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A NON-LINEAR MODEL FOR STUDYING THE MOTION OF A HUMAN BODY

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Abstract

In our paper we developed a non-linear model for studying the motions of a human body. For our model we obtained the equations of motion using the Newton's equations. We also treated a few aspects concerning the stability of the equilibrium and motion.

1. INTRODUCTION

The mechanical model is captured in figure 1.

We made the following notations: m_i, $i=\overline{1,4}$ - mass of the body i; C_i, $i=\overline{1,4}$ - weight center of the body i; J_i, $i=\overline{3,4}$ - inertial moment of the body i with respect to the weight center C_i; h - distance between C_1 and C_2 measured in the direction of the axis y; $C_2C_3=l_3'$; $D_4E_4=L_4$; $C_2D_3=l_3''$; $D_4C_4=l_4$; $C_2D_4=L_3$; $F_{e_{01}}$ - elastic force in the spring between the bodies 0 and 1; $M_{e_{23}}$ - elastic moment in the spring between the bodies 2 and 3; $M_{e_{34}}$ - elastic moment in the spring between the bodies 3 and 4; $F_{e_{12}}$ elastic force in the spring between the bodies 1 and 2; $F_{e_{13}}$ elastic force in the spring between the bodies 1 and 3; d - distance between C_1 and C_2 measured in the direction of the axis x and for unstressed springs; F_f - coulombian friction force between the bodies 1 and 2; N_{01} - normal reaction between the bodies 0 and 1; N_{12} - normal reaction between the bodies 1 and 2. We shall consider that all elastic forces and moments are equal to zero for zero strains.

2. ISOLATION OF THE BODIES

For the fourth body we obtain the following equations (fig. 2):

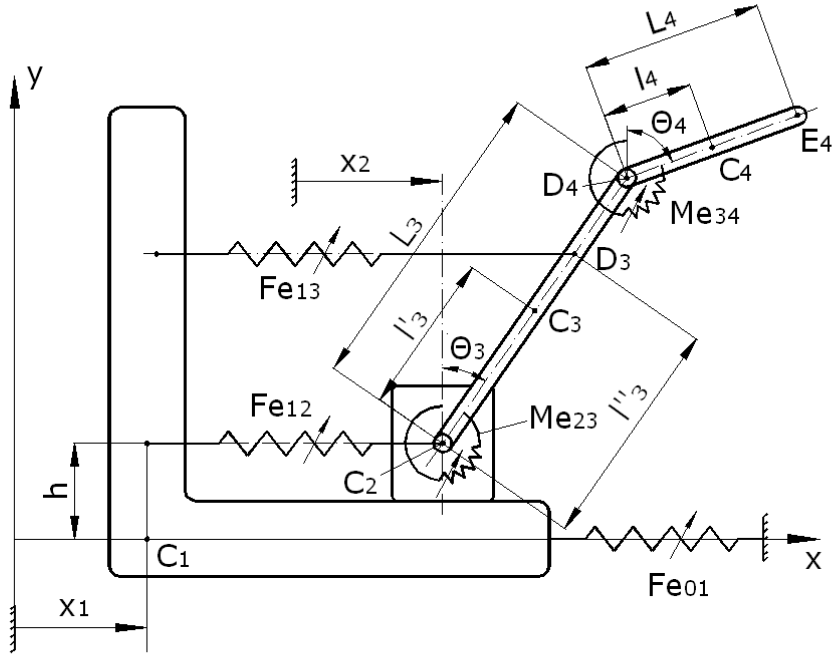

Figure 1. Mechanical model.

