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ZERO-POLE ANALYSIS OF A TENSIONED STRING WITH PERIODIC STIFFENERS

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Abstract

This paper analyzes the zero-pole locations of an infinite length, tensioned string that has attached periodic stiffeners. The dynamic response of the system is derived for distributed wavenumber forcing and discrete point forcing acting on the string. These wavenumber-frequency transfer functions are then written in zero-pole format by a mathematical transformation of their infinite series. Once this is accomplished, the location of the system poles and zeros becomes apparent and they can be plotted in the wavenumber-frequency plane. It is shown that there are specific regions where an infinite number of poles can exist and specific regions where poles cannot exist. For the system with wavenumber forcing, the system zeros correspond very closely to the system poles except in the area of the fundamental string resonance. For the system with point forcing, the zeros can exist in the entire wavenumber-frequency plane except at the fundamental resonance.

1. INTRODUCTION

Motion of tensioned strings is typically studied to understand the dynamic response of the system. Using a number of assumptions, the underlying mathematical model of this problem is typically a second order wave equation written as a function of space and time. The problem of an unreinforced string is a classical continuous media problem and has been discussed by numerous authors [1,2,3]. The tensioned string with periodic stiffeners has been analyzed in the frequency domain with a moving harmonic force [4] and a suddenly applied concentrated force [5]. In these papers, the stability of the system is discussed, especially with respect to the value of the stiffeners. There is no extension of the problem into the wavenumber domain. Other continuous systems, specifically thin plates [6,7], thick plates [8], and beams [9] have been modeled with periodic masses, dampers, and/or springs attached to the medium. These papers typically involve modeling a continuous system with a differential equation(s), using various boundary conditions to represent the mechanical elements attached to the system, and then deriving a solution technique to find the corresponding displacements of the system.

This paper presents a zero-pole analysis of a tensioned string in wavenumber-frequency space when it is loaded with a continuous wavenumber forcing function and a discrete point forcing function. The equations of motion and corresponding solution to both mechanical loads are found using a previously derived analytical method. These equations are then transferred from an infinite summation series into a continuous analytical expression using a series to trigonometric mathematical formula. Once this is accomplished, the dynamic response of the system is now in zero-pole format. This allows an examination of the stability of the system from a parametric standpoint. Additionally, the locations of the transfer function poles and zeros are apparent. The dynamic response of this system is discussed.

2. EQUATION OF MOTION

The system model is that of an infinite length, tensioned string attached to periodically spaced discrete stiffeners, as shown in Figure 1. The string is under a tension of T (N), has a constant mass per unit length ρ (kg/m), and an external load per unit length of $f(x,t)$ (N/m). The stiffeners are equally spaced at a distance of L (m) in the x-direction and each has a stiffness of K (N/m). The model uses the following assumptions: (1) the forcing function acting on the string is at a definite wavenumber and frequency or is a discrete point function at a definite frequency, (2) motion is normal to the string in the transverse direction (one-dimensional system), (3) the string has infinite spatial extent in the x-direction, (4) the particle motion is linear, (5) the string offers no resistance to bending, and (6) the string is perfectly elastic. The motion of the system is governed by the equation [10]

$$T\frac{\partial^2 w(x,t)}{\partial x^2} - \rho\frac{\partial^2 w(x,t)}{\partial t^2} - K\sum_{n=-\infty}^{n=\infty} w(x,t)\delta(x-nL) = f(x,t) \ , \tag{1}$$

where $w(x,t)$ is the transverse displacement of the string (m), x is the position on the string (m), t is time (seconds), and $\delta(x-nL)$ is the spatial Dirac delta function (1/m).

