DYNAMIC BEHAVIOUR OF MECHANICAL VARIATORS WITH HALF BALL AS NON-HOLONOMIC SYSTEMS

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ABSTRACT
Different kinds of mechanical variators are used as a part of complex machine system. Variators, as transmission system , with a changeable transmission ration, are involved in a lot of complex machines. They are used for changing speed in agricultural machines, industries of cable, carpet and paper industries, mining machines, account machines, etc. Seeing this large use of mechanical variators in industry, the aim of this paper is to describe a dynamical behavior of the general example of the variators of speed as non-holonomic system. In this paper a frontal variator of speed with two discs, half ball and regulator will be analyzed. The non-holonomic connection is in points of physical contact between the discs and half ball. To this system is added mechanical regulator for regulation of variable transmission relation between input and output elements. A damper is added for stabilizing movement. Differential equations of moving will be solved by using Appell's equations and by resolving the numerical method. In this way, we are getting an answer to a dynamical and kinematical behavior of mechanical systems under the give us the answer to a working stability a system being observed.
Keywords*: dynamic, kinematic, non-holonomic system, variator*

1. INTRODUCTION

Different kinds of mechanical speed variators are as part of complex machine systems. They are used for changing speed in agricultural machines, cutting machines the cable, carpet and paper industries, mining machines, account machines, etc.
Seeing this large use of mechanical variations in industry, the aim of this paper is to give the dynamic analysis of a general example of this class of machine element. These mechanical systems are non-holominc, because the connections is differential. Holonomic systems have connections which are functions of speed and acceleration. In variators the constraints are

described by differential equations with which it is not possible by integration to deduce geometric characteristics. This is the basis of non-holonomic mechanics and gives the difference from holonomic mechanics where are all connections have geometric characteristic and there are no limits of speed and acceleration for the system. The practical use of mechanical non-holonomic systems is only beginning but the theory is very developed.
In this paper a frontal frictional variator of speed with two discs, half ball and Watt`s regulator will be analyzed. The non-holonomic connection is in points of physical contact between the discs and half ball. To this system is added Watt`s regulator for regulation of the variable transmission relation between input and output elements. A damper is added for stabilizing movement. During dynamically analysis Appell`s equations are used. As a result differential equations of movement are obtained which describe the mechanical non-holonomic system. In many cases it is not possible to solve these differential equations and they must be solved on digital computers using numerical methods.

2. METHOD FOR DYNAMIC ANALYSIS OF NON-HOLONOMIC SYSTEMS

Appell`s equations, which are very suitable for dynamic analysis are given in the well known form:

$$\frac{\partial S^*}{\partial \ddot{q}_\nu} = Q_\nu^* \qquad (\nu = 1,2, ..., p) \tag{1}$$

were:
S* - energy of acceleration (a function of $\ddot{q}_\nu$ only)
$\ddot{q}_\nu$ - generalized acceleration
Q_ν^* - generalized force
The energy of acceleration is given by the relations:

$$S = \sum_{j=1}^{N} \frac{m_j \vec{a}_j^{\,2}}{2} = \frac{1}{2}\sum_{j=1}^{N} m_j (\ddot{\vec{r}}_j)^2 \tag{2}$$

where:
$\ddot{\vec{r}}_j$ - acceleration of material points j with mass m_j

$$Q_\nu^* = \sum_{j=1}^{N} \vec{F}_j \vec{A}_{j\nu} \tag{3}$$

where:
$\vec{F}_j$ - external force in material point j

$$\vec{A}_{j\nu} = \frac{\partial \ddot{\vec{r}}_j}{\partial \ddot{q}_\nu} \tag{4}$$

$\vec{A}_{j\nu}$ - Appell`s vector
The relationship between functional and nonfunctional generalized accelerations and velocities is given as:

$$\dot{q}_h = \sum_{\nu=1}^{p} b_{h\nu} \dot{q}_\nu + b_h \cdot dt;$$

$$\ddot{q}_h = \sum_{\nu=1}^{p} b_{h\nu} \ddot{q}_\nu + E_h(q_i, \dot{q}_i, t); \tag{5}$$

where:

