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VARIATIONAL FORMULATION BY INTEGRAL EQUATIONS FOR THE RESOLUTION OF VIBRO-ACOUSTIC PROBLEMS IN A VISCO-THERMAL FLUID

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ABSTRACT

A total variational formulation of vibro-acoustic interaction between vibrating membrane and a visco-thermal fluid layer was investigated. This formulation combines a new variational formulation by integral equations of the fluid, which taking into account of acoustic and entropic waves coupling, with a classical variational formulation of the membrane. This formulation has been implemented numerically for the problems with revolution geometry. The obtained numerical results are compared to analytical ones developed for three models : perfect fluid model, visco-thermal fluid model which coupling acoustic and entropic waves (developed in this work) and visco-thermal fluid model which coupling acoustic, entropic and shear waves. These comparisons showed the validity of our formulation proposed in this work and the importance of the effects of entropic and shear waves on the damping of modes of coupling fluid-structure system and the decrease of their natural frequencies in the case of the micro-cavities.

1. INTRODUCTION

The study of acoustic propagation in a visco-thermal fluid has received a lot of interest during the last twenty years, with the development of miniaturised transducers. When the thickness of the gap which separates the membrane from a rigid electrode is comparable with the size of the viscous and thermal boundary layer, the account of viscosity and thermal conductivity of the fluid filling the cavity becomes necessary [1-3].

For simple geometry, an analytic solution can be established [4]. However for complex geometries, the recourse to numerical solutions proves necessary.

This work presents the study of the acoustic propagation problem in the gap between the two electrodes. We consider that the rotational velocity is parallel to the frontiers, thus we decoupled the problem of calculation of acoustic and entropic pressures from that of rotational velocity. To solve this problem, we have developed a total variational formulation by coupling a variational formulation by integral equations of the fluid with a classical variational formulation of the membrane. This formulation has been implemented numerically for the problems with revolution geometry.

2. EQUATIONS OF COUPLED PROBLEM

We consider a visco-thermal fluid domain Ω_f (Figure 1) surrounded by a surface Sp where the pressure is imposed to (0), a surface Sv where the velocity is imposed to (0) and an elastic membrane which is stretched and clamped at its periphery and excited by a mechanical charge p_e. We are in presence of a coupling vibro-acoustic problem where we have to resolve the equation governing the motion of the membrane and the linear acoustic equations in a visco-thermal fluid.

2.1. Equation of motion of membrane

The equation governing the motion of the membrane is written for harmonic excitation as :

$$-T\,\Delta w - \rho_s\,\omega^2 w = p - p_e \tag{1}$$

$$w = 0 \qquad \text{on } \partial\Gamma \tag{2}$$

T is the tension per unit length of the membrane, w is the transverse displacement of the membrane, ρ_s is the mass per unit area of the membrane, p is the inside pressure, p_e is the outside known pressure and $\partial\Gamma$ is the periphery of the membrane.

2.2. Equations of the linear acoustic in a visco-thermal fluid

For harmonