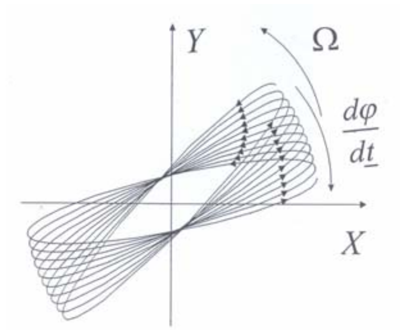


Figure 2 The precession of the orbit of a gyro with non-zero rotation

3. NONIDEAL VIBRATORY GYROSCOPE

A linear error model due to imperfections of materials or manufacturing tolerance of the gyro in terms of damping *c*, gyroscopic *g*, stiffness *k* and circulation *h* in the governing equations are written as

$$\begin{aligned}\ddot{X} + X - 2\varepsilon G\Omega^*\dot{Y} = -2\varepsilon(c_s\dot{X} - c_a\dot{X} + g_s\dot{Y} - g_a\dot{Y} + k_sX - k_aX + h_sY - h_aY), \\ \ddot{Y} + Y + 2\varepsilon G\Omega^*\dot{X} = -2\varepsilon(c_s\dot{Y} + c_a\dot{Y} + g_s\dot{X} + g_a\dot{X} + k_sY + k_aY + h_sX + h_aX).\end{aligned} \tag{13}$$

Errors are assumed relatively small and are treated through ε as small perturbations. Subscripts *c* and *s* are representing symmetrical and anti-symmetrical terms, respectively. Parameters c_s 、c_a 、g_s 、g_a 、k_s 、k_a 、h_s and h_a are assumed constants in time. Each term in the linear equation Eq. (13) can be considered separately and then superposed. Equations of error due to the symmetric part of damping c_s on non-rotating (Ω=0) gyro are

$$\ddot{X} + X = -2\varepsilon c_s\dot{X}, \quad \ddot{Y} + Y = -2\varepsilon c_s\dot{Y}. \tag{14}$$

$$A' = -Ac_s, \qquad B' = -Bc_s, \qquad \varphi' = 0, \qquad \vartheta' = 0. \tag{15}$$

Equations for error due to the anti-symmetric part of damping c_a, we have

$$\ddot{X} + X = 2\varepsilon c_a \dot{X}, \quad \ddot{Y} + Y = -2\varepsilon c_a \dot{Y}. \tag{16}$$

$$A' = c_a A\cos 2\varphi, \;\; B' = -c_a B\cos 2\varphi, \;\; \varphi' = -\frac{(A^2+B^2)\sin 2\varphi}{A^2-B^2}c_a, \;\; \vartheta' = \frac{2AB\sin 2\varphi}{A^2-B^2}c_a. \tag{17}$$

Similarly, we obtain orbit change rate due to errors of stiffness k_s and k_a , respectively,

$$A' = 0, \qquad B' = 0, \qquad \varphi' = 0, \qquad \vartheta' = k_s, \tag{18}$$

and $A' = k_a B\sin 2\varphi, \;\; B' = -k_a A\sin 2\varphi, \;\; \varphi' = \frac{2AB\cos 2\varphi}{A^2-B^2}k_a, \;\; \vartheta' = -\frac{(A^2+B^2)\cos 2\varphi}{A^2-B^2}k_a.$ (19)

Orbit change rate due to errors of gyroscopic g_s and g_a, respectively,

$$A' = -g_s A\sin 2\varphi, \;\; B' = g_s B\sin 2\varphi, \;\; \varphi' = -\frac{(A^2+B^2)\cos 2\varphi}{A^2-B^2}g_s, \;\; \vartheta' = \frac{2AB\cos 2\varphi}{A^2-B^2}g_s. \tag{20}$$

and $A' = 0, \qquad B' = 0, \qquad \varphi' = -g_a, \qquad \vartheta' = 0.$ (21)

