FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

EQUATIONS IN A MIXED REPRESENTATION AND A NEW FORMULATION BY FINITE ELEMENTS AND BOUNDARY ELEMENTS FOR THE RESOLUTION OF VIBRO-ACOUSTIC PROBLEM IN THE PRESENCE OF NON UNIFORM MEAN FLOW

BEN TAHAR Mabrouk and GOY Eugénie

Laboratoire de Génie Mécanique et des Structures (LGmS)
Division Acoustique et Vibrations Industrielles
Université de Technologie de Compiègne, B.P.629, 60205 Compiègne cedex, France

One of the most important question arises when studying the vibro-acoustic phenomenon in the presence of non uniform mean flow is the justification of the boundary conditions at the structure-fluid interface.

Primarily, we are presenting the linear acoustic equations in a mixed representation for a heterogeneous moving medium and associated boundary conditions. In the mixed representation, the equations verified by Lagrangian perturbations are written in Eulerian variables in association with mean flow of the medium concerned. The choice of this approach allows us to represent - in the sense of distribution theory - the acoustic field equations provided that one uses normal acoustic displacement continuity, and to deduce boundary conditions at the interface between two moving media.

Secondly, we are proposing a new formulation for the elasto-acoustic coupling problem using the previous results. In this approach the acoustic domain is divided in two sub-domains : The first one, coupling the structure, where the flow of fluid is non uniform, is discretised by finite elements. The second one, bounding the previous one, where the flow is supposed to be zero, is discretised by boundary elements. The association of functional structure, discretised by finite element, gives us the final vibro-acoustic system to solve.

1. VIBRO-ACOUSTIC EQUATIONS

The acoustic equations are obtained using perturbation of the mechanics equations of continuous media. When the considered medium is initially at rest, the linear acoustical equations can be established in Eulerian as well as Lagrangian representation. However, in our elasto-acoustic problem, characteristics of the medium in its initial state are not constant throughout the space : flow is non uniform. The choice of the representation and that of the boundary conditions is therefore more complex and we have to clearly distinguish the different descriptions.

It is in the mixed representation with the work of H.GALBRUN and B.POIREE that we have found the most suitable response to our problem. In 1931, H.GALBRUN has shown that all acoustic variables can be deduced from the Lagrangian acoustic displacement which verifies the partial differential equation called "Galbrun's equation". The work of B.POIREE is based on the same principle. In the mixed representation, equations verified by Lagrangian perturbations are written in Eulerian variables in association with the mean flow. B.POIREE has shown the interest of such a representation for a complex problem like a heterogeneous fluid flow : it allows to represent -in the sense of distributions theory- the acoustic field equation provided one uses the continuity of the normal Lagrangian acoustic displacement. Then he determines the jump conditions for Lagrangian perturbations across a flow discontinuity in the particularly case of a plan acoustic dioptre.

We summarise linear acoustic equations, for the general case, in the mixed representation for a heterogeneous moving medium and associated boundary conditions across a surface discontinuity for a given geometry. Then we study the particular case of the vibro-acoustic coupling with non uniform mean flow. The appropriate use of the boundary conditions allows us to establish the complete system of equations for our problem.

1.1. SOME DEFINITIONS AND NOTATIONS

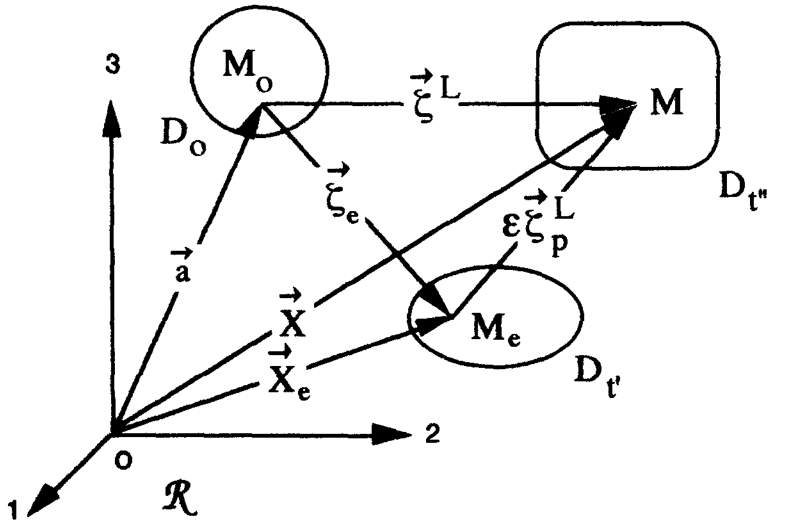

Figure 1 - Perturbed and non perturbed working configuration.

