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SPLIT REGION ACOUSTICAL WAVE PROPAGATOR AND ITS APPLICATIONS

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Abstract

In this paper, we present a recently developed split region method that solves the time-dependent acoustic wave equation with greatly increased efficiency. This method uses a Chebyshev propagation scheme in areas where there are interfaces and medium variations, and a simple free space propagator where the medium is homogenous. It is proven to be highly accurate and effective. It can easily incorporate variations and boundaries in the propagation medium to simulate a "real-life" wave passing through air, liquids, and solids.

1. INTRODUCTION

Previously, Pan and Wang [1,2] developed an explicit acoustic wave propagator (AWP) to describe the time-domain evolution of mechanical waves in various media. This method was based on a similar scheme that was developed by Tal-Ezer and Kosloff [3,4], who studied seismic wave propagation and a variety of gas-phase reactive scattering and related chemical processes. The AWP method has been successfully applied to study both the propagation of a flexural wave in a thin plate by Peng, Pan, and Sum [5] and an acoustic wave in a room by Sun, Wang, and Pan [6]. However, this method requires significant computer resources since the AWP propagation scheme utilizes a large set of modified Chebyshev polynomials with Bessel functions of the first kind as the expansion coefficients.

In this paper we develop and implement a split region technique that uses the sophisticated Chebyshev propagation scheme in areas where there are interfaces and medium variations, but a simple and much more efficient free space propagator where the medium is homogenous. We then demonstrate the application of this method in one- and two-dimensional sound diffraction around a wave trapping barrier and sound propagation in enclosed spaces.

2. THEORY

2.1 Acoustic Wave Propagator

The motion of acoustical waves in air and solids can be described by a partial differential equation known as the acoustic wave equation:

$$\frac{\partial}{\partial t}\Phi(r,t) = -\hat{H}\Phi(r,t). \tag{1}$$

Integrating this with respect to time gives the formal solution:

$$\Phi(r,t) = e^{-(t-t_0)\hat{H}}\Phi(r,t_0), \tag{2}$$

where r denotes the spatial co-ordinates collectively and t stands for time, with t_0 being the initial starting time. Φ is a state vector, while $\hat{H}$ is the system Hamiltonian that describes the physical properties of the propagation and the boundary medium. The acoustic wave propagator, or AWP, is defined as:

$$\hat{U} = e^{-(t-t_0)\hat{H}}. \tag{3}$$

In a two-dimensional space, Φ describes the sound pressure $p(x, y, t)$ and the particle velocities $v_x(x, y, t)$ and $v_y(x, y, t)$ in the x- and y-directions:

$$\Phi = \begin{pmatrix} p(x,y,t) \\ v_x(x,y,t) \\ v_y(x,y,t) \end{pmatrix}, \tag{4}$$

and $\hat{H}$ is of the form:

$$\hat{H} = \begin{pmatrix} 0 & \rho c^2 \frac{\partial}{\partial x} & \rho c^2 \frac{\partial}{\partial y} \\ \frac{1}{\rho}\frac{\partial}{\partial x} & 0 & 0 \\ \frac{1}{\rho}\frac{\partial}{\partial x} & 0 & 0 \end{pmatrix}, \tag{5}$$

where c is the speed of sound within the medium and ρ is its density.

2.2 J-Chebyshev Expansion

The exponential operator of Eq 3 is impractical in its current form, and cannot be evaluated exactly, so it needs to be expanded as a finite polynomial. We shall use a Chebsyhev polynomial expansion since it allows for long time-steps and its expansion coefficients decay exponentially when the order is sufficiently larger than the argument, both of which contribute to make the scheme faster and more accurate. To ensure the convergence of this expansion, the system Hamiltonian $\hat{H}$ needs to be normalized as:

$$\hat{H}' = \frac{\hat{H}}{\sqrt{\lambda_{\max}}}, \tag{6}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the system operator $\hat{H}$. In the case of sound pressure in a two-dimensional space:

$$\lambda_{\max} = \left(\frac{c\pi}{\Delta x}\right)^2. \tag{7}$$

If we let $R = (t - t_0)\sqrt{\lambda_{\max}}$, then the AWP of Eq 3 can be expressed as:

$$\hat{U} = e^{-(t-t_0)\hat{H}} = e^{-R\hat{H}'}. \tag{8}$$

The next step is to make a simple, if somewhat non-intuitive, change of variables. We let $X' = iX$, then expand the exponential operator in terms of the Chebyshev polynomials $T_n(X')$, which gives us:

