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PHASE CHANGE ANALYSIS OF PLANAR AND NONPLANAR VIBRATION IN ONE-TO-ONE RESONANCE OF NONLINEAR CANTILEVER BEAM

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Abstract

A symmetric long slender cantilever beam with nonlinearities shows many nonlinear dynamic phenomena. One-to-one resonance of the beam is well shown in the nonlinear vibration. Among the nonlinear factors of the flexible symmetric cantilever beam, nonlinear inertia term and nonlinear spring term are the most important. When base harmonic excitation is applied to the beam, planar vibration and nonplanar vibration occur in the beam due to one-to-one resonance. When one-to-one resonance occurs, the planar vibration is different from the nonlinear one. When one-to-one resonance is developed in the first and the second mode of the nonlinear beam, the beam has different amplitudes and phase values. The phase value changes according to the excitation frequency. Thus, the phase change and the phase difference between the planar vibration and the nonlinear vibration are investigated both theoretically and experimentally when the symmetric nonlinear beam shows one-to-one resonance.

1. INTRODUCTION

A flexible circular cantilever beam subject to an external force shows a very complex nonlinear response due to its nonlinear characteristics. This non-linearity appears in helicopter rotor blades, spacecraft antennas, flexible large space structures, and many other systems. When the amplitude of the beam becomes large due to forced vibration, such effects as shear deformation, warping, rotational inertia, and gravity, which affects dynamic characteristics of a cantilever beam, become important, and thus various nonlinear phenomena derives from nonlinear factors. In theory, it is almost impossible to derive nonlinear equations of motion that include all the beam effects. In the existing literature, a particular effect is introduced that it has a certain form of response. In nonlinear vibration, a flexible circular cantilever beam has nonlinear terms of inertia, spring, damping, gravity, warping, and so on. When the external excitation force is weak, the beam shows a linear motion, but as the force increases, the linear motion transforms to a nonlinear motion due to those nonlinear terms in the beam. As a result, the beam shows such nonlinear phenomena that appear only in nonlinear systems as superharmonic, subharmonic, super-subharmonic vibration, quasi-periodic phenomenon, jumping, and phase change. It is needed to study the cause of those phenomena in the view of dynamics by investigating the characteristics of nonlinear terms from the analysis of nonlinear responses. Both theoretical and experimental methods are used to investigate a flexible circular cantilever beam when the beam shows nonlinear responses for the base harmonic excitation. Both theoretical and experimental analyses are conducted for its phase change and phase difference [1], [3], [5].

2. NONLINERAITIES OF A CANTILEVER BEAM IN ONE-TO-ONE RESONANCE

To investigate the responses of the beam subject to a forced vibration, integro-differential equations derived by Crespo da Silva and Glynn are used. All variables are non-dimensionalized with the length of the beam L and the characteristic time $L^2\sqrt{m/D_\eta}$. This yields the following equations of motion [6].

$$\begin{aligned}\ddot{v}+c\dot{v}+\beta_y v^{iv} &= \left(1-\beta_y\right)\left[w''\int_1^s v''w''ds-w'''\int_0^s v''w'ds\right]' \\ &-\frac{\left(1-\beta_y\right)^2}{\beta_y}\left[w''\int_0^s\int_1^s v''w''dsds\right]''-\beta_y\left[v'\left(v'v''+w'w''\right)'\right]' \\ &-\frac{1}{2}\left\{v'\int_1^s\left[\int_0^s\left(v'^2+w'^2\right)ds\right]^{\cdot\cdot}ds\right\}'+\left[v''\left(s-1\right)+v'\right]\frac{L^3}{D_\eta}mg\end{aligned} \tag{1a}$$

$$\begin{aligned}\ddot{w}+c\dot{w}+w^{iv} &= -\left(1-\beta_y\right)\left[v''\int_1^s v''w''ds-v'''\int_0^s w''v'ds\right]' \\ &-\frac{\left(1-\beta_y\right)^2}{\beta_y}\left[v''\int_0^s\int_1^s v''w''dsds\right]''-\left[w'\left(w'w''+v'v''\right)'\right]' \\ &-\frac{1}{2}\left\{w'\int_1^s\left[\int_0^s\left(v'^2+w'^2\right)ds\right]^{\cdot\cdot}ds\right\}'+\left[w''\left(s-1\right)+w'\right]\frac{L^3}{D_\eta}mg \\ &+F\Omega^2\cos\left(\Omega t\right)+cF\Omega^2\sin\left(\Omega t\right)\end{aligned} \tag{1b}$$