$\dot{\pi}_h, \ddot{\pi}_h$ - functional generalized velocity and acceleration
E_h - all terms without second derivative of generalized coordinates q_i

Then for all h and v:

$$b_h = \frac{\partial \ddot{\pi}_h}{\partial \ddot{\pi}_v} \qquad (6)$$

The value for $\ddot{\vec{r}}_j$ is:

$$\ddot{\vec{r}}_j = \sum_{v=1}^{p} \vec{A}_{jv}\, \ddot{\pi}_v + \vec{E}^*(q_i, \dot{q}_i, t); \qquad (p = n - l) \qquad (7)$$

where vector E* is without differential of generalized coordinates ($\ddot{\pi}_v$).

$$\sum_{j=1}^{n} A_{\rho i}\dot{q}_i + A_\rho = 0 \;\; (\rho = 1,2,\ \ldots,\ l) \qquad (8)$$

where $A_{\rho i}$ and A_ρ are coefficients in function of q_i and t.
In practice it is necessary to write the energy of accelerations S* as a function of the nonfunctional generalized accelerations $\ddot{\pi}_v$ and then to find Q_v^* and put it in eq. (1).

3. MECHANICAL VARIATOR WITH HALF BALL AS NON-HOLONOMIC SYSTEM

The frictional variator with half ball is shown in fig. 1. Variation of angular speed by this variator is due to a change of position of contact point A or B caused by rotation of the half ball trough on angle in one or other direction. In this way the distance between the contact point and rotating axis is changed and gives the variable transmission relation between elements 1 and 2. The closed automatic regulation is made such that the slider is connected at point D. The torsion elastic element c_1 is located at point O_3 where the half ball rotates. The work of elastic element c_1 is proportional to that of elastic element c_2. This type of variator is good for large ranges of regulation.

Symbols in fig. 1 are:

r_1 and r_2	- radiuses of leading and leaded disc r_1-r_2
r	- radius of half ball 3
r_3	- radius of gear z_3
x and x_o	- position and start position of slider D of regulator
α and α_o	- angle and start angle of half ball 3 around the axle O_3
γ	- angular position of contact points A and B to vertical axle
δ	- angle of position for regulators bar

Δl - start load of elastic element
c - elasticity of elastic element
z_1, z_2, z_3, z_4 and z_L - number of teeth of gears and lath
i_r - transmission relation of reductor
$\dot\varphi_1, \dot\varphi_2, \dot\varphi_3, \dot\alpha$ - angular speeds of leading, leaded discs, balls, and rotating around axle O_3
m_D - mass of slider in point D
m_N - mass of ball of regulator
J_1, J_2 - inertial moments of leading and leaded discs
J_3 - inertial moment of half ball
J_{o3} - inertial moment for axle O_3 normal to the plane of drawing
J_r - reduced inertial moment of all rotating masses of working elements on the axle O_2 of element 2
M_1 and M_2 - moments on leading and leaded elements $M_1 = M_e$ $M_2 = M_k\ i_o$
i_o - transmission relation to shaft O_2 and working element $i_o = i_{12}\ \ i_r$
i_{12} and i_r - transmission relation between gears z_1 and z_2 and reductor
$\dot x$ - speed of regulation, change of position for slider D