Orbit change rate due to errors of circulatory h_s and h_a, respectively,

$$A' = h_s B\cos 2\varphi, \;\; B' = -h_s A\cos 2\varphi, \;\; \varphi' = -\frac{2AB\sin 2\varphi}{A^2-B^2}h_s, \;\; \vartheta' = \frac{(A^2+B^2)\sin 2\varphi}{A^2-B^2}h_s. \tag{22}$$

and $A' = -Bh_a, \qquad B' = -Ah_a, \qquad \varphi' = 0, \qquad \vartheta' = 0.$ (23)

We obtain

$$\begin{Bmatrix} A' \\ B' \\ \varphi' \\ \vartheta' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -g_a \\ k_s \end{Bmatrix} + \begin{bmatrix} -c_s & -h_a & 0 & 0 \\ -h_a & -c_s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ \varphi \\ \vartheta \end{Bmatrix}$$
$$+ \begin{Bmatrix} c_a A\cos 2\varphi - g_s A\sin 2\varphi + k_a B\sin 2\varphi + h_s B\cos 2\varphi \\ -c_a B\cos 2\varphi + g_s B\sin 2\varphi - k_a A\sin 2\varphi - h_s A\cos 2\varphi \\ -\frac{(A^2+B^2)\sin 2\varphi}{A^2-B^2}c_a - \frac{(A^2+B^2)\cos 2\varphi}{A^2-B^2}g_s + \frac{2AB\cos 2\varphi}{A^2-B^2}k_a - \frac{2AB\sin 2\varphi}{A^2-B^2}h_s \\ \frac{2AB\sin 2\varphi}{A^2-B^2}c_a + \frac{2AB\cos 2\varphi}{A^2-B^2}g_s - \frac{(A^2+B^2)\cos 2\varphi}{A^2-B^2}k_a + \frac{(A^2+B^2)\sin 2\varphi}{A^2-B^2}h_s \end{Bmatrix} \tag{24}$$

The constant terms in the RHS of Eq. (24); the gyroscopic anti-symmetric error g_a changes only φ but not A, B and the phase angle ϑ, the other constant term k_s changes the frequency only as shown in Figures 3a and 3b. For zero rate input (Ω=0), and without k_s and g_a， we have：

$$\begin{aligned} &\ddot{X} + X = -2\varepsilon(c_s \dot{X} - c_a \dot{X} + g_s \dot{Y} - k_a X + h_s Y - h_a Y), \\ &\ddot{Y} + Y = -2\varepsilon(c_s \dot{Y} + c_a \dot{Y} + g_s \dot{X} + k_a Y + h_s X + h_a X). \end{aligned} \tag{25}$$

A perturbation technique based on the methods of two time scales is used to solved above equations to obtain

$$\begin{aligned} X &= \exp[-(c_s - \zeta)T_1](X_{CN}\cos(T_0 - \kappa T_1) + X_{SN}\sin(T_0 - \kappa T_1)) \\ &\quad + \exp[-(c_s + \zeta)T_1](X_{CP}\cos(T_0 + \kappa T_1) + X_{SP}\sin(T_0 + \kappa T_1)), \\ Y &= \exp[-(c_s - \zeta)T_1](Y_{CN}\cos(T_0 - \kappa T_1) + Y_{SN}\sin(T_0 - \kappa T_1)) \\ &\quad + \exp[-(c_s + \zeta)T_1](Y_{CP}\cos(T_0 + \kappa T_1) + Y_{SP}\sin(T_0 + \kappa T_1)). \end{aligned} \tag{26}$$

The subscripts in Eq. (26) are: $_C$ for *cosine*，$_S$ for *sine*, $_N$ for $-\kappa$ ，$_P$ for $+\kappa$ ，and

$$\zeta=\sqrt{\frac{E_r}{2}+\frac{1}{2}\sqrt{E_i^{\,2}+E_r^{\,2}}}\geq 0,\qquad \kappa=\sqrt{-\frac{E_r}{2}+\frac{1}{2}\sqrt{E_i^{\,2}+E_r^{\,2}}}\geq 0.$$

$$E_r=c_a^{\,2}+g_s^{\,2}+h_a^{\,2}-h_s^{\,2}-k_a^{\,2},\qquad E_i=g_sh_s+c_ak_a. \tag{27}$$

$$X_{CP}=\frac{[g_s^{\,2}+(h_a+h_s)^2](c_1\gamma_2+c_2\gamma_1)+c_3\gamma_3-c_4\gamma_4}{(\zeta^2+\kappa^2)[g_s^{\,2}+(h_a+h_s)^2]},\ X_{SP}=\frac{[g_s^{\,2}+(h_a+h_s)^2](c_1\gamma_1-c_2\gamma_2)-c_3\gamma_4-c_4\gamma_3}{(\zeta^2+\kappa^2)[g_s^{\,2}+(h_a+h_s)^2]},$$

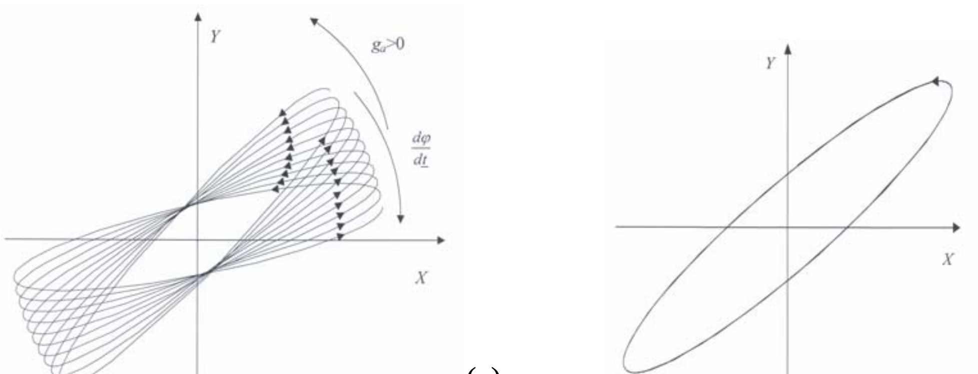
$$X_{CN}=\frac{[g_s^{\,2}+(h_a+h_s)^2](c_1\gamma_5-c_2\gamma_1)-c_3\gamma_3+c_4\gamma_4}{(\zeta^2+\kappa^2)[g_s^{\,2}+(h_a+h_s)^2]},\ X_{SN}=\frac{[g_s^{\,2}+(h_a+h_s)^2](-c_1\gamma_1-c_2\gamma_5)+c_3\gamma_4+c_4\gamma_3}{(\zeta^2+\kappa^2)[g_s^{\,2}+(h_a+h_s)^2]},$$

$$Y_{CP}=\frac{c_1\gamma_6-c_2\gamma_7+c_3\gamma_5-c_4\gamma_1}{(\zeta^2+\kappa^2)},\qquad Y_{SP}=\frac{c_1\gamma_7+c_2\gamma_6+c_3\gamma_1+c_4\gamma_5}{(\zeta^2+\kappa^2)},$$

$$Y_{CN}=\frac{-c_1\gamma_6+c_2\gamma_7+c_3\gamma_2+c_4\gamma_1}{(\zeta^2+\kappa^2)},\qquad Y_{SN}=\frac{c_1\gamma_7+c_2\gamma_6+c_3\gamma_1-c_4\gamma_2}{(\zeta^2+\kappa^2)}. \tag{28}$$

$$\begin{aligned}&\gamma_1=\kappa c_a-\zeta k_a,\qquad \gamma_2=\zeta^2+\kappa^2-\zeta c_a-\kappa k_a,\\&\gamma_3=(h_a+h_s)[\kappa(\zeta^2+\kappa^2+c_a^{\,2}-k_a^{\,2})-2\zeta c_ak_a]+g_s[\zeta(\zeta^2+\kappa^2-c_a^{\,2}+k_a^{\,2})-2\kappa c_ak_a],\\&\gamma_4=(h_a+h_s)[\zeta(\zeta^2+\kappa^2-c_a^{\,2}+k_a^{\,2})-2\kappa c_ak_a]-g_s[\kappa(\zeta^2+\kappa^2+c_a^{\,2}-k_a^{\,2})-2\zeta c_ak_a],\\&\gamma_5=\zeta^2+\kappa^2+\zeta c_a+\kappa k_a,\qquad \gamma_6=\zeta g_s+\kappa(h_a+h_s),\quad \gamma_7=\kappa g_s-\zeta(h_a+h_s).\end{aligned} \tag{29}$$

c_1，c_2，c_3 and c_4 are integral constants，determining by the initial conditions.



(a) (b)

Figure 3 Orbit change due to gyroscopic error g_a (a) and stiffness error $k_s>0$ (b)

X and Y are combined with four harmonic vibrations, Eq. (26). Angular frequencies of the vibration are varied from 1(normalized) to 1+εκ and 1-εκ. For free vibration, $c_s \geq \zeta \geq 0$.

3.1 Effect of angular frequency varies only ($\kappa\neq 0$，$\zeta=0$)

For angular frequency varies only case, Eq. (27) becomes

$$E_r=c_a^{\,2}+g_s^{\,2}+h_a^{\,2}-h_s^{\,2}-k_a^{\,2}<0.\qquad E_i=g_sh_s+c_ak_a=0. \tag{30}$$

Then we have

$$\zeta=0,\qquad \kappa=\sqrt{-(c_a^{\,2}+g_s^{\,2}+h_a^{\,2}-h_s^{\,2}-k_a^{\,2})}\geq 0. \tag{31}$$

This is the case of angular frequency varies only.

Substitute Eq. (30) into Eq.(27) and Eq. (26) to obtain

$$\begin{aligned}&X=\exp(-c_sT_1)(X_{CN}\cos(T_0-\kappa T_1)+X_{SN}\sin(T_0-\kappa T_1)+X_{CP}\cos(T_0+\kappa T_1)+X_{SP}\sin(T_0+\kappa T_1)),\\&Y=\exp(-c_sT_1)(Y_{CN}\cos(T_0-\kappa T_1)+Y_{SN}\sin(T_0-\kappa T_1)+Y_{CP}\cos(T_0+\kappa T_1)+Y_{SP}\sin(T_0+\kappa T_1)),\end{aligned} \tag{32}$$

where

$$X_{CN}=\frac{c_1(\kappa+k_a)+c_3(h_a-h_s)-c_2c_a+c_4g_s}{\kappa},\quad X_{SN}=\frac{-c_1c_a+c_3g_s-c_2(\kappa+k_a)-c_4(h_a-h_s)}{\kappa},$$

$$X_{CP}=\frac{c_1(\kappa-k_a)-c_3(h_a-h_s)+c_2c_a-c_4g_s}{\kappa},\quad X_{SP}=\frac{c_1c_a-c_3g_s-c_2(\kappa-k_a)+c_4(h_a-h_s)}{\kappa},$$
$$Y_{CN}=\frac{-c_1(h_a+h_s)+c_3(\kappa-k_a)+c_2g_s+c_4c_a}{\kappa},\quad Y_{SN}=\frac{c_1g_s+c_3c_a+c_2(h_a+h_s)-c_4(\kappa-k_a)}{\kappa},\quad (33)$$
$$Y_{CP}=\frac{c_1(h_a+h_s)+c_3(\kappa+k_a)-c_2g_s-c_4c_a}{\kappa},\quad Y_{SP}=\frac{-c_1g_s-c_3c_a-c_2(h_a+h_s)-c_4(\kappa+k_a)}{\kappa}.$$

X and Y are combined motion of four harmonic oscillations with amplitudes of linear combinations of c_1，c_2，c_3 and c_4 , and also of function of the coefficients of errors. For each given initial value there are two elliptic orbits having same orientation angle correspond to 1+εκ and 1-εκ, as shown in Fig. 4.

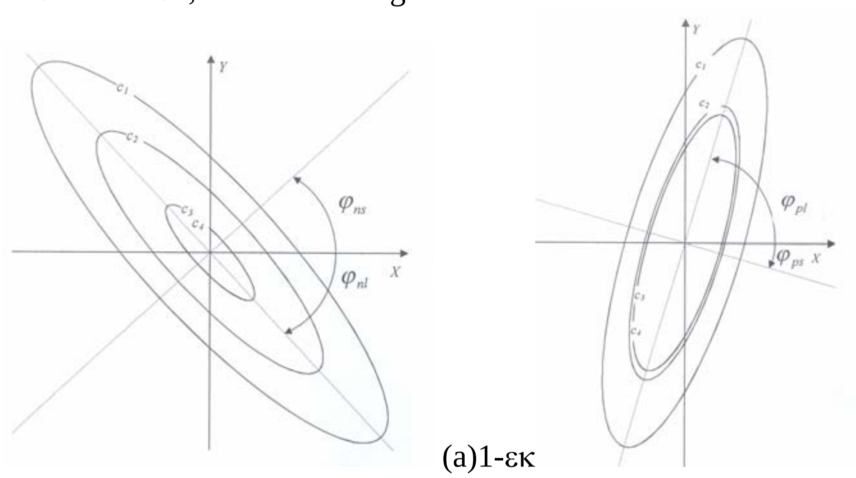

(a)1-εκ (b) 1+εκ

Figure 4 Orbits for angular frequencies (a) 1-εκ and (b) 1+εκ with initial values of 1s.

For a set of given c_1，c_2，c_3 and c_4 , there are two orbits of frequencies 1-εκ and 1+εκ with same orbit angle ϑ as that of individual c_1，c_2，c_3 and c_4 and these two orbits combine to form the trajectory in XY plane as shown in Fig. 5.

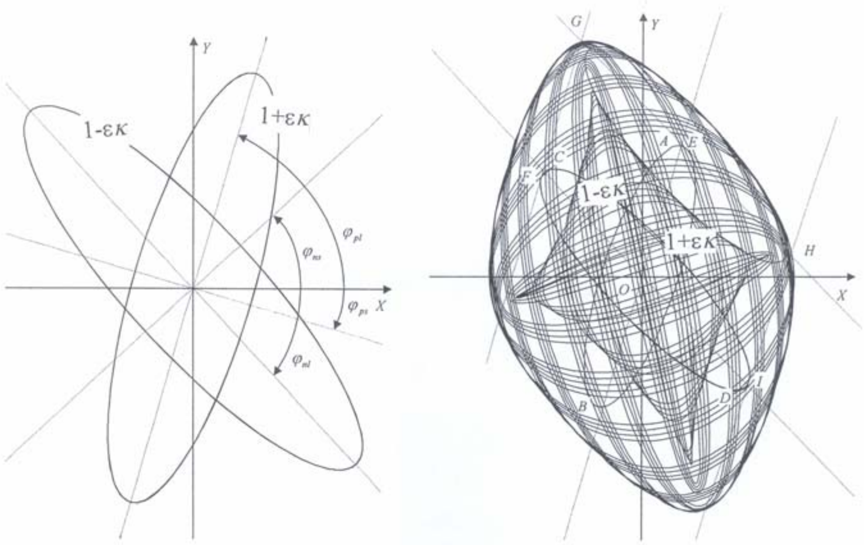


Figure 5 Orbits with frequencies 1-εκ and 1+εκ and trajectories in XY plane

In Figure 5, when the orbit with angular frequency 1+εκ reached the apogee A, the orbit of 1-εκ is lag behind and with origin at A rather than O. The trajectory is formed by the orbits of angular frequency 1-εκ with origins at each point of the trajectories of angular frequency 1+εκ. Major axes of the orbit of angular frequencies 1-εκ and1+εκ are referred as the "stiff" axis and the "soft" axis, respectively. The orientation angles of orbits of 1-εκ and1+εκ are given by

$$\varphi_{nl} = \tan^{-1}\left(\frac{Y_{CN}\cos\vartheta_{nl} + Y_{SN}\sin\vartheta_{nl}}{X_{CN}\cos\vartheta_{nl} + X_{SN}\sin\vartheta_{nl}}\right). \qquad \varphi_{pl} = \tan^{-1}\left(\frac{Y_{CP}\cos\vartheta_{pl} + Y_{SP}\sin\vartheta_{pl}}{X_{CP}\cos\vartheta_{pl} + X_{SP}\sin\vartheta_{pl}}\right). \tag{34}$$

The orientation angle of the ellipse is oscillating between the stiff and the soft axes.