All function Ψ can be written as $\Psi^L = \Psi(a,t)$ in Lagrangian variables (a,t). (X_e,t') are Eulerian variables linked to the working movement (non perturbed and supposed to be known) of particles and (X ,t") are Eulerian variables linked to the particle in its perturbed state. The perturbation of Ψ at the first order is made by posing $\Psi = \Psi_e + \varepsilon\Psi_p$, where Ψ_e is the non perturbed part of Ψ and is known. The linearisation of equations of conservation implies the definition of a Lagrangian displacement as $X_i = X_{e_i} + \varepsilon\zeta^L_{p_i}$ (1)

1.2. EQUATIONS OF PERTURBATIONS IN THE MIXED REPRESENTATION THE GOVERNING EQUATIONS OF THE STRUCTURE AND THE FLUID

The equation of perturbation of the $\rho\mathcal{B}_i$ quantity can be generally written in the mixed representation as :

$$\left[\frac{\partial}{\partial t'} + \frac{\partial}{\partial X_{e_j}} u_{e_j}\right]\left[\rho_e \mathcal{B}^L_{p_i}\right] + \frac{\partial}{\partial X_{e_j}}\left[b^L_{p_{ji}} + b_{e_{li}}\left[\frac{\partial \zeta^L_{p_k}}{\partial X_{e_k}}\delta_{lj} - \frac{\partial \zeta^L_{p_j}}{\partial X_{e_l}}\right]\right] = \rho_e B^L_{p_i} \qquad (2)$$

where $\mathcal{B}_i$ is the specific density, B_i the rate of specific density which represents sources of the considered variable and b_{ji} the flux density tensor of the considered variable.

The equation of the structure is deduced from this last equation. The structure is supposed to be elastic, without any working movement and volumic forces. Assuming the initial state stationary and a harmonic time dependence $(\exp(-i\omega t))$, we determine :

$$\rho_e\omega^2\zeta^L_{p_i} + \frac{\partial}{\partial X_{e_j}}\left[\sigma^L_{p_{ij}} + \sigma_{e_{li}}\left[\frac{\partial\zeta^L_{p_k}}{\partial X_{e_k}}\delta_{lj} - \frac{\partial\zeta^L_{p_j}}{\partial X_{e_l}}\right]\right] = 0 \qquad (3)$$

In the same way but after few transformations, we deduce the equation of propagation verified by the Lagrangian acoustical displacement. The fluid is supposed to be perfect, compressible, isentropic, in the presence of non uniform mean flow and not subjected to any conservative forces. The initial state of the fluid is considered stationary. We assume a low Mach number so the speed of sound can be considered constant and a harmonic time dependence $(\exp(-i\omega t))$: :

$$\rho_e c_e^2\frac{\partial^2\zeta_{p_k}}{\partial X_{e_i}\partial X_{e_k}} + \rho_e\omega^2\zeta_{p_i} + 2i\omega\rho_e u_{e_j}\frac{\partial\zeta_{p_i}}{\partial X_{e_j}} - \rho_e u_{e_j}\frac{\partial}{\partial X_{e_j}}\left(u_{e_k}\frac{\partial\zeta_{p_i}}{\partial X_{e_k}}\right) + \frac{\partial p_e}{\partial X_{e_j}}\frac{\partial\zeta_{p_j}}{\partial X_{e_i}} = 0 \qquad (4)$$

We recognise the 'Galbrun's equation".