$$e^{-RX} = e^{iRX'} = \sum_n b_n(R) T_n(X'). \tag{9}$$

If we use the orthogonality relationship for the Chebyshev polynomials, the coefficients $b_n(R)$ are then:

$$b_n(R) = \frac{c_n}{\pi}\int_{-1}^{1}\frac{e^{iRX'}T_n(X')}{\sqrt{1-X'^2}}dX' = \frac{c_n}{2\pi}\int_{-\pi}^{\pi}e^{iR\cos\theta + in\theta}d\theta = i^n c_n J_n(R), \tag{10}$$

where $c_0 = 1$ and $c_n = 2$ for $n > 0$, and $J_n(R)$ is a Bessel function of the first kind. However, there are complex numbers involved in this expansion, while the state vector and operator are real. Hence, we shall define a new set of modified Chebyshev polynomials, as given by:

$$\widetilde{T} = i^n T_n(iX). \tag{11}$$

It can be shown that the modified Chebyshev polynomials satisfy the following recursion relation:

$$\widetilde{T}_{n+1}(X) = -2X\widetilde{T}(X) + \widetilde{T}_{n-1}(X), \tag{12}$$

with $\widetilde{T}_0(X) = 1$ and $\widetilde{T}_1(X) = -X$. It is now possible for us to write the acoustic wave propagator, $\hat{U}$, in the form:

$$\hat{U} = e^{-(t-t_0)\hat{H}} = e^{-R\hat{H}'} = \sum_{n=0}^{\infty} c_n J_n(R)\widetilde{T}(\hat{H}'), \tag{13}$$

which only involves real-valued operations. We now obtain our state vector with an expanded acoustic wave propagator:

$$\Phi(r,t) = \hat{U}\Phi(r,t_0) = \sum_{n=0}^{\infty} c_n J_n(R)\widetilde{T}(\hat{H}')\Phi(r,t_0), \tag{14}$$

As this scheme uses the Bessel functions, $J_n(R)$, as its expansion coefficients, we name this method the J-Chebyshev expansion. The benefit of this method is that both the Chebyshev polynomials and the Bessel functions are bounded in [-1,1] and decay exponentially with the coefficient index n when n > R. These properties are very useful for numerical computation, as they allow expansions of the exponential function to be accurately calculated for arbitrarily large values of R, that is arbitrarily large time steps.

2.3 Free Space Solution

An exact solution to the propagation of a sound wave through free two-dimensional space is available, and is applicable to any situation that is free of medium changes and interfaces. It can be shown that the pressure $p(x, y, t)$ and the velocities $v_x(x, y, t)$ and $v_y(x, y, t)$ that satisfy Eq 1 also satisfy the second order wave equation:

$$\frac{\partial^2 p}{\partial t^2} - c^2\nabla^2 p = 0, \tag{15}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and the initial conditions are $p(x,y,0) = \varphi(x,y)$ and $\frac{\partial}{\partial t} p(x,y,0) = \psi(x,y)$.

The solution to the second order wave equation is of the form:

$$\begin{aligned} p(x,y,t) = {} & \frac{1}{2\pi c}\frac{\partial}{\partial t}\iint_{\sum_{ct}^{x,y}} \frac{\varphi(x',y')}{\sqrt{c^2t^2 - (x'-x)^2 - (y'-y)^2}}\,dx'dy' \\ & + \frac{1}{2\pi c}\iint_{\sum_{ct}^{x,y}} \frac{\psi(x',y')}{\sqrt{c^2t^2 - (x'-x)^2 - (y'-y)^2}}\,dx'dy' \end{aligned}, \tag{16}$$

where $\sum_{ct}^{x,y}$ represents a two-dimensional disc centred at (x, y) with a radius of ct that the integral is performed over.

2.4 Split Region Implementation

The Chebyshev expansion is sufficient to perform any calculation that is necessary in simulating sound diffraction and propagation. this situation. However, it requires significant computer resources. For a large grid, the computational effort that is required becomes prohibitive. On the other hand, the free space solution is much faster, but only applicable in a homogenous free space.

Many acoustical problems involve large amounts of free space interspersed with medium variations, interfaces, and barriers. It is readily apparent that using both solutions in the appropriate regions would be much more efficient. We have developed a split region technique that will use the exact solution in regions of free space, and the Chebyshev expansion in areas where there are variations. This leads to increased computational efficiency and speed.

The splitting technique involves multiplying the total acoustic wave distributions by a step function, where the step occurs at the intended boundary between regions. This method divides the wave-distribution into two sections that can be propagated separately, by the most appropriate technique for each region. Due to the linear nature of the splitting, these two sections can be easily recombined by adding the two resulting distributions together. The whole process can then be repeated if more than one time step is desired.