The boundary conditions for the beam are given as the following.

$$v=w=v'=w'=0 \quad \text{at} \quad s=0 \tag{2a}$$

$$v'''=w'''=v''=w''=0 \quad \text{at} \quad s=1. \tag{2b}$$

In the equations (1a) and (1b), the terms in the first and second square brackets represent the torsion and the axial vibration. The second term in the third square bracket represents the coupling between the axial vibrations in the y and x directions. The terms in the curly bracket are inertia terms, and the terms in the last square bracket represent gravitational term and the boundary conditions for the inextensionality. $F\Omega^2\cos\Omega t$ includes the excitation force at the base and its frequency. All the nonlinear terms are cubic. And if inertia ration $\beta_y=1$, then the coupling between the torsion and the axial motion disappear.

3. SOLUTIONS OF THE NONLINEAR EQUATIONS OF MOTION

To analyze the nonlinear equations of motion (1a) and (1b), method of multiple scales is used. And for the perturbation analysis of the equations of motion, a small parameter ε is used. The approximations in the planar and non-planar directions can be expressed as the following.

$$v(s,T_0,T_2;\varepsilon) = \varepsilon v_1(s,T_0,T_2,\ldots) + \varepsilon^3 v_3(s,T_0,T_2,\ldots) + \ldots \tag{3}$$

$$w(s,T_0,T_2;\varepsilon) = \varepsilon w_1(s,T_0,T_2,\ldots) + \varepsilon^3 w_3(s,T_0,T_2,\ldots) + \ldots \tag{4}$$

Where $T_0 = t$ is a fast scale that has Ω and natural frequency of ω_{mn} and $T_2 = \varepsilon^2 t$ is a slow scale characterizing the modulations of amplitudes and phases. And it is assumed that $c = \varepsilon^2 \mu$, $F = \varepsilon^3 f$ and $\beta_y = 1 + \delta_0 + \varepsilon^2 \delta_2$. By substituting equations (3) and (4) into equations (1a) and (1b), the equations are rearranged for ε [6], [7].

In this paper, we consider the primary parametric excitation of one mode in the x direction such as that of the n-th mode. This excites the other mode in the y direction such as the m-th mode through a one-to-one internal or parametric resonance. Thus, the former is directly excited and the latter is indirectly excited by the parametric resonance.

$$v_1(s,T_0,T_2,\ldots) = \Phi_m(s) A_1(T_2) e^{i\omega_{1m}T_0} + cc \tag{5a}$$

$$w_1(s,T_0,T_2,\ldots) = \Phi_n(s) A_2(T_2) e^{i\omega_{2n}T_0} + cc \tag{5b}$$

Where cc means complex conjugate.

$$\omega_{1m} = z_m^2 \sqrt{1+\delta_0}, \quad \omega_{2n} = z_n^2 \,, \tag{6}$$

$$\Phi_i(s) = \cosh z_i s - \cos z_i s + \frac{\cos z_i + \cosh z_i}{\sin z_i + \sinh z_i}\left(\sin z_i s - \sinh z_i s\right) \tag{7}$$

Where Zi is the root of $1 + \cos z \cdot \cosh z = 0$ and three roots of the equation (7) are 1.8751, 4.6941, and 7.8548. The eigenfunction Φ_i satisfies the following condition.

$$\int_0^1 \Phi^2{}_i(s)ds = 1 \tag{8}$$

We note that ω_{2n} is the linearized natural frequency of the nth mode of vibration in the x direction and ω_{1m} is approximately equal to the linearized natural frequency of the m-th mode in the y direction. The actual frequency of the latter is $Z^2{}_m\sqrt{1+\delta_0+\varepsilon^2\delta_2}$. The above equations are reduced to ordinary differential equations by the Galerkin procedure as the following.