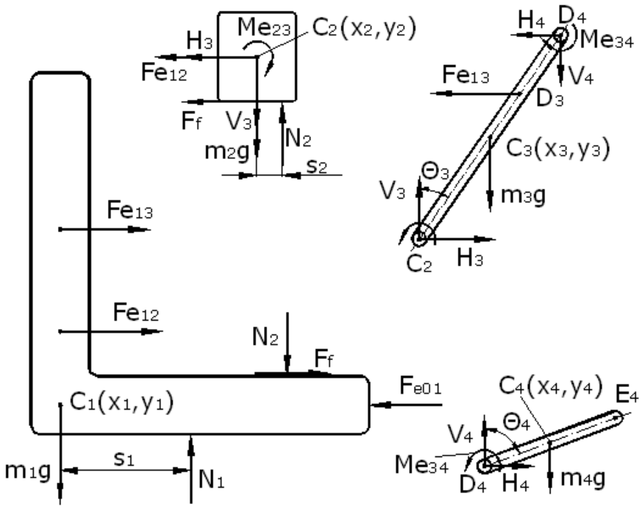

Figure 2. Isolation of the bodies.

$$m_4\ddot{x}_4 = H_4;\ m_4\left(\ddot{y}_4 + g\right) = V_4;\ J_4\ddot{\theta}_4 = -H_4 l_4 \cos\theta_4 + V_4 l_4 \sin\theta_4 - M_{e_{34}}\left(\theta_4 - \theta_3\right). \qquad (1)$$

For the third body we obtain:

$$\begin{gathered} m_3\ddot{x}_3 = H_3 - H_4 - F_{e_{13}}\left(x_2 + l_3''\sin\theta_3\right);\ m_3\left(\ddot{y}_3 + g\right) = V_3 - V_4; \\ J_3\ddot{\theta}_3 = -H_3 l_3' \cos\theta_3 - H_4\left(L_3 - l_3'\right)\cos\theta_3 + V_3 l_3' \sin\theta_3 + V_4\left(L_3 - l_3'\right)\sin\theta_3 + \\ + M_{e_{34}}\left(\theta_4 - \theta_3\right) - M_{e_{23}}\left(\theta_3\right) - F_{e_{13}}\left(x_2 + l_3''\sin\theta_3\right)\left(l_3'' - l_3'\right)\cos\theta_3 = 0. \end{gathered} \qquad (2)$$

For the second body and the first body it follows

$$m_2\ddot{x}_2 = -H_3 - \mu N_2 - F_{e_{12}}(x_2);\ 0 = N_2 - V_3 - m_2 g\,, \tag{3}$$

respectively

$$m_1\ddot{x}_1 = \mu N_2 + F_{e_{12}}(x_2) + F_{e_{13}}(x_2 + l_3''\sin\theta_3) - F_{e_{01}}(x_1);\ 0 = N_1 - m_1 g - N_2\,. \tag{4}$$

3. OBTAINING THE SECOND ORDER DIFFERENTIAL EQUATION SYSTEM

It is easy to obtain the following relations:

$$J_4\ddot{\theta}_4 = -m_4\ddot{x}_4 l_4\cos\theta_4 + m_4(\ddot{y}_4 + g)l_4\sin\theta_4 - M_{e_{34}}(\theta_4 - \theta_3) \tag{5}$$

$$\begin{aligned} J_3\ddot{\theta}_3 = &-\left[m_3\ddot{x}_3 + m_4\ddot{x}_4 + F_{e_{13}}(x_2 + l_3''\sin\theta_3)\right]l_3'\cos\theta_3 - m_4\ddot{x}_4(L_3 - l_3')\cos\theta_3 + \\ &+ (m_3\ddot{y}_3 + m_3 g + m_4\ddot{y}_4 + m_4 g)l_3'\sin\theta_3 + (m_4\ddot{y}_4 + m_4 g)(L_3 - l_3')\sin\theta_3 + \\ &+ M_{e_{34}}(\theta_4 - \theta_3) - M_{e_{23}}(\theta_3) - F_{e_{13}}(x_2 + l_3''\sin\theta_3)(l_3'' - l_3')\cos\theta_3 = 0\,, \end{aligned} \tag{6}$$

$$\begin{aligned} m_2\ddot{x}_2 = &-\left[m_3\ddot{x}_3 + m_4\ddot{x}_4 + F_{e_{13}}(x_2 + l_3''\sin\theta_3)\right] - \\ &- \mu\left[m_3\ddot{y}_3 + m_4\ddot{y}_4 + (m_2 + m_3 + m_4)g\right] - F_{e_{12}}(x_2)\,, \end{aligned} \tag{7}$$