Ideally, the splitting function would be a step function. However, the discontinuities that this would introduce into the wave distribution would create large inaccuracies in the numerical techniques that are to be used. To avoid this consequence, a more gentle splitting function is used, where the step function is convolved with a Gaussian curve, so that the splitting actually occurs over some width. This eliminates the discontinuities of the pure step function and the associated difficulties and errors.

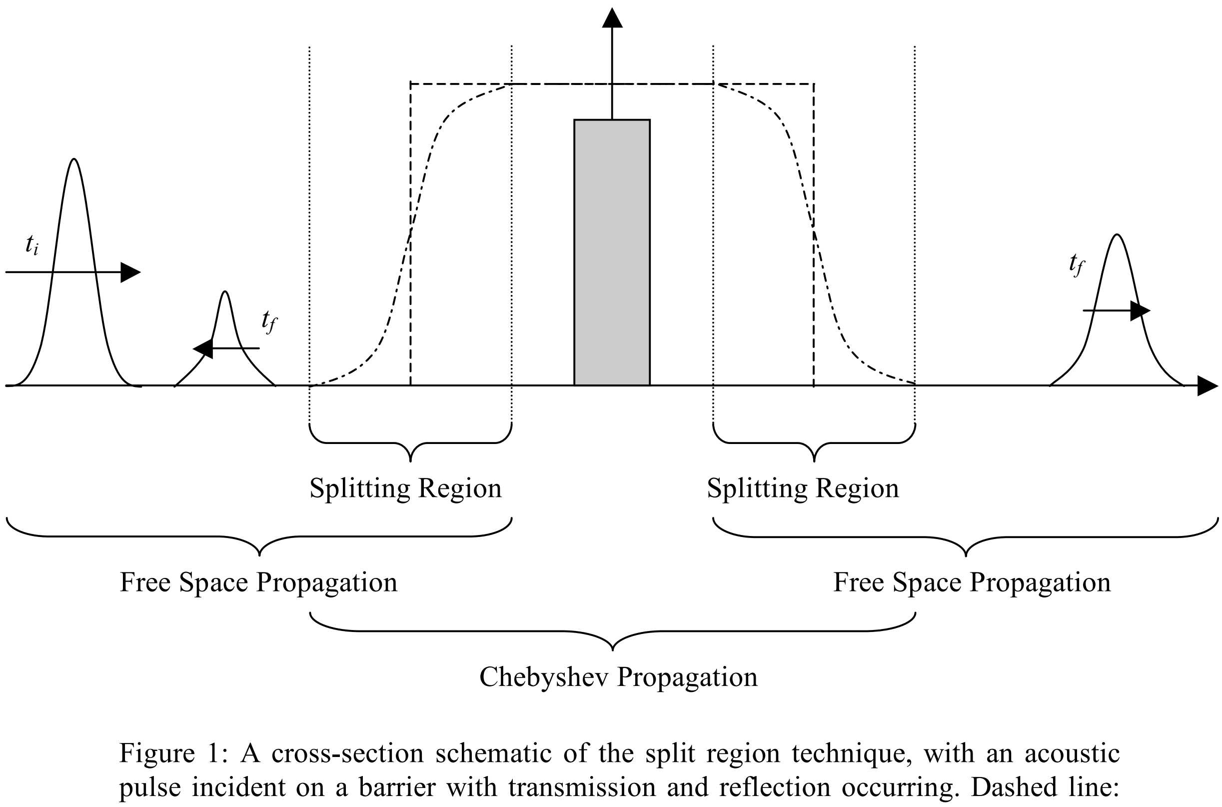

Figure 1: A cross-section schematic of the split region technique, with an acoustic pulse incident on a barrier with transmission and reflection occurring. Dashed line: step-function. Dot-dashed line: smoothed step-function. Dotted line: region boundaries.

The essence of the split region implementation is the division of the space into homogenous and inhomogeneous regions. An example of this can be seen in Figure 1, where an incident wave is made to travel to and across an obstruction. The wave starts off in the homogenous region on the left at time t_i, and will be moved across this region with the free space propagator. As the wave approaches the vicinity of the obstacle, the algorithm will then pass the wave onto the Chebyshev expansion propagator, which is operating in the central region. When the acoustic wave emerges from the central region, the free space propagator can again be employed to model its propagation in the homogenous region.

Note that the sizes of the splitting region, the interaction zone, and the time-step need careful determination. The buffer zone in the interaction region surrounding the split region needs to be large enough that portions of the wave that are at the edge of the split region at the start of the time-step do not travel further than the edge of the buffer region, otherwise they are artificially wrapped and thus provide inaccurate results. As a general rule, the size of the buffer region needs to be greater than the distance travelled by the wave in the length of the time-step.