1.3. ASSOCIATED BOUNDARY CONDITIONS

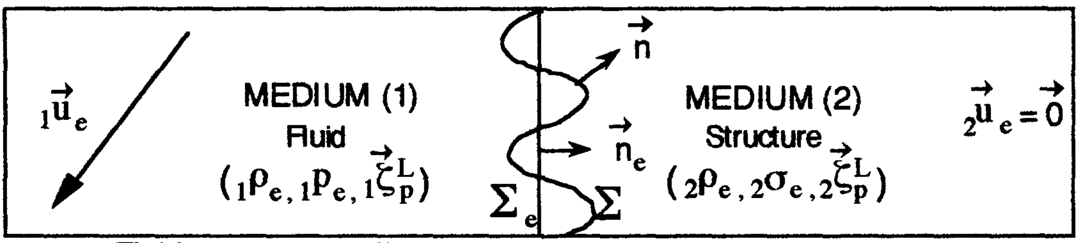

Figure 2 - Fluid-structure coupling

We assume that every variables attached to the considered medium are discontinuous across a surface Σ. The writing of the equation of perturbation (2) - in the sense of the distributions theory - requires the supplementary assumption of normal Lagrangian acoustical displacement continuity :

$$\left[\zeta^L_{p_i} n_{e_i}\right]_{\Sigma_e} = 0\text{, where } [\ ]_{\Sigma_e} \text{ represents the jump across } \Sigma_e. \qquad (5)$$

With this condition, one achieves the boundary condition across Σ_e, whose equation is $S_e(X_e, t') = 0$:

$$\left[-\sigma^L_{p_{ji}}\right]_{\Sigma_e} n_{e_j} + \left[\sigma_{e_{li}}\frac{\partial\zeta^L_{p_j}}{\partial X_{e_l}}\right]_{\Sigma_e} n_{e_j} + \frac{1}{|\nabla S_e|}\left[\zeta^L_{p_j}\right]_{\Sigma_e}\frac{\partial\left(\sigma_{e_{li}} n_{e_l}|\nabla S_e|\right)}{\partial X_{e_j}} = 0 \qquad (6)$$

This last result is essential because it is a generalisation of the particularly case treated by B.POIREE : this one deals with every continuous heterogeneous medium. In our vibro-acoustic case and with a light fluid, (6) gives :

$${}_2\sigma^L_{p_{ji}}\, n_{e_j} - {}_2\sigma_{e_{li}}\frac{\partial\, {}_2\zeta^L_{p_j}}{\partial X_{e_l}} n_{e_j} = -{}_1p^L_p\, n_{e_i} \quad \text{across } \Sigma_e \qquad (7)$$

with : - the acoustical pressure $\quad {}_1p^L_p = -{}_1\rho_e c_e^2 \operatorname{div}\left({}_1\zeta^L_p\right) \qquad (8)$

- the stress tensor $${}_2\sigma^L_{p_{ji}} = \frac{1}{2}\left(\frac{\partial_2 \zeta^L_{p_i}}{\partial X_{e_j}} + \frac{\partial_2 \zeta^L_{p_j}}{\partial X_{e_i}}\right) \qquad (9)$$

2. THE VARIATIONAL FORMULATION

In the approach that we have adopted, the acoustic domain is divided into two sub-domains : Ω_1 called the "internal" domain and Ω_2 called the "external" domain.

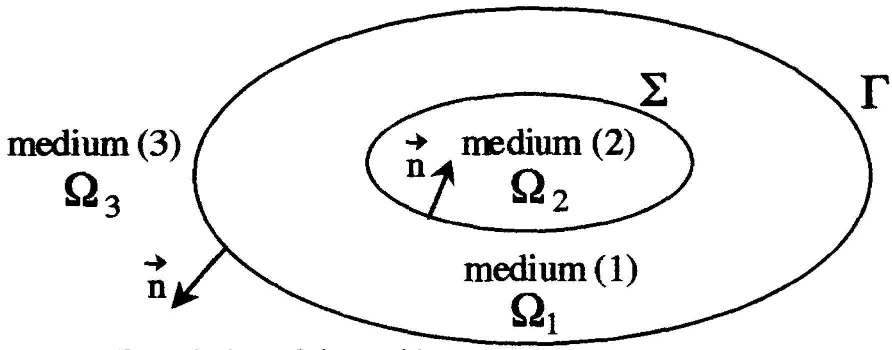

Figure 3 - Description of the problem

where
- Ω_1 is the acoustic domain where the flow is non uniform,
- Ω_2 is the domain representing the structure,
- Σ is the elastic surface,
- Ω_3 is the infinite acoustic domain where the flow is supposed to be zero,
- Γ is the interface between Ω_1 and Ω_3.