$$\begin{aligned} D_0^2 v_{3t} + \omega_{1m}^2 v_{3t} &= -2D_0 D_2 v_{1t} - \mu D_0 v_{1t} - \delta_2 z^4 v_{1t} \\ &+ \left(1-\beta_y\right)\alpha_1 v_{1t} w_{1t}^2 - \frac{\left(1-\beta_y\right)^2}{\beta_y}\alpha_2 v_{1t} w_{1t}^2 - \beta_y \alpha_3 v_{1t} w_{1t}^2 - \beta_y \alpha_4 v_{1t}^3 \\ &- \frac{1}{2}\alpha_5 v_{1t} D_0^2 v_{1t}^2 - \frac{1}{2}\alpha_6 v_{1t} D_0^2 w_{1t}^2 + \alpha_7 g_0 v_{1t} \end{aligned} \tag{9}$$

$$\begin{aligned} D_0^2 w_{3t} + \omega_{2n}^2 w_{3t} &= -2D_0 D_2 w_{1t} - \mu D_0 w_{1t} \\ &- \left(1-\beta_y\right)\beta_1 v_{1t}^2 w_{1t} - \frac{\left(1-\beta_y\right)^2}{\beta_y}\beta_2 v_{1t}^2 w_{1t} - \beta_3 v_{1t}^2 w_{1t} - \beta_4 w_{1t}^3 \\ &- \frac{1}{2}\beta_5 w_{1t} D_0^2 w_{1t}^2 - \frac{1}{2}\beta_6 w_{1t} D_0^2 v_{1t}^2 + \beta_7 g_0 w_{1t} + \beta_8 f \Omega^2 \cos\Omega T_0 \end{aligned} \tag{10}$$

Where α_i and β_i are defined in the reference [6] (Appendix A). To investigate the system response subject to an excitation with a frequency such as the natural frequency of the first mode, a detuning parameter $\hat{\sigma}$ is introduced and $\omega_{1m} = \omega_{2n}$ is assumed.

$$\Omega = \omega_{2n}(1 + \varepsilon^2 \hat{\sigma}) \tag{11}$$

Substituting the equations into (9) and (10), and eliminating the terms that produce secular terms, the conditions for the solutions can be obtained as the following.

$$\begin{aligned}&-i\left(2\omega_{1m}A_1' + \mu\omega_{1m}A_1\right) - \delta_2 z^4 A_1 - \left[\delta_0\alpha_1 + \frac{\delta_0^2}{\beta_y}\alpha_2 + (1+\delta_0)\alpha_3\right]\left(2A_1A_2\bar{A}_2 + \bar{A}_1A_2^2\right)\\ &-3(1+\delta_0)\alpha_4 A_1^2\bar{A}_1 + 2\alpha_5\omega_{1m}^2\bar{A}_1A_1^2 + 2\alpha_6\omega_{2n}^2\bar{A}_1A_2^2 + \alpha_7 g_0 A_1 = 0\end{aligned} \tag{12}$$

$$\begin{aligned}&-i\left(2\omega_{2n}A_2' + \mu\omega_{2n}A_2\right) - \left(\delta_0\beta_1 - \frac{\delta_0^2}{\beta_y}\beta_2 - \beta_3\right)\left(2A_2A_1\bar{A}_1 + \bar{A}_2A_1^2\right) - 3\beta_4 A_2^2\bar{A}_2\\ &+2\beta_5\omega_{2n}^2\bar{A}_2A_2^2 + 2\beta_6\omega_{1m}^2\bar{A}_2A_1^2 + \beta_7 g_0 A_2 + \frac{1}{2}\beta_8 f\omega_{2n}^2 e^{i\omega_{2n}\hat{\sigma}T_2} = 0\end{aligned} \tag{13}$$

Converting the above equations into polar form,

$$A_1 = \frac{1}{2}a_1(T_2)e^{i\theta_1(T_2)} \quad \text{and} \quad A_2 = \frac{1}{2}a_2(T_2)e^{i\theta_2(T_2)} \tag{14}$$