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 + m_3\ddot{x}_3 + m_4\ddot{x}_4 = -F_{e_{01}}(x_1)\,. \tag{8}$$

On the other hand

$$\begin{aligned} &\ddot{x}_3 = \ddot{x}_1 + \ddot{x}_2 + l_3'\ddot{\theta}_3\cos\theta_3 - l_3'\dot{\theta}_3^2\sin\theta_3;\ \ddot{y}_3 = -l_3'\ddot{\theta}_3\sin\theta_3 - l_3'\dot{\theta}_3^2\cos\theta_3; \\ &\ddot{x}_4 = \ddot{x}_1 + \ddot{x}_2 + L_3\ddot{\theta}_3\cos\theta_3 - L_3\dot{\theta}_3^2\sin\theta_3 + l_4\ddot{\theta}_4\cos\theta_4 - l_4\dot{\theta}_4^2\sin\theta_4; \\ &\ddot{y}_4 = -L_3\ddot{\theta}_3\sin\theta_3 - L_3\dot{\theta}_3^2\cos\theta_3 - l_4\ddot{\theta}_4\sin\theta_4 - l_4\dot{\theta}_4^2\cos\theta_4 \end{aligned} \tag{9}$$

and replacing in the equations (5)-(8) we obtain the system

$$\begin{aligned} &A_{11}\ddot{x}_1 + A_{12}\ddot{x}_2 + A_{13}\ddot{\theta}_3 + A_{14}\ddot{\theta}_4 = B_1;\ A_{21}\ddot{x}_1 + A_{22}\ddot{x}_2 + A_{23}\ddot{\theta}_3 + A_{24}\ddot{\theta}_4 = B_2; \\ &A_{31}\ddot{x}_1 + A_{32}\ddot{x}_2 + A_{33}\ddot{\theta}_3 + A_{34}\ddot{\theta}_4 = B_3;\ A_{41}\ddot{x}_1 + A_{42}\ddot{x}_2 + A_{43}\ddot{\theta}_3 + A_{44}\ddot{\theta}_4 = B_4\,, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &A_{11} = m_4 l_4\cos\theta_4;\ A_{12} = m_4 l_4\cos\theta_4; \\ &A_{13} = m_4 L_3\cos\theta_3 l_4\cos\theta_4 + m_4 L_3\sin\theta_3 l_4\sin\theta_4 = m_4 L_3 l_4\cos(\theta_4 - \theta_3); \\ &A_{14} = J_4 + m_4 l_4^2\cos^2\theta_4 + m_4 l_4^2\sin^2\theta_4 = J_4 + m_4 l_4^2\,, \end{aligned} \tag{11}$$

$$\begin{aligned} &A_{21} = (m_3 + m_4)l_3'\cos\theta_3 + m_4(L_3 - l_3')\cos\theta_3; \\ &A_{22} = (m_3 + m_4)l_3'\cos\theta_3 + m_4(L_3 - l_3')\cos\theta_3;\ A_{23} = J_3 + m_3(l_3')^2 + m_4 L_3^2; \\ &A_{24} = m_4 l_4 l_3'\cos(\theta_4 - \theta_3) + m_4 l_4(L_3 - l_3')\cos(\theta_4 - \theta_3)\,, \end{aligned} \tag{12}$$

$$\begin{gathered} A_{31} = m_3 + m_4\,;\; A_{32} = m_2 + m_3 + m_4\,; \\ A_{33} = m_3 l_3' \cos\theta_3 + m_4 L_3 \cos\theta_3 - \mu m_3 l_3' \sin\theta_3 - \mu m_4 L_3 \sin\theta_3\,; \\ A_{34} = m_4 l_4 \cos\theta_4 - \mu m_4 l_4 \sin\theta_4\,, \end{gathered} \tag{13}$$

$$\begin{gathered} A_{41} = m_1 + m_3 + m_4\,;\; A_{42} = m_2 + m_3 + m_4\,;\; A_{43} = m_3 l_3' \cos\theta_3 + m_4 L_3 \cos\theta_3\,; \\ A_{44} = m_4 l_4 \cos\theta_4\,, \end{gathered} \tag{14}$$