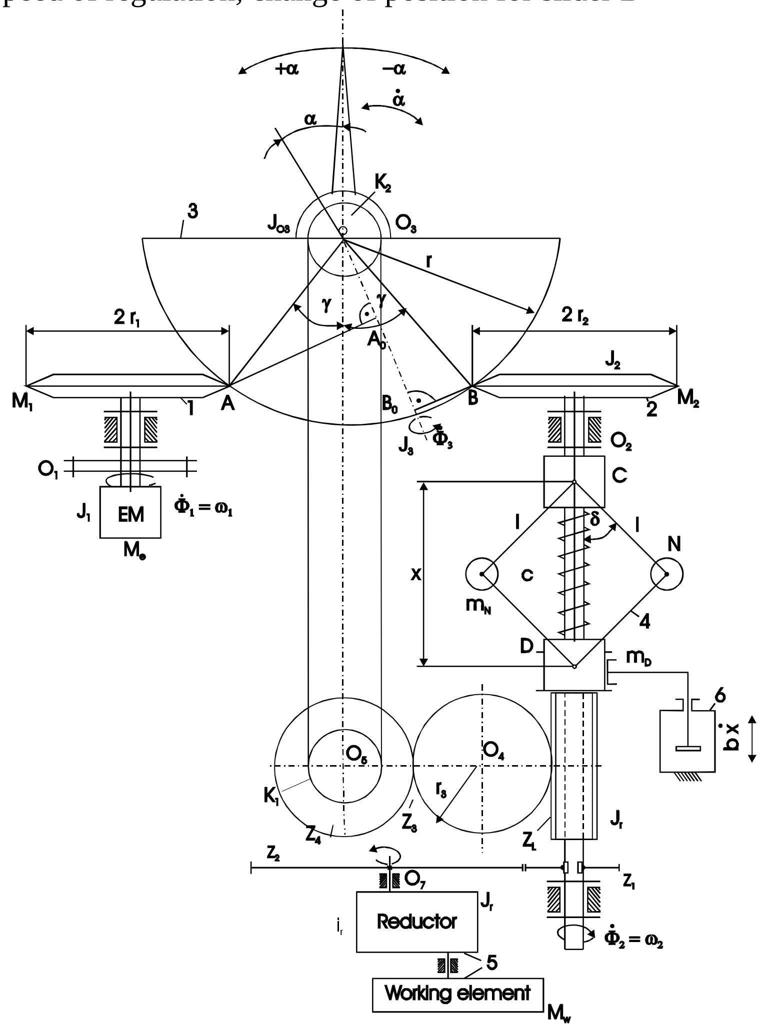

Fig. 1. Close mechanical regulating system with variator with half ball
1. leading disc; 2. leaded disc; 3. half ball; 4. regulator;
5. reductor with working element; 6. damper

Dynamically analysis will be done by Appell's equations for non-holonomic systems. This system has three generalized coordinates ϕ_1, ϕ_2 and x. The general form of Appell's equations is:

$$\frac{\partial S}{\partial \ddot{\phi}_2} = Q_{\phi_2} \quad ; \quad \frac{\partial S}{\partial \ddot{x}} = Q_x \qquad (9)$$

where:

S - energy of acceleration in function of non-functional generalized accelerations

$\ddot{\phi}_2$ and $\ddot{x}$ - non-functional generalized accelerations

Q_{ϕ_2} and Q_x - generalized forces in function of non-functional generalized coordinates and speeds.

Non-holonomic connection is:

$$\phi = \dot{\phi}_1 - i_v \cdot \dot{\phi}_2 = 0 \qquad (10)$$

where:

$$i_v = \frac{\sin(\gamma + \alpha)}{\sin(\gamma - \alpha)} \qquad (11)$$

The energy of acceleration is:

$$S = \frac{1}{2} J_1 \ddot{\phi}_1^2 + \frac{1}{2} J_3 \ddot{\phi}_3^2 + \frac{1}{2} J_{o3} \ddot{\alpha}^2 + \frac{1}{2} (J_2 + J_r) \ddot{\phi}_2^2 + \frac{1}{2} m_D \ddot{x}^2 + 2 S_k \qquad (12)$$

where:

$\ddot{\phi}_1, \ddot{\phi}_2, \ddot{\phi}_3, \ddot{\alpha}$ - angular acceleration for axles (fig. 1)

$\ddot{x}$ - acceleration of point D

S_k - energy of acceleration of ball

$$S_K = \frac{1}{2} a_N^2 m_N ; \qquad \vec{a}_N = \vec{a}_{pN} + \vec{a}_{pT} + \vec{a}_{rN} + \vec{a}_{rT} + \vec{a}_K \qquad (13)$$

where:

$\vec{a}_{pN}$ and $\vec{a}_{pT}$ - normal and tangential transmission acceleration of point N

$\vec{a}_{rN}$ and $\vec{a}_{rT}$ - relative normal and tangential acceleration of point N

$\vec{a}_k$ - Coriolis`s acceleration

$$a_{pN} = l \, \dot{\phi}_2^2 \sin\delta \qquad a_{rN} = l \, \dot{\delta}^2 \qquad a_k = 2 \cdot \dot{\phi}_2 \, l \, \dot{\delta} \cos\delta$$

$$a_{pT} = l \, \ddot{\phi}_2 \sin\delta \qquad a_{rT} = l \, \ddot{\delta} \qquad (14)$$

Square or acceleration for point N is:

$$a_N^2 = (a_{pN} \sin\delta + a_{rN})^2 + (a_{rT} - a_{pN} \cos\delta)^2 + (a_{pT} + a_K)^2 \qquad (15)$$

Geometrical connections are:

$$x - x_o = (\alpha - \alpha_o) r_3 \quad ; \quad \cos\delta = \frac{x}{2l} \qquad (16)$$

Differenting eq. (16) is:

$$\dot{\alpha} = \frac{\dot{x}}{r_3}$$

$$\dot{\delta} = - \frac{\dot{x}}{2 l \sin\delta} = - \frac{\dot{x}}{(4 l^2 - x^2)^{1/2}} \qquad (17)$$

Now it can be written:

$$a_N^2 = l^2[\dot\phi_2^2(1 - \frac{x^2}{4\,l^2}) + \frac{x^2}{4\,l^2 - x^2}]^2 + l^2[-\frac{\ddot x}{(4\,l^2 - x^2)^{1/2}} - \frac{\dot x^2\, x}{(4\,l^2 - x^2)^{3/2}} -$$

$$- \dot\phi_2 \frac{x}{2\,l}(1 - \frac{x^2}{4\,l^2})^{1/2}]^2 + l^2[\ddot\phi_2(1 - \frac{x^2}{4\,l^2})^{1/2} - \dot\phi_2 \frac{\dot x x}{l(4\,l^2 - x^2)^{1/2}}]^2 \qquad (18)$$

where:

$$\ddot\delta = -\frac{\ddot x}{(4\,l^2 - x^2)^{1/2}} - \frac{\dot x^2 x}{(4\,l^2 - x^2)^{3/2}}$$

If eq. (18) is put in the first eq. (13) then S_K is defined. Connection between $\ddot\phi_1$ and $\ddot\phi_2$ is got by differenting of non-holonomic connection (10).

$$\ddot\phi_1 = i_v\,\ddot\phi_2 + \frac{d\,i_v}{dt}\,\dot\phi_2 \qquad (19)$$

From the first eq. (17) is

$$\ddot\alpha = \frac{\ddot x}{r_3} \qquad (20)$$

From relation:

$$\dot\phi_3 = \dot\phi_2 \frac{r_2}{r \cdot \sin(\gamma - \alpha)} \qquad (21)$$

is:

$$\ddot\phi_3 = \ddot\phi_2 \frac{r_2}{r \cdot \sin(\gamma - \alpha)} + \dot\phi_2 \dot\alpha \frac{r_2 \cos(\gamma - \alpha)}{r \cdot \sin^2(\gamma - \alpha)}$$

Eq. for energy of acceleration in function of non-functional generalized accelerations is:

$$S = \frac{1}{2}\{J_1(i_v\ddot\phi_2 + \frac{di_v}{dt}\dot\phi_2)^2 + J_3[\ddot\phi_2 \frac{r_2}{r \cdot \sin(\gamma - \alpha)} + \frac{\dot\phi_2 \dot x r_2 \cos(\gamma - \alpha)}{r_3 \cdot r \sin^2(\gamma - \alpha)}]^2 +$$

$$+ (J_2 + J_r)\ddot\phi_2^2 + J_{o3}\frac{\ddot x^2}{r_3^2} + m_D\,\ddot x^2 + 2 \cdot m_N\, a_N^2\} \qquad (22)$$

Generalized forces are defined from virtual work and potential energy.

$$\delta A = M_1 \cdot \delta\phi_1 - M_2 \cdot \delta\phi_2 - b\,\dot x \cdot \delta x = Q\phi_2 \cdot \delta\phi_2 + Q_x' \cdot \delta x$$

and

$$Q_{\phi_2} = M_1 \cdot i_v - M_2; \qquad Q_x' = -b\,\dot x \qquad (23)$$

$M_1 = A - B\dot\phi_1$; $M_2 = i_o(D - Ct)$

If the change of positions for elastic element is:

$$\Delta l + (x - x_o) \qquad (24)$$

Using the eq. (24) and start load of elastic elements Δl we can write potential energy in form:

$$\Pi = \frac{1}{2}\,c\,(\Delta l + x - x_o)^2 \qquad (25)$$

Generalized force with force for damping ($-b\,\dot x$) is:

$$Q_x = -(x - x_o)c - c\Delta l - b\,\dot x \qquad (26)$$

Differential equations of movement are:

$$[J_1 i_v^2 + J_3 \frac{r_2^2}{r^2 \sin^2(\gamma - \alpha)} + J_2 + J_r + 2\,l^2 m_N(1 - \frac{x^2}{4\,l^2})]\,\ddot\phi_2 +$$

$$+ [J_1 i_v \frac{di_v}{dt} + \frac{\dot x\, r_2^2 \cos(\gamma - \alpha)}{r_3 \cdot r^2 \cdot \sin^3(\gamma - \alpha)} J_3 - m_N \cdot x \cdot \dot x + B \cdot i_v^2]\,\dot\phi_2 = A \cdot i_v - i_o(D - C \cdot t)$$

$$[J_{03}\frac{1}{r_3^2} + m_D + \frac{2\,l^2 \cdot m_N}{(4\,l^2 - x^2)}]\,\ddot{x} + \dot{x}^2 x\frac{2\,l^2 m_N}{(4\,l^2 - x^2)^2} + b\dot{x} + \tag{27}$$

$$+ \frac{1}{2}\dot{\varphi}_2 \cdot x \cdot m_N + c\,(x - x_o) = -c\,\Delta l$$

where:

$$\alpha = (x - x_o)\frac{1}{r_3} + \alpha_o \quad ; \qquad \frac{di_v}{dt} = \dot{\alpha}\frac{\sin 2\gamma}{\sin^2(\gamma - \alpha)} = \frac{\dot{x}}{r_3}\frac{\sin 2\gamma}{\sin^2(\gamma - \alpha)} \tag{28}$$

Putting $\dot{x} = V$ and $\dot{\varphi}_2 = \omega_2$ and solving the system of non-linear differential equations (27) on electrical computer using numerical method of integration it is possible to denote the parameters x, V and ω_2 in function of time t.

4. EXAMPLE

For example depicted on Fig. 1 values of parameters are:
$J_1 = 0.08$; $J_2 = 0.08$; $J_3 = 0.10$; $J_r = 0{,}42$; $J_{03} = 0.15$ (kg m^2); $m_N = 0.3$; $m_D = 0.8$ (kg); c = 10 (N/m); b = 5000 (Ns/m); $\Delta l = 7$ (mm); l = 200; r = 80; $r_1=r_2 = 30$; $r_3 = 20$ (mm); $\gamma = 45^0$; $z_1=20$; $z_2=50$;$z_3=z_4=30$; $i_r = 3$; $M_1 = A\text{-}B\dot{\varphi}_1$; $M_2 = M_k \cdot i_o = i_o\,(D\text{-}C\cdot t)$; A = 557,9 (Nm); B = 5.33 (Nms); D = 10 (Nm); C = 0.524 (Nm/s); to = O(s); $\alpha_o = 35^o$; $\omega_{20} = 10$ (s^{-1});
$x_o = 0.28$(m); $i_o = i_{12} \cdot i_r = 7.5$.

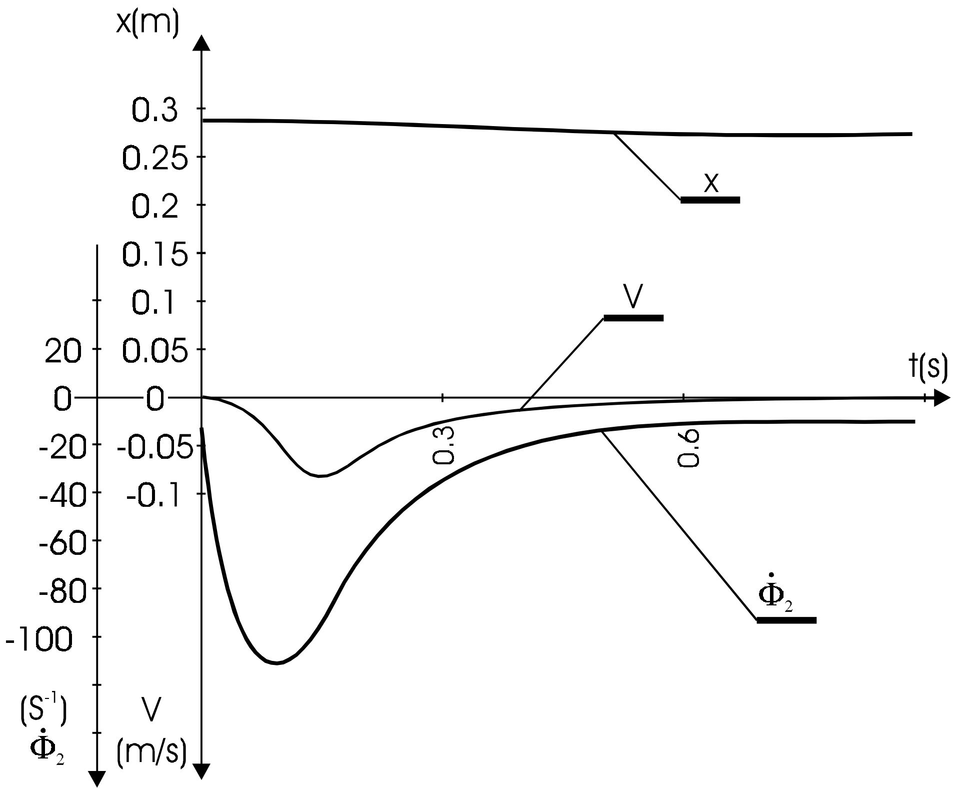

Figure 2. Time histories $\dot{\varphi}_2$, $\dot{x}$=V, x

5. CONCLUSION

Using the dynamically analyze of frontal frictional variator with two discs and half ball with contact in point it is got answer about the manner of this mechanical non-holonomic system. In dynamical analyze are used Appell`s equations.

Solving the system of differential equations it is possible to determine the parameters x, V, ω_2 as ω_1 too, in function of time t. Then it is possible very simple to denote the stability of this system.

The motion becomes stable often 0.7 second (See Fig. 2). This example is very good for presentation of dynamically behavior of non-holonomic mechanical transmission as regulation system.

6. REFERENCES

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