Substituting the above variables into equations (12) and (13) and separating the real parts and the imaginary parts, the following equations of modulation can be derived.

$$2\omega_{1m}a_1' + \left[R_1 + R_2a_2^2\sin 2(\gamma_1 - \gamma_2)\right]a_1 = 0 \tag{15}$$

$$\left[2\omega_{1m}\gamma_1' - R_3 - R_4a_1^2 + R_5a_2^2 + R_2a_2^2\cos 2(\gamma_1 - \gamma_2)\right]a_1 = 0 \tag{16}$$

$$2\omega_{2n}a_2' + \left[E_1 + E_2a_1^2\sin 2(\gamma_1 - \gamma_2)\right]a_2 - \beta_8 f\omega_{2n}^2\sin\gamma_2 = 0 \tag{17}$$

$$\begin{aligned}&\left[2\omega_{2n}\gamma_2' - E_3 - E_4a_2^2 - E_5a_1^2 - E_2a_1^2\cos 2(\gamma_1 - \gamma_2)\right]a_2\\ &-\beta_8 f\omega_{2n}^2\cos\gamma_2 = 0\end{aligned} \tag{18}$$

Where R_i and E_i are defined in the reference [6] (Appendix B).

$$\gamma_1 = \omega_{2n}\hat{\sigma}T_2 - \theta_1, \quad \gamma_2 = \omega_{2n}\hat{\sigma}T_2 - \theta_2 \tag{19}$$

Periodic solutions of the beam represent the fixed points of (15)-(18). Those points correspond to $a_1' = a_2' = \gamma_1' = \gamma_2' = 0$. From (19), it follows that $\theta_1' = \omega_{2n}\hat{\sigma}$ and $\theta_2' = \omega_{2n}\hat{\sigma}$. Algebraic equations for the equations (15)-(18) can be solved using a numerical method. The first order approximation of the beam response is given by

$$v(s,t) = \varepsilon\Phi_m(s)a_1\left(\varepsilon^2 t\right)\cos\left(\Omega t - \gamma_1\right) + \ldots \tag{20}$$

$$w(s,t) = \varepsilon\Phi_n(s)a_2\left(\varepsilon^2 t\right)\cos\left(\Omega t - \gamma_2\right) + \ldots \tag{21}$$

The numerical analysis is performed in MATLAB 7.0 using the approximations (20) and (21). In the numerical analysis of the algebraic equations, the nonlinear response is investigated in the planar and the non-planar directions, and then the phase analysis is performed. The analysis of the nonlinear response and the phase difference is performed in the second mode of the beam. Figure 1 shows frequency responses of the planar and the nonplanar vibrations of the beam in the second mode due to one-to-one resonance. In figure 2, the phase change and the phase difference are shown for the planar and the nonplanar vibrations of the beam in the second mode. When the excitation force is $F = \varepsilon^3 f = 0.1980$, the phase difference the planar and the nonplanar vibrations is 90^0.

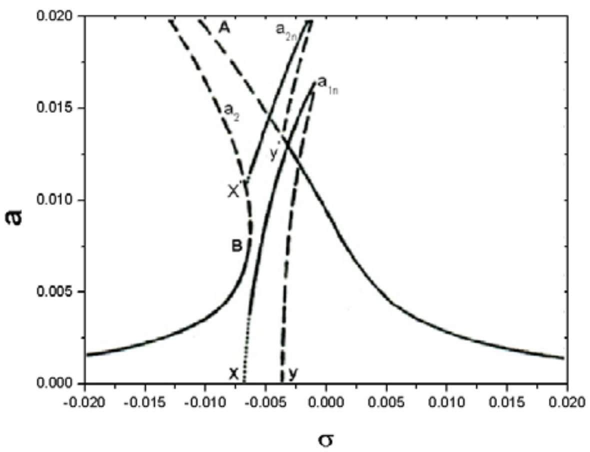

Figure 1 – Response curves of the second mode for the cantilever beam: a_2=planar response amplitudes; a_{1n}, a_{2n}= nonplanar response amplitudes