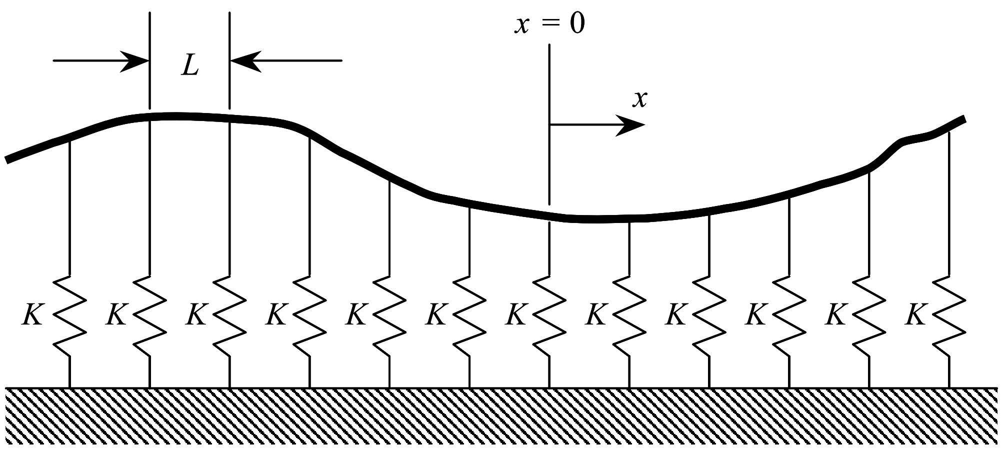

Figure 1. Tensioned and reinforced string with coordinate system

3. SOLUTION WITH A WAVENUMBER FORCING FUNCTION

The problem is first solved with a wavenumber forcing function, which is a continuous exponential harmonic function in space and time, written as

$$f(x,t) = F\exp(\mathrm{i}\omega t)\exp(\mathrm{i}kx) \ , \tag{2}$$

where ω is frequency (rad/s), k is wavenumber with respect to the x-axis (rad/m), and F is the magnitude of the distributed force (N/m). The problem is transformed into the wavenumber-frequency domain (k, ω) by using the functional form where the string displacement is equal to an unknown function of wavenumber and frequency multiplied by an exponential harmonic function in the x-direction multiplied by an exponential harmonic function in time. The displacement becomes

$$w(x,t) = W(k,\omega)\exp(\mathrm{i}\omega t)\exp(\mathrm{i}kx) \ , \tag{3}$$

Substitution of equations (2) and (3) into equation (1), and use of the Fourier relationship between the delta function and the exponential function [6,7], transforms the governing equation into the wavenumber-frequency domain as

$$(-k^2 T + \rho\omega^2)W(k,\omega) = \frac{K}{L}\sum_{n=-\infty}^{n=\infty} W(k+\frac{2\pi n}{L},\omega) + F\delta(k-k_0) \ , \tag{4}$$

or

$$A(k,\omega)W(k,\omega) = \frac{K}{L}\sum_{n=-\infty}^{\infty} W(k+\frac{2\pi n}{L},\omega) + F\delta(k-k_0) \ , \tag{5}$$

where

$$A(k,\omega) = -Tk^2 + \rho\omega^2 \ , \tag{6}$$

and k_0 is the wavenumber of excitation (rad/m). Applying the solution method previously derived for a thin plate to this problem [6,7], the solution of the displacement divided by the input force is

$$\frac{W(k,\omega)}{F} = A^{-1}(k,\omega)\left[\frac{\left(\frac{K}{L}\right)A^{-1}(k,\omega)}{1-\left(\frac{K}{L}\right)\sum_{m=-\infty}^{\infty} A^{-1}(k+\frac{2\pi m}{L},\omega)}\right] + A^{-1}(k,\omega) \ , \tag{7}$$

which, after some rearranging, becomes

$$\frac{W(k,\omega)}{F}=\frac{1}{(-Tk^2+\rho\omega^2)}\left\langle\frac{\dfrac{(K/L)}{(-Tk^2+\rho\omega^2)}+1-\displaystyle\sum_{n=-\infty}^{n=\infty}\frac{(K/L)}{\{-T[k+(2n\pi/L)]^2+\rho\omega^2\}}}{1-\displaystyle\sum_{n=-\infty}^{n=\infty}\frac{(K/L)}{\{-T[k+(2n\pi/L)]^2+\rho\omega^2\}}}\right\rangle . \quad (8)$$