This technique should be useful for any acoustic problem that can be easily divided into homogenous and inhomogenous regions, in any number of dimensions, though the algorithms for the splitting will become increasingly complex. Figure 2 depicts a simple two-dimensional situation (or a cross-section of a three-dimensional one) using square and circular splitting regions around a small round obstruction amidst free space:

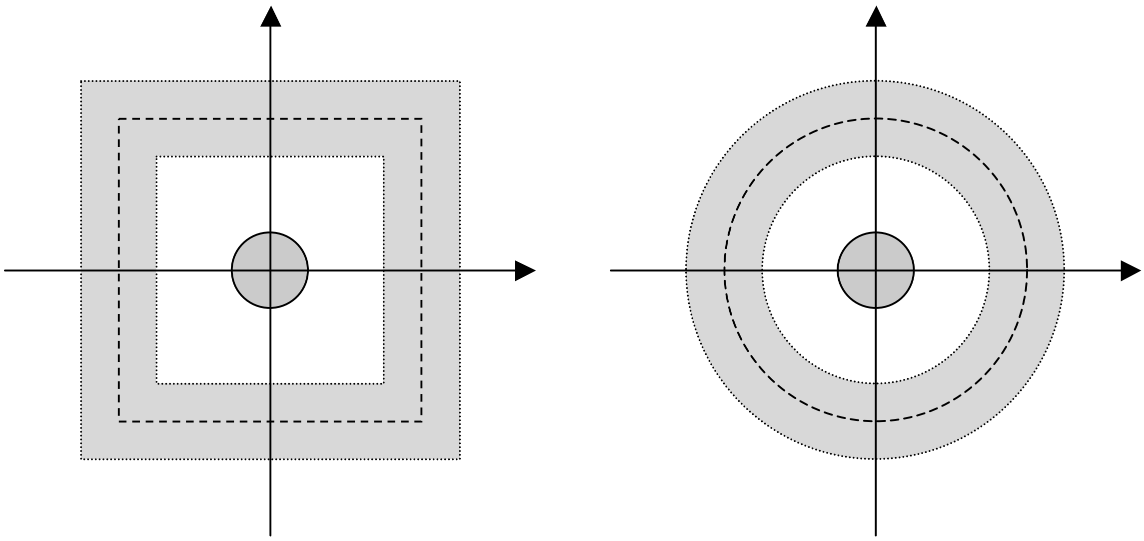

Figure 2: A schematic of the split region technique in two-dimensions, with square and circular splitting regions shaded, around a small circular obstruction. Dashed line: step-function. Dot-dashed line: smoothed step-function. Dotted line: region boundaries.

3. RESULTS

In Figure 3, an acoustic wave is propagated towards a small barrier (a) using both a full J-Chebsyhev propagation (b), and a split region implementation of both free space and J-Chebshev solutions (c). The absolute difference between the two results is quite small, on the order of 10^{-6} and less. The computational saving is significant especially when the interaction region is small in comparison with the entire space under consideration.

The J-Chebyshev scheme and the split region implementation have both proven to be highly accurate methods in modelling the propagation of acoustic waves. The split region implementation builds on the J-Chebyshev scheme becomes invaluable especially when solving acoustical problems in two or three dimensions with various barriers and interfaces, where the time and complexity of calculations rises sharply.

As an example, a two-dimensional room with a barrier in the middle of the room can be modelled by the scheme to investigate the effects of the barrier. Figure 4 shows the sound diffraction around the barrier at various times after an initial sound pulse emitted at one of the corners.