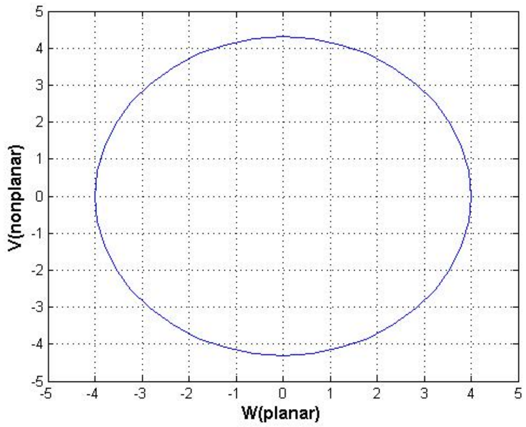

Figure 2 – Phase change in one to one resonance of the second mode

4. EXPERIMENT OF NONLINEARITIES

4.1 Experiment Equipment

For the experiment, a circular cantilever beam of aluminum alloy was used as a uniform elastic material. The dimensions of the beam were: the modulus of elasticity E=72GPa, the coefficient of stiffness G=27GPa, Poisson's ratio $\nu = 0.3333$, mass per unit length of the beam $m = 0.0336 Kg/m$, diameter $\phi = 5mm$, length L=675mm. For the excitation of the beam, base harmonic excitation was applied to the fixed part of the base in the form of a sine wave with constant amplitude. The jig that held the beam was made of aluminum AL2024 and was designed so that it satisfied the boundary conditions of the beam and so that it was subject to transverse (in the planar direction) excitation [8].

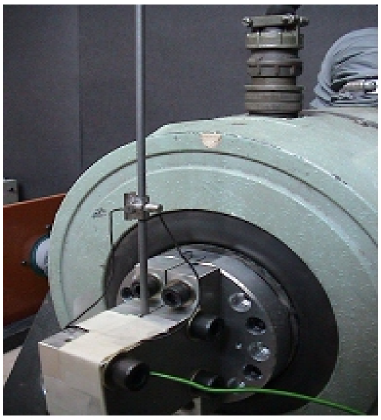

Figure 3 – Accelerometer position on the circular cantilever beam

Figure 4 – One to one internal resonance of the circular cantilever beam on the second mode

Table 1 Measured *natural frequency and damping coefficient of the circular cantilever beam*

Mode	λ	Theory(Hz)	Meas.(Hz)	Damping
1	1.8751	7.94	7.63	1.510%
2	4.6941	49.78	48.25	0.736%
3	7.8548	139.40	135.13	0.341%

4.2 Experiment Method

A flexible beam was fixed to the shaker to satisfy the boundary conditions of the cantilever beam. Keeping the voltage applied to the shaker constant, the experiment was performed by increasing or decreasing the excitation frequency. When the voltage to the shaker is kept constant, the speed component in the harmonic vibration has a constant value regardless of the change of excitation frequency. The increase and decrease of excitation frequency were in the form of sine sweeping and the rate of change was 0.030Hz/s. The experiment was performed by increasing the amplitude of the excitation with the frequency fixed in the second mode of the beam. The process of change in the beam from the linear vibration to the nonlinear vibration was investigated as the amplitude of the excitation increases. To investigate the phase change in the vibration, the phase portrate and the phase difference were analyzed. To investigate the response of the beam, each B&K 4374 accelerometer was attached to the beam in the planar (-x) and nonplanar (-y) directions. The mass of each accelerometer was about 0.65g, the range of frequency for measurement 1-25KHz, and the level of measurable acceleration 250,000 m/s^2. The accelerometers were attached to the surface of the beam 100mm above the base with strong adhesives. The signals of the planar and nonplanar vibrations due to one-to-one resonance were respectively measured by each accelerometer and transformed to voltage signals in a charge amplifier charge amplifier B&K2635. The signals from the amplifier were displayed as lissajous figures on an oscilloscope. Using the obtained lissajous figures, the phase change and the phase difference were analyzed.

5. EXPERIMENTAL RESULTS AND DISCUSSION

In the experiment, the planar and nonplanar vibrations due to one-to-one were well observed in the second mode of the beam (48.25Hz). It can be seen that the response of the planar motion in one-to-one resonance showed the decrease of the amplitude when nonplanar motion occurred due to nonlinearities. That is to say, it can be seen that the vibration energy was transferred from the planar vibration to the nonplanar vibration (figure 5, figure 6). As a result, it is easy to analyze the responses of the planar and nonplanar vibrations in one-to-one resonance and the phase. The planar and nonplanar vibrations have different vibrations in the region where they occur simultaneously. In figure 7, it can be seen that when the level of excitation is 25m/s^2, the phase angle between the planar and nonplanar vibrations varies up to 87^0.

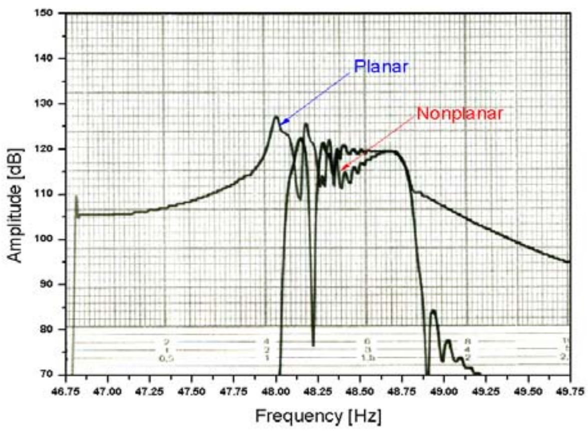

Figure 5 – Frequency response curves for plane and nonplane on the second mode (forward direction).

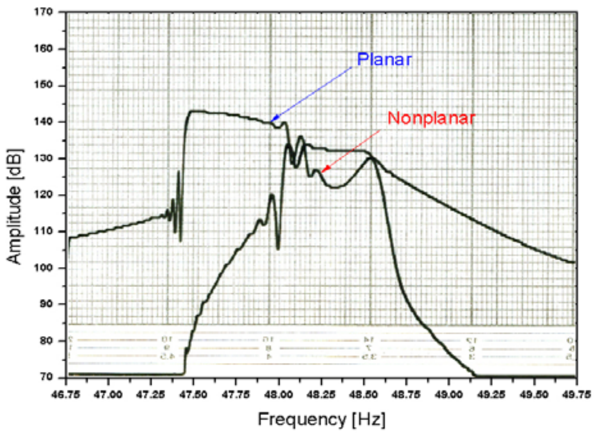

Figure 6 – Frequency response curves for plane and nonplane on the second mode (backward direction)

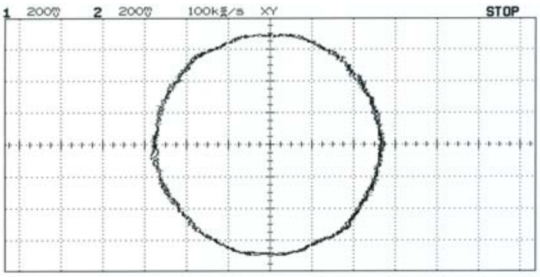

Figure 7 – Phase change in phase portrait of the second mode (x-planar, y-nonplanar)

6. CONCLUSIONS

The frequency response and the phase difference due to one-to-one resonance which occurs in the nonlinear vibration of a flexible circular cantilever beam subject to base harmonic excitation were investigated. The integro-differential equations derived by Crespo da Silva and Glynn were used. It can be seen that thee are changes in frequency response between planar and nonplanar vibrations due to one-to-one resonance in the second mode of the beam and phase difference. In the second mode of the beam, the phase difference was shown to be up to 90^0. For the experimental analysis, base harmonic excitation was applied to a flexible circular cantilever beam and then the responses of the planar and nonplanar vibrations were investigated. The frequency responses show that when the nonlinear vibration occurs due to nonlinearities, the amplitude of the planar vibration decreases. That is, the vibration energy of the planar motion is transferred to the out-of-plane to cause the nonplnar vibration. In the region where the planar and nonplanar vibrations occur simultaneously, the mutual phase difference occurs. It can be seen that the phase difference between the planar and nonplanar vibrations is up to 87^0. Therefore it can be seen that there is a phase difference between the planar and nonplanar vibrations due to one-to-one resonance.

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