$$\begin{gathered} B_1 = m_4 L_3 l_4 \dot\theta_3^2 \sin(\theta_3 - \theta_4) + m_4 g l_4 \sin\theta_4 - M_{e34}(\theta_3 - \theta_4)\,; \\ B_2 = m_4 l_4 L_3 \dot\theta_4^2 \sin(\theta_4 - \theta_3) - F_{e13}(x_2 + l_3'' \sin\theta_3) l_3'' \cos\theta_3 + \\ + m_3 g l_3' \sin\theta_3 + m_4 g L_3 \sin\theta_3 + M_{e34}(\theta_4 - \theta_3) - M_{e23}(\theta_3)\,; \\ B_3 = m_3 l_3' \dot\theta_3^2 (\sin\theta_3 - \mu\cos\theta_3) + m_4 L_3 \dot\theta_3^2 (\sin\theta_3 + \mu\cos\theta_3) + \\ + m_4 l_4 \dot\theta_4^2 (\sin\theta_4 + \mu\cos\theta_4) - F_{e13}(x_2 + l_3'' \sin\theta_3) - \\ - \mu(m_2 + m_3 + m_4) g - F_{e12}(x_2)\,; \\ B_4 = m_3 l_3' \dot\theta_3^2 \sin\theta_3 + m_4 L_3 \dot\theta_3^2 \sin\theta_3 + m_4 l_4 \dot\theta_4^2 \sin\theta_4 - F_{01}(x_1)\,; \end{gathered} \tag{15}$$

hence $A_{ij} = A_{ji}(x_1, x_2, \theta_3, \theta_4, \dot x_1, \dot x_2, \dot\theta_3, \dot\theta_4)$, $B_i = B_i(x_1, x_2, \theta_3, \theta_4, \dot x_1, \dot x_2, \dot\theta_3, \dot\theta_4)$, $i = \overline{1,4}$, $j = \overline{1,4}$.

The system (10) leads to

$$\begin{gathered} \ddot x_1 = f_1(x_1, x_2, \theta_3, \theta_4, \dot x_1, \dot x_2, \dot\theta_3, \dot\theta_4);\; \ddot x_2 = f_2(x_1, x_2, \theta_3, \theta_4, \dot x_1, \dot x_2, \dot\theta_3, \dot\theta_4); \\ \ddot\theta_3 = f_3(x_1, x_2, \theta_3, \theta_4, \dot x_1, \dot x_2, \dot\theta_3, \dot\theta_4);\; \ddot\theta_4 = f_4(x_1, x_2, \theta_3, \theta_4, \dot x_1, \dot x_2, \dot\theta_3, \dot\theta_4). \end{gathered} \tag{16}$$

Denoting

$$x_1 = \xi_1\,;\; x_2 = \xi_2\,;\; \theta_3 = \xi_3\,;\; \theta_4 = \xi_4\,;\; \dot x_1 = \xi_5\,;\; \dot x_2 = \xi_6\,;\; \dot\theta_3 = \xi_7\,;\; \dot\theta_3 = \xi_8\,, \tag{17}$$

it results the first order non-linear differential equations system

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \xi_{i+4}\,;\; i = \overline{1,4}\,;\; \frac{\mathrm{d}\xi_i}{\mathrm{d}t} = f_{i-4}(\xi_1, \ldots, \xi_8);\; i = \overline{5,8}\,. \tag{18}$$

Let us observe that the system (18) is valuable for $N_2 \geq 0$, otherwise the system has five degrees of freedom.

Our system has four degrees of freedom : x_1, x_2, θ_3 and θ_4.