4. SOLUTION WITH A POINT FORCING FUNCTION

The problem is next solved with a point forcing function, which is a continuous exponential harmonic function in time and a delta function in space, written as

$$f(x,t)=F\exp(\mathrm{i}\omega t)\delta(x-x_f) \ , \quad (9)$$

where x_f is the spatial location of the point force (m) and F is the magnitude of the point force (N). To differentiate this solution from the solution found in Section 3, the displacement variable is changed from $w(x,t)$ to $u(x,t)$. The system equation in the wavenumber-frequency domain, with $x_f=0$, becomes

$$(-k^2T+\rho\omega^2)U(k,\omega)=\frac{K}{L}\sum_{n=-\infty}^{n=\infty}U(k+\frac{2\pi n}{L},\omega)+F \ , \quad (10)$$

This equation is solved in the same manner as equations (4) – (8) and the result is

$$\frac{U(k,\omega)}{F}=\frac{1}{(-Tk^2+\rho\omega^2)}\left\langle\frac{1}{1-\displaystyle\sum_{n=-\infty}^{n=\infty}\frac{(K/L)}{\{-T[k+(2n\pi/L)]^2+\rho\omega^2\}}}\right\rangle . \quad (11)$$

5. TRANSFORMATION INTO ZERO-POLE FORM

To facilitate an understanding of this system, it is desirable to transform equations (8) and (11) into a single analytical expression that does not contain summations. This transformation is accomplished by rewriting the infinite series solution term as [11]

$$\sum_{n=-\infty}^{n=\infty}\frac{1}{\{-T[k+(2n\pi/L)]^2+\rho\omega^2\}}=\frac{-L\cos\left(\dfrac{\omega L}{2c}\right)\sin\left(\dfrac{\omega L}{2c}\right)}{2\omega\sqrt{T\rho}\left[\sin^2\left(\dfrac{kL}{2}\right)-\sin^2\left(\dfrac{\omega L}{2c}\right)\right]} \ , \quad (12)$$

where $c=\sqrt{T/\rho}$. Inserting equation (12) into equation (8) results in

$$\frac{W(k,\omega)}{F} = \frac{z_w(k,\omega)}{p(k,\omega)}, \tag{13}$$

where $z_w(k,\omega)$ is an expression that contains the system zeros for a continuous forcing function and $p(k,\omega)$ is an expression that contains the system poles in the wavenumber-frequency plane. These expressions are

$$z_w(k,\omega) = 2\omega\sqrt{T\rho}\left[\frac{K}{L(-Tk^2+\rho\omega^2)}+1\right]\left[\sin^2\left(\frac{kL}{2}\right)-\sin^2\left(\frac{\omega L}{2c}\right)\right]+K\cos\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega L}{2c}\right), \tag{14}$$

and

$$p(k,\omega) = (-Tk^2+\rho\omega^2)\left\{(2\omega\sqrt{T\rho})\left[\sin^2\left(\frac{kL}{2}\right)-\sin^2\left(\frac{\omega L}{2c}\right)\right]+K\cos\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega L}{2c}\right)\right\}, \tag{15}$$

Inserting equation (12) into equation (11) results in

$$\frac{U(k,\omega)}{F} = \frac{z_u(k,\omega)}{p(k,\omega)}, \tag{16}$$

where $z_u(k,\omega)$ is an expression that contains the system zeros for a point forcing function and $p(k,\omega)$ is an expression that contains the system poles in the wavenumber-frequency plane. The expressions for $z_u(k,\omega)$ is

$$z_u(k,\omega) = 2\omega\sqrt{T\rho}\,\sin\left(\frac{kL}{2}+\frac{\omega L}{2c}\right)\sin\left(\frac{kL}{2}-\frac{\omega L}{2c}\right). \tag{17}$$

The system pole expression for the discrete point loaded string is identical to the system pole expression for the continuous wavenumber loaded string.

The terms for the poles are now examined. The denominator in equation (13) has two distinct terms and both are set equal to zero. The first term is

$$-Tk^2+\rho\omega^2 = 0, \tag{18}$$

and this corresponds to the pole location of the string without stiffeners. This is referred to as the fundamental pole or resonance of the (unstiffened) string. When *K* is not equal to zero, the numerator *and* the denominator in equations (13) and (16) both approach zero when the relationship between wavenumber and frequency in equation (18) are satisfied, and thus the term given by equation (18) does *not* correspond to a pole of the stiffened system. The second term is

$$(2\omega\sqrt{T\rho})\left[\sin^2\left(\frac{kL}{2}\right)-\sin^2\left(\frac{\omega L}{2c}\right)\right]+K\cos\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega L}{2c}\right) = 0, \tag{19}$$

which can be rewritten as

$$\sin\left(\frac{kL}{2}\right) = \sqrt{\sin^2\left(\frac{\omega L}{2c}\right) - \frac{K}{2\omega\sqrt{T\rho}}\cos\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega L}{2c}\right)} = \sqrt{\theta} \ , \tag{20}$$

Note that for the equality in equation (20) to hold true, it is required that

$$0 \le \sin^2\left(\frac{\omega L}{2c}\right) - \frac{K}{2\omega\sqrt{T\rho}}\cos\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega L}{2c}\right) \le 1 \ , \tag{21}$$

and this occurs when

$$\tan\left(\frac{\omega L}{2c} - \frac{m\pi}{2}\right) \le \frac{K}{2\omega\sqrt{T\rho}} \ , \tag{22}$$

where m is the largest integer such that

$$0 \le \frac{\omega L}{2c} - \frac{m\pi}{2} \le \frac{\pi}{2} \ . \tag{23}$$

Therefore, the inequality given by equation (22) is a necessary condition for poles to exist in the wavenumber-frequency plane for the stiffened system. This expression is not a function of wavenumber. Figure 2 is a plot of equation (22) where the x-axis is the term $K/(2\omega\sqrt{T\rho})$ and the y-axis is nondimensional frequency $\omega L/2c$. The grey area of the plot is a region that corresponds to a location where poles may be present in the system response and the white area is the region where poles cannot exist. When the inequality in equation (22) is satisfied, the poles of the stiffened system will reside at

$$\hat{k}_n^p = \pm\frac{2}{L}\left| \arcsin\left(\sqrt{\theta}\right) + n\pi \right| \qquad n = \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \ , \tag{24}$$

where $\hat{k}_n^p$ is the location of the n indexed poles (rad/m).

The zeros of the system with a wavenumber forcing function are found by setting the numerator of equation (13) equal to zero. This results in

$$z_w(k,\omega) = 2\omega\sqrt{T\rho}\left[\frac{K}{L(-Tk^2+\rho\omega^2)} + 1\right]\left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{\omega L}{2c}\right)\right] + K\cos\left(\frac{\omega L}{2c}\right)\sin\left(\frac{\omega L}{2c}\right) = 0 \tag{25}$$

which is transcendental in wavenumber and frequency. The zeros of the system with a discrete forcing function are found by setting the numerator of equation (16) equal to zero. This results in

$$\hat{k}_n^u = \pm\left|\frac{\omega}{c} + \frac{2n\pi}{L}\right| \qquad n = \cdots, -3, -2, -1, 1, 2, 3, \cdots \ , \tag{26}$$

where $\hat{k}_n^u$ is the location of the n indexed zeros (rad/m) for the system with point forcing. The value of n = 0 is not a zero because the fundamental pole exists at this location which negates the effects of this specific zero. Note that the zero location given in equation (26) is independent of the value of the stiffener, *K*.

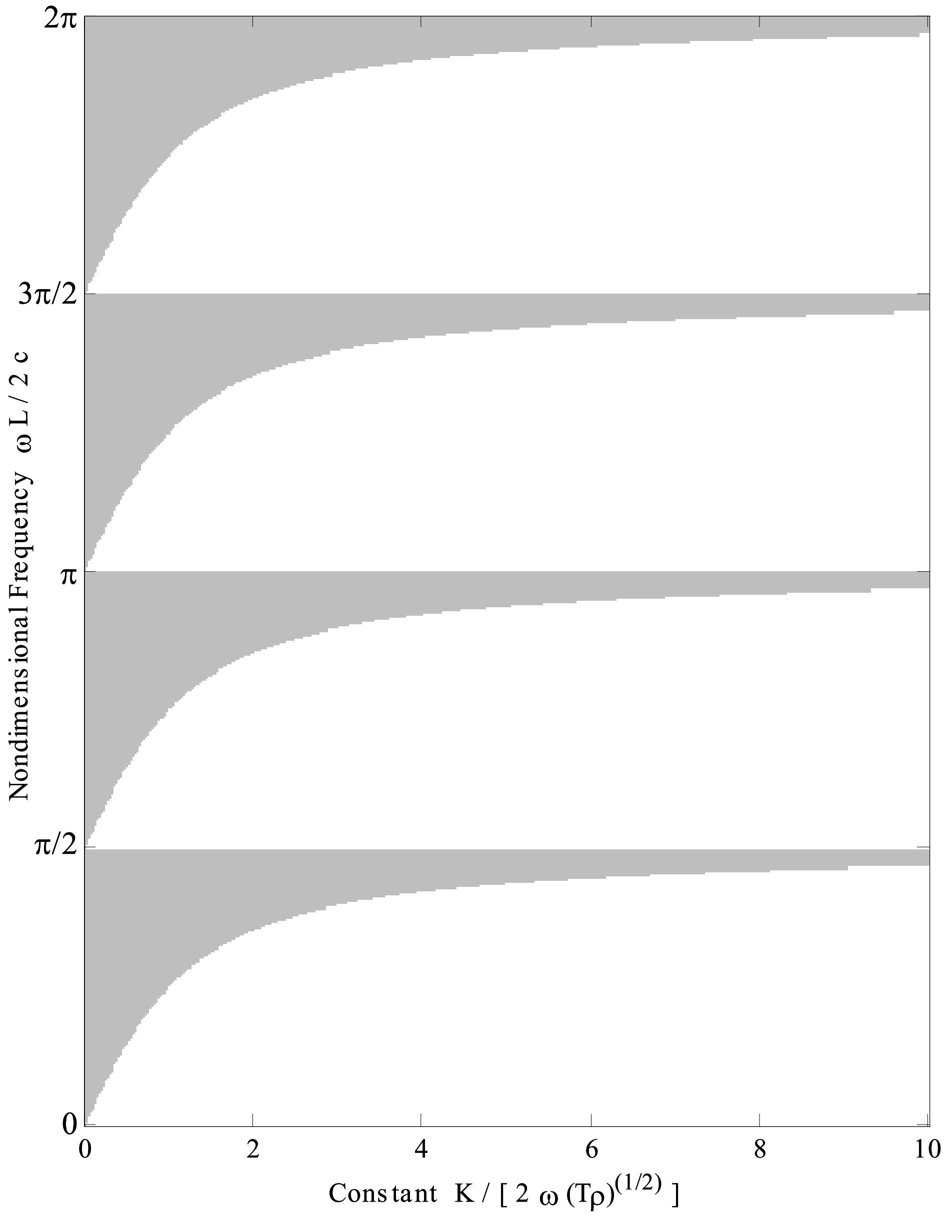

Figure 2. Regions of pole existence (grey) and non-existence (white).

6. CONCLUSIONS

The zero-pole response of a tensioned, reinforced string has been derived in the wavenumber-frequency domain for wavenumber and point forcing functions. The stability of the system for various stiffener values has been demonstrated. It was shown that there are regions where the poles can and cannot exist. It was also shown that zeros for wavenumber forcing are almost collocated with the poles and the zeros for point forcing can exist almost everywhere.

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