3.2 Effect of anisoelasticity on orbit

Anisoelasticity effect is a particular case of angular frequency changing only ($\kappa \neq 0$，$\zeta = 0$). The stiffness matrix is written in terms of isoelasticity and anisoelasticity as

$$\begin{bmatrix} k_s & 0 \\ 0 & k_s \end{bmatrix} + \begin{bmatrix} -k_a & h_s - h_a \\ h_s + h_a & k_a \end{bmatrix} \tag{35}$$

The isoelasticity effect is considered in the constant term in Eq. (26). For the anisoelasticity

$$c_s = 0, \quad c_a = 0, \quad g_s = 0. \tag{36}$$

Equation (27) becomes

$$E_r = h_a^2 - h_s^2 - k_a^2, \qquad E_i = 0. \tag{37}$$

When $E_r<0$, $\quad \zeta = 0, \quad \kappa = \sqrt{-(h_a^2 - h_s^2 - k_a^2)} \geq 0.$ (38)

$$\begin{aligned} X &= X_{CN}\cos(T_0 - \kappa T_1) + X_{SN}\sin(T_0 - \kappa T_1) + X_{CP}\cos(T_0 + \kappa T_1) + X_{SP}\sin(T_0 + \kappa T_1), \\ Y &= Y_{CN}\cos(T_0 - \kappa T_1) + Y_{SN}\sin(T_0 - \kappa T_1) + Y_{CP}\cos(T_0 + \kappa T_1) + Y_{SP}\sin(T_0 + \kappa T_1). \end{aligned} \tag{40}$$

where
$$\begin{aligned} X_{CN} &= \frac{c_1(\kappa + k_a) + c_3(h_a - h_s)}{\kappa}, & X_{SN} &= -\frac{c_2(\kappa + k_a) + c_4(h_a - h_s)}{\kappa}, \\ X_{CP} &= \frac{c_1(\kappa - k_a) - c_3(h_a - h_s)}{\kappa}, & X_{SP} &= -\frac{c_2(\kappa - k_a) - c_4(h_a - h_s)}{\kappa}, \\ Y_{CN} &= -\frac{c_1(h_a + h_s) - c_3(\kappa - k_a)}{\kappa}, & Y_{SN} &= \frac{c_2(h_a + h_s) - c_4(\kappa - k_a)}{\kappa}, \\ Y_{CP} &= \frac{c_1(h_a + h_s) + c_3(\kappa + k_a)}{\kappa}, & Y_{SP} &= -\frac{c_2(h_a + h_s) + c_4(\kappa + k_a)}{\kappa}. \end{aligned} \tag{41}$$

We have
$$\begin{aligned} X_{CN}Y_{SN} - X_{SN}Y_{CN} &= \frac{(c_2c_3 - c_1c_4)(\kappa^2 + h_a^2 - h_s^2 - k_a^2)}{\kappa^2} = 0, \\ X_{CP}Y_{SP} - X_{SP}Y_{CP} &= \frac{(c_2c_3 - c_1c_4)(\kappa^2 + h_a^2 - h_s^2 - k_a^2)}{\kappa^2} = 0. \end{aligned} \tag{42}$$

The oscillations of angular frequencies 1-εκ and 1+εκ are become a line orbits with directions of $\boldsymbol{b_1}=(X_{CN},Y_{CN})$ and $\boldsymbol{b_2}=(X_{CP},Y_{CP})$, respectively, as shown in Figure 6

The incline angles of the line oscillations of angular frequencies 1-εκ and 1+εκ are