In Ω_1, we use a discretisation by finite elements of the acoustical displacement. In Ω_3 , which is infinite and without flow, we keep the simplified integral formulation written in terms of the acoustic pressure and acoustic pressure gradient. The coupling of the two media Ω_1 and Ω_3 is obtained not only by applying the pressure and the pressure gradient continuity, but also by associating the two integral formulations (if we keep in mind that the pressure can be written in terms of acoustic displacement).

The association of functional structure, discretised by finite elements, is made by using appropriate boundary conditions, presented in the first part of this work, and gives us the final vibro-acoustic system to solve.

We summarise now the integral formulation associated to the "internal" problem and the one corresponding to the "external" problem. Then we write the coupling between the two acoustic formulations. At the end, we present the integral form associated to the structure and the global coupling of the vibro-acoustic problem.

2.1. THE INTEGRAL FORMULATION ASSOCIATED TO THE INTERNAL DOMAIN Ω_1

The weak integral formulation is obtained by weighting the equation of propagation (4) by a test function ${}_1W^*$, integration over the domain Ω_1 and integration by parts. Since the working velocity is zero in the domain Ω_2 attached to the structure and Ω_3 the one corresponding to the "external" fluid, we will obtain :

$$\int_{\Omega_1}\left\{\rho_e\omega^2\zeta_{p_i}+\frac{\partial p_e}{\partial X_{e_j}}\frac{\partial\zeta_{p_j}}{\partial X_{e_i}}+2i\omega\rho_e u_{e_j}\frac{\partial\zeta_{p_i}}{\partial X_{e_j}}\right\}{}_1W_i^*\,d\Omega_1$$

$$-\int_{\Omega_1}\left\{c_e^2\frac{\partial\zeta_{p_k}}{\partial X_{e_k}}\frac{\partial_1\left(\rho_e W_i^*\right)}{\partial X_{e_i}}-u_{e_k}\frac{\partial\zeta_{p_i}}{\partial X_{e_k}}\frac{\partial}{\partial X_{e_j}}\left(\rho_e u_{e_j}\,{}_1W_i^*\right)\right\}d\Omega_1 \tag{10}$$

$$+\int_{\Sigma}{}_1\rho_e\,c_e^2\frac{\partial_1\zeta_{p_k}}{\partial X_{e_k}}\,{}_1W_i^*\,n_i\,d\Sigma\;+\int_{\Gamma}{}_1\rho_e\,c_e^2\frac{\partial_1\zeta_{p_k}}{\partial X_{e_k}}\,{}_1W_i^*\;n_i\,d\Gamma=0\qquad\forall\,{}_1W^*$$

2.2. THE INTEGRAL FORMULATION ASSOCIATED TO THE "EXTERNAL" DOMAIN Ω_3

In Ω_3, the equation of propagation, written in terms of pressure and for a harmonic time dependence $(\exp(-i\omega t))$, is given by the Helmholtz's equation (which is a particular case of (4)) and by the conditions of pressure continuity and/or gradient pressure continuity (since it is the same fluid in Ω_1 and Ω_3):

$$\begin{cases}\Delta_3 p_p+k^2{}_3p_p=0 \text{ in } \Omega_3\\ \dfrac{\partial_3 p_p}{\partial n}-\dfrac{\partial_1 p_p}{\partial n}=\omega^2\rho_e\;{}_1\zeta_{p_i}n_i \qquad \text{on } \Gamma\\ {}_3p_p-{}_1p_p=-\rho_e c_e^2\dfrac{\partial\;{}_1\zeta_{p_i}}{\partial X_{e_i}}=\mu_\Gamma \qquad \text{on } \Gamma\\ +\;\text{Sommerfeld's radiation condition}\end{cases} \tag{11}$$