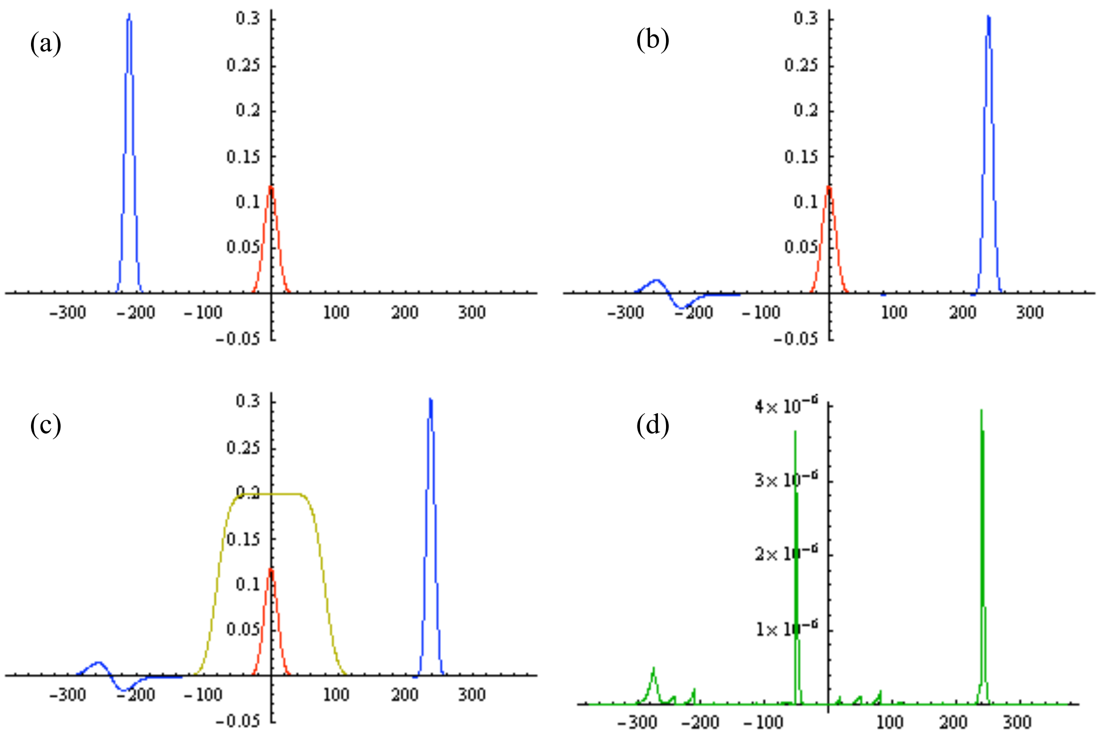

Figure 3: The propagation of an acoustical wave through a small barrier (central peak, ~0.1x denser than surrounding free space). (a) The incident wave at time ti. (b) The reflected and transmitted waves at time tf, propagated entirely using the J-Chebyshev scheme. (c) The reflected and transmitted waves at time tf, propagated using the split region implementation, with the smoothed top-hat (scaled to fit) as the splitting function. (d) The absolute difference between the results of the full J-Chebyshev and split region implementations.

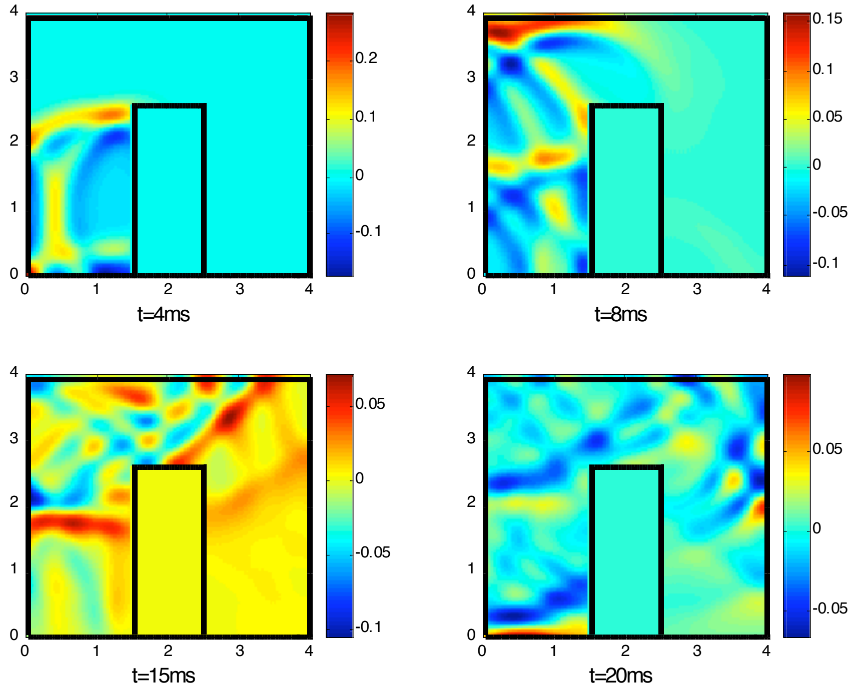

Figure 4: Sound pressure distributions at different times in a square room with a barrier, initial sound field at the corner (1m, 1m).

6. CONCLUSIONS

In this paper, we have presented an efficient and highly accurate time dependent propagation scheme for simulating the propagation of one- and two-dimensional acoustic waves. In particular, the use of the wave split region technique, which can be readily extnded to higher dimensional grids, allows for significant reduction in computation time whilst not compromising accuracy.

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