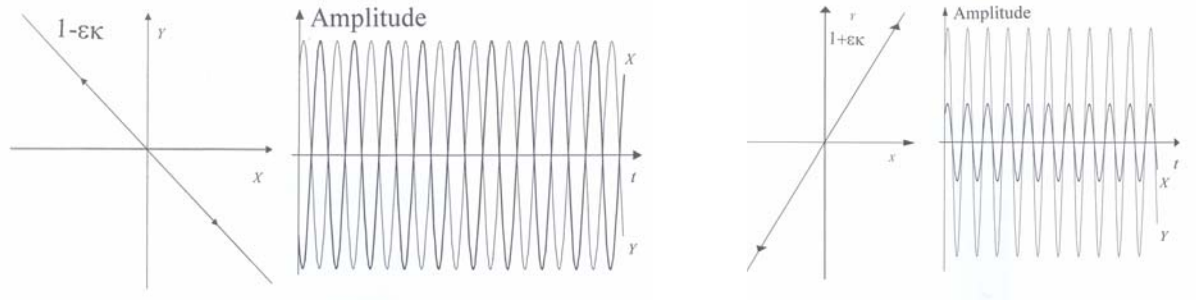
$$\tan\varphi_{nl} = -(h_a + h_s)/(\kappa + k_a), \qquad \tan\varphi_{nl} = -(h_a + h_s)/(k_a - \kappa). \tag{43}$$

In general, the initial values are not limit the motions on the principal axes $\boldsymbol{b}_{a1}$ and $\boldsymbol{b}_{a2}$, the vibration is the combination of the vibrations on the two major axes determined by the initial values of c_1，c_2，c_3 and c_4 as shown in Figure 8.

$$\begin{aligned} X &= [X(0)\cos(\kappa T_1)]\cos(T_0) + [\frac{X(0)k_a + Y(0)(h_a - h_s)}{\kappa}\sin(\kappa T_1)]\sin(T_0), \\ Y &= [Y(0)\cos(\kappa T_1)]\cos(T_0) + [\frac{-Y(0)k_a - X(0)(h_a + h_s)}{\kappa}\sin(\kappa T_1)]\sin(T_0). \end{aligned} \tag{44}$$

The amplitude is a slowly varying periodic function, when $\kappa T_1 = n\pi,\ n = 0,1,2,\ldots$ the orbit becomes a line with incline angle Y/X=Y(0)/X(0). This is one of the major axes of motion, A. when κT_1 increasing, the amplitudes of $\cos T_0$ decreases and $\sin T_0$ increases, and the orbit becomes a line again when $\kappa T_1 = n\pi + \pi/2,\ n = 0,1,2,\ldots$ with incline angle

$\frac{Y}{X} = \frac{-Y(0)k_a - X(0)(h_a + h_s)}{X(0)k_a + Y(0)(h_a - h_s)}$. This is also one of the major axes of motion, E. The incline angle of the orbit is oscillating between those two major axes.



(a) 1-εκ (b) 1+εκ

Figure 6 Orbit of angular frequencies of (a) 1-εκ and (b) 1+εκ

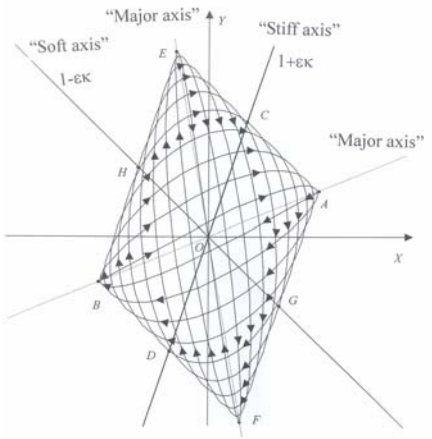


Figure 8 Orbit with anisoelasticity

SUMMARY

The constitutive equations of a homogenous and anisotropic thin shell are derived in an invariant form. Multiple time scale method is used to derive the precession of the free vibrating shell with non-zero rotation. A reference point on the vibrating shell relative to a coordinate system fixed on the shell supporting frame moves in an elliptical orbit with period inversely proportional to the rotating speed of the shell. Linear error model due to imperfections of material or manufacturing tolerance of the shell in terms of damping, gyroscopic, stiffness and circulatory are established to derive the governing equations. The effects of the resulting angular frequency varying only and the anisoelasticity are investigated.

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