Introducing $G(X_e,Y_e)$ the elementary solution of the Helmholtz equation which verifies the Sommerfeld radiation condition, we obtain the integral representation of the acoustic pressure ${}_3p_p$ and of the gradient pressure $\dfrac{\partial_3 p_p}{\partial n}$ on Γ. The first one is used in the boundary integral of equation (10). The ponderation of the second one by a test function μ_Γ^* and the integration over the contour Γ give :

$$\frac{1}{2}\int_\Gamma\mu_\Gamma^*(X_e)\frac{\partial_3 p_p(X_e)}{\partial n(X_e)}\,d\Gamma(X_e)+\int_{\Gamma\times\Gamma}\mu_\Gamma^*(X_e)\,{}_3p_p(Y_e)\frac{\partial^2 G(X_e,Y_e)}{\partial n(X_e)\partial n(Y_e)}\,d\Gamma(Y_e)\,d\Gamma(X_e)$$

$$-\int_{\Gamma\times\Gamma}\mu_\Gamma^*(X_e)\frac{\partial_3 p_p(Y_e)}{\partial n(Y_e)}\frac{\partial G(X_e,Y_e)}{\partial n(X_e)}\,d\Gamma(Y_e)\,d\Gamma(X_e)\;=0\qquad\forall\,\mu_\Gamma^* \tag{12}$$

2.3. COUPLING OF THE FLUID INTEGRAL FORMULATIONS

In order to couple the two fluid integral formulations, we apply the continuity of pressure (11) on the boundary Γ between the "internal" domain Ω_1, discretised by finite elements, and the external domain Ω_3 , discretised by boundary finite elements.

2.4. INTEGRAL FORMULATION OF THE STRUCTURE

By ponderating equation (3) by a arbitrary weight ${}_2W^*$, integrating over Ω_2, and using the first Green's formulation, we obtain :

$$\int_{\Omega_2} \left\{ \rho_e \, \omega^2 \zeta_{pi} \, {}_2W_i^* - \sigma_{pij}\left(\zeta_{pi}\right)\varepsilon_{ij}\left({}_2W_i^*\right) - \sigma_{eij} \frac{\partial \zeta_{pk}}{\partial X_{ek}} \frac{\partial_2 W_i^*}{\partial X_{ej}} + \sigma_{eli} \frac{\partial \zeta_{pj}}{\partial X_{el}} \frac{\partial_2 W_i^*}{\partial X_{ej}} \right\} d\Omega_2$$

$$- \int_{\Sigma} \left\{ {}_2\sigma_{pij} \, {}_2W_i^* \, n_j - {}_2\sigma_{eli} \frac{\partial_2 \zeta_{pj}}{\partial X_{el}} \, {}_2W_i^* \, n_j \right\} d\Sigma = 0 \qquad \forall \, {}_2W^* \qquad (13)$$

2.5. ELASTO-ACOUSTIC JUMP CONDITION

In the functional of the structure (13) as in the one of the fluid flow (10), we use the boundary condition (7) noting ${}_1p_p^L = \mu_\Sigma$:

$$\int_{\Sigma} \left\{ {}_2\sigma_{pij} \, {}_2W_i^* \, n_j - {}_2\sigma_{eli} \frac{\partial_2 \zeta_{pj}}{\partial X_{el}} \, {}_2W_i^* \, n_j \right\} d\Sigma = - \int_{\Sigma} \mu_\Sigma \, {}_2\vec{W}^* . \vec{n} \, d\Sigma \qquad (14)$$

$$\int_{\Sigma} {}_1\rho_e \, c_e^2 \frac{\partial_1 \zeta_{pk}}{\partial X_{ek}} \, {}_1W_i^* \, n_i \, d\Sigma = - \int_{\Sigma} \mu_\Sigma \, {}_1\vec{W}^* \vec{n} \, d\Sigma \qquad (15)$$