4. THE STUDY OF THE EQUILIBRIUM

In this case, we obtain

$$\dot x_1 = 0\,;\; \dot x_2 = 0\,;\; \dot\theta_3 = 0\,;\; \dot\theta_4 = 0\,. \tag{19}$$

We study the equilibrium for the elastic forces and moments given by linear relations

$$F_{e_{12}}(x_2) = k_{12}x_2;\ F_{e_{13}}(x_2 + l_3''\sin\theta_3) = k_{13}(x_2 + l_3''\sin\theta_3);$$
$$M_{e_{34}}(\theta_4 - \theta_3) = k_{34}(\theta_4 - \theta_3);\ M_{e_{23}}(\theta_3) = k_{23}(\theta_3), \tag{20}$$

the bars are homogeneous, and $l_3' = \dfrac{L_3}{2} = l_3''$, $l_4 = \dfrac{L_4}{2}$, $\mu = 0$.

It follows the system

$$A_{11}\ddot{x}_1 + A_{11}\ddot{x}_2 + A_{13}\ddot{\theta}_3 + A_{14}\ddot{\theta}_4 = B_1;\ A_{21}\ddot{x}_1 + A_{21}\ddot{x}_2 + A_{23}\ddot{\theta}_3 + A_{13}\ddot{\theta}_4 = B_2;$$
$$A_{31}\ddot{x}_1 + A_{32}\ddot{x}_2 + A_{21}\ddot{\theta}_3 + A_{34}\ddot{\theta}_4 = B_3;\ A_{41}\ddot{x}_1 + A_{32}\ddot{x}_2 + A_{21}\ddot{\theta}_3 + A_{34}\ddot{\theta}_4 = B_4. \tag{21}$$

Subtracting the third and the fourth equation (21) term by term, it results

$$(A_{31} - A_{41})\ddot{x}_1 = B_3 - B_4 \Rightarrow \ddot{x}_1 = \frac{B_3 - B_4}{A_{31} - A_{41}}. \tag{22}$$

We obtain a system of three equations with three unknowns

$$A_{11}\ddot{x}_2 + A_{13}\ddot{\theta}_3 + A_{14}\ddot{\theta}_4 = B_1 - A_{11}\frac{B_3 - B_4}{A_{31} - A_{41}} = D_1;$$
$$A_{21}\ddot{x}_2 + A_{23}\ddot{\theta}_3 + A_{13}\ddot{\theta}_4 = B_2 - A_{21}\frac{B_3 - B_4}{A_{31} - A_{41}} = D_2; \tag{23}$$
$$A_{32}\ddot{x}_2 + A_{21}\ddot{\theta}_3 + A_{34}\ddot{\theta}_4 = B_3 - A_{31}\frac{B_3 - B_4}{A_{31} - A_{41}} = D_3.$$

Denoting

$$\Delta = \begin{vmatrix} A_{11} & A_{13} & A_{14} \\ A_{21} & A_{23} & A_{13} \\ A_{32} & A_{21} & A_{34} \end{vmatrix};\ \Delta_{x_2} = \begin{vmatrix} D_1 & A_{13} & A_{14} \\ D_2 & A_{23} & A_{13} \\ D_3 & A_{21} & A_{34} \end{vmatrix};\ \Delta_{\theta_3} = \begin{vmatrix} A_{11} & D_1 & A_{14} \\ A_{21} & D_2 & A_{24} \\ A_{32} & D_3 & A_{34} \end{vmatrix};$$
$$\Delta_{\theta_4} = \begin{vmatrix} A_{11} & A_{13} & D_1 \\ A_{21} & A_{23} & D_2 \\ A_{32} & A_{21} & D_3 \end{vmatrix}, \tag{24}$$

one obtains the equilibrium conditions

$$\Delta_{x_2} = 0;\ \Delta_{\theta_3} = 0;\ \Delta_{\theta_4} = 0 \tag{25}$$

or, equivalently,

$$D_1(A_{23}A_{34} - A_{21}A_{13}) - D_2(A_{13}A_{34} - A_{21}A_{14}) + D_3(A_{13}A_{13} - A_{23}A_{14}) = 0;$$
$$-D_1(A_{21}A_{34} - A_{32}A_{13}) + D_2(A_{11}A_{34} - A_{32}A_{14}) - D_3(A_{11}A_{13} - A_{21}A_{14}) = 0; \tag{26}$$
$$D_1(A_{21}^2 - A_{23}A_{32}) - D_2(A_{11}A_{21} - A_{32}A_{13}) + D_3(A_{11}A_{32} - A_{21}A_{13}) = 0.$$