We have to use now the condition of continuity of the normal displacement (5). We write :

$$\int_{\Sigma} \mu_\Sigma^* \left({}_2\zeta_{pj} n_j - {}_1\zeta_{pj} n_j \right) d\Sigma = 0 \qquad \forall \, \mu_\Sigma^* \qquad (16)$$

3. CONCLUSION

We have established, in terms of Lagrangian acoustic displacement, the complete equations of a vibro-acoustic problem in the presence of non uniform mean flow. This equations are written in the mixed representation where Lagrangian perturbations are written in Eulerian variables linked to the working flow. The choice of this representation has been justified because it allows us the suitable writing of the jump conditions at the interface between the structure and the fluid flow provided we use the continuity of the normal Lagrangian acoustic displacement.

Those equations have been used in the writing of global numeric formulation in order to solve our vibro-acoustic problem : The variational formulation associated to the global coupled system (figure 4) is obtained by assembling the three formulations (10) (13), discretised by finite elements, and (12), discretised by boundary finite elements. The unknowns of the system are the acoustical Lagrangian displacement, for the structure and the fluid flow, and the acoustical pressure for the "external fluid without flow.

Our formulation is now associated to a software of acoustic modelisation but computational results are not the purpose of this paper.

${}_2W^*$	μ_Σ^*	${}_1W^*$	μ_Γ^*	
$\int_{\Omega_2} \rho_e \omega^2 \zeta_{p_i}\, {}_2W_i^*$ $-\int_{\Omega_2} \sigma_{p_{ij}}(\zeta_{p_i})\,\varepsilon_{ij}({}_2w_i^*)$ $-\int_{\Omega_2}\sigma_{e_{ij}}\frac{\partial\zeta_{p_k}}{\partial X_{e_k}}\frac{\partial_2 W_i^*}{\partial X_{e_j}}$ $+\int_{\Omega_2}\sigma_{e_{li}}\frac{\partial\zeta_{p_j}}{\partial X_{e_l}}\frac{\partial_2 W_i^*}{\partial X_{e_j}}$	$\int_\Sigma \mu_\Sigma^*\, {}_2\vec\zeta_p\cdot\vec n$	0	0	${}_2\zeta_p$
$\int_\Sigma \mu_\Sigma\, {}_2\vec W^*\cdot\vec n$	0	$-\int_\Sigma \mu_\Sigma\, {}_1\vec W^*\cdot\vec n$	0	μ_Σ
0	$-\int_\Sigma \mu_\Sigma^*\,{}_1\vec\zeta_p\cdot\vec n$	$\int_{\Omega_1}\rho_e\omega^2\zeta_{p_i}\,{}_1W_i^*$ $+\int_{\Omega_1}\frac{\partial p_e}{\partial X_{e_j}}\frac{\partial\zeta_{p_j}}{\partial X_{e_i}}\,{}_1W_i^*$ $+\int_{\Omega_1}2i\omega\rho_e u_{e_j}\frac{\partial\zeta_{p_i}}{\partial X_{e_j}}\,{}_1W_i^*$ $-\int_{\Omega_1}c_e^2\frac{\partial\zeta_{p_k}}{\partial X_{e_k}}\frac{\partial_1(\rho_e W_i^*)}{\partial X_{e_i}}$ $+\int_{\Omega_1}u_{e_k}\frac{\partial\zeta_{p_i}}{\partial X_{e_k}}\frac{\partial}{\partial X_{e_j}}(\rho_e u_{e_j}\,{}_1W_i^*)$ $-\int_{\Gamma\times\Gamma}\omega^2\rho_e\,{}_1\vec W^*\cdot\vec n\,{}_1\vec\zeta_p\cdot\vec n\, G$	$-\frac{1}{2}\int_\Gamma \mu_\Gamma^*\,{}_1\vec\zeta_p\cdot\vec n$ $+\int_{\Gamma\times\Gamma}\mu_\Gamma^*\,{}_1\vec\zeta_p\cdot\vec n\,\frac{\partial G}{\partial n}$	${}_1\zeta_p$
0	0	$-\frac{1}{2}\int_{\Gamma\times\Gamma}\mu_\Gamma\,{}_1\vec W^*\cdot\vec n$ $+\int_{\Gamma\times\Gamma}\mu_\Gamma\,{}_1\vec W^*\cdot\vec n\,\frac{\partial G}{\partial n}$	$\int_{\Gamma\times\Gamma}\frac{-1}{\rho_e\omega^2}\mu_\Gamma^*\mu_\Gamma\frac{\partial^2 G}{\partial n\partial n}$	μ_Γ

Figure 4 - Matrix representing the coupled global system

4. BIBLIOGRAPHY

[1] H.GALBRUN : "Propagation d'une Onde Sonore dans l'Atmosphère et Théorie des Zones de Silence". Éditions Gauthier-Villars, Paris, 1931.

[2] J.W.MILES : "On the Reflection of Sound at an Interface of Relative Motion". J.Acous.Soc.Am., vol.29(2), p.226-228, 1957.

[3] H.S.RIBNER : "Reflection, Transmission and Amplification of Sound by a Moving Medium". J.Acous.Soc.Am., vol.29(19), p.435-441, 1957.

[4] U.INGARD : "Influence of Fluid Motion Past a Plane Boundary on Sound Reflection, Absorption and Transmission". J.Acous.Soc.Am., vol.31(22), p.1035-1036, 1959.

[5] P.GOTTLIEB : "Sound Transmission Trough a Velocity Discontinuity". J.Acous.Soc.Am., vol.31(22), p.1036-1037, 1959.

[6] P.MUNGUR et H.E.PLUMBLEE : "Propagation and Attenuation of Sound in a Soft-walled Annular Duct Containing a Sheared Flow". N.A.S.A. SP-207, vol.71(22), p.305-327, 1969, Basic Aerodynamic Noise Research.

[7] S.D.SAVKAR : "Propagation of Sound in Ducts with Shear Flow". J.Sound Vib., vol.19(22), p.355-372, 1971.

[8] W.EVERSMAN and R.J.BECKMEYER : "Transmission of sound in Ducts With Thin Shear Layers-Convergence to the Uniform Flow Case". J.Acous.Soc.Am. vol.52(1), p.216-220, 1972.

[9] B.J.TESTER : "Some Aspects of Sound Attenuation in Lined Ducts Containing Inviscid Mean Flow With Boundary Layers". J.Sound Vib., vol.28(2), p.217-245, 1973.

[10] A.H.NAYFEH, J.E.KAISER, and D.TELIONIS : "Acoustics of Aircraft Engine-Duct Systems". A.I.A.A., vol.13(2), p.130-153, 1975.

[11] M.K.MYERS : "On the Acoustic Boundary Condition in the Presence of Flow". J.Sound Vib., vol.71(22), p.429-434, 1980.

[12] B.POIRÉE : "Les Equations de l'Acoustique Linéaire et Non Linéaire dans un Ecoulement de Fluide Parfait". Acustica, vol.57, p.5-25, 1985.

[13] M.BEN TAHAR : "Formulation Variationnelle par Equations Intégrales pour le Rayonnement Acoustique en Présence d'un Ecoulement Non Uniforme". Thèse d'État, Université de Technologie de Compiègne, Décembre 1991.

[14] M. BEN TAHAR : " Formulation Variationnelle par Equations Intégrales pour un Problème de Couplage Elasto-acoustique en Présence d'un Ecoulement Non Uniforme". Rapport de Recherche (DRET/ 90-198), Université de Technologie de Compiègne, Février 1993.

[15] E.GOY et M.BEN TAHAR : "Equations de Perturbation en variables d'Euler, de Lagrange, mixtes et Conditions de Passage Associées". Rapport de Recherche (DRET/ 94-107), Université de Technologie de Compiègne, Janvier 1996.

[16] E.GOY et M.BEN TAHAR : "Rayonnement des Structures Elastiques dans un Milieu Acoustique en Présence d'un Ecoulement Non Uniforme - Représentations Eulérienne, Lagrangienne et mixte et Conditions de Passage Associées - Application au Couplage Structure-Fluide". Rapport de Recherche, Université de Technologie de Compiègne, Juillet 1996.