It results two situations: either $D_i = 0$, $i = \overline{1,3}$, or the determinant of the system in the unknowns D_i, $i = \overline{1,3}$ is equal to zero. The second situation would imply that the equations (23) are not linear independent which is absurd. It remains

$$D_i = 0,\ i = \overline{1,3} \tag{27}$$

From the definition of the parameters D_i, $i = \overline{1,3}$, it follows the equilibrium positions given by

$$x_1 = 0;\ x_2 = 0;\ \theta_3 = 0;\ \theta_4 = 0;\ \dot{x}_1 = 0;\ \dot{x}_2 = 0;\ \dot{\theta}_3 = 0;\ \dot{\theta}_4 = 0. \tag{28}$$

The study of the stability of this equilibrium position will be made numerically, considering a deviated position characterized by ξ_i, $i = \overline{1,8}$ with $|\xi_i|$ sufficiently small.

We selected the following values:

$$\begin{gathered} m_1 = 1000[\text{kg}];\ m_2 = 26.68[\text{kg}];\ m_3 = 46.06[\text{kg}];\ m_4 = 5.52[\text{kg}];\ L_3 = 0.427[\text{m}]; \\ L_4 = 0.24[\text{m}];\ k_{34} = 180\left[\text{Nm}/\text{rad}\right] \text{ for } \theta_4 \geq \theta_3, \text{ otherwise } k_{34} = 300\left[\text{Nm}/\text{rad}\right]; \\ k_{23} = 350\left[\text{Nm}/\text{rad}\right] \text{ for } \theta_3 \geq 0, \text{ otherwise } k_{23} = 1000\left[\text{Nm}/\text{rad}\right];\ k_{12} = 600000\left[\text{N}/\text{m}\right]; \\ k_{13} = 600000\left[\text{N}/\text{m}\right];\ k_{01} = 800000\left[\text{N}/\text{m}\right]. \end{gathered} \tag{29}$$

The step time is $\Delta t = 0.001[\text{s}]$, and the initial values are

$$\theta_4^0 = \frac{\pi}{360}[\text{rad}];\ \dot{\theta}_3^0 = 0.001\left[\text{rad}/\text{s}\right]. \tag{30}$$

The graphics are plotted in the next figures.

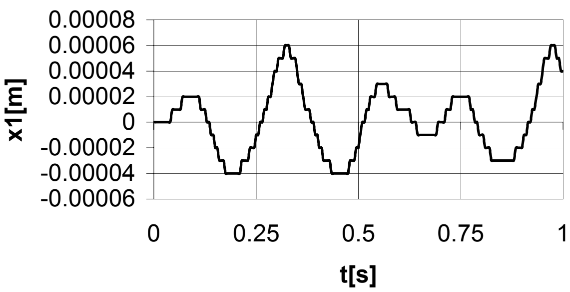

Figure 3. Time history for x_1.

It is easy to observe the quasi periodicity of the diagrams, so the equilibrium is a simply stable one.

The equilibrium becomes if $k_{12} = 0$ for $x_2 < 0$.

The domain of stability increases around the equilibrium position (28) if the friction coefficient has a non-zero value.

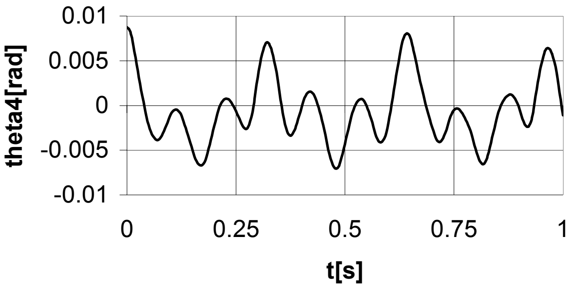

Figure 4. Time history for θ_4.

5. CONCLUSIONS

In our paper we presented a four degrees of freedom model for study the motion oh a human body. This model characterizes the behavior of the human body in a car. We obtained the equations of motion and we study the stability of the equilibrium position in a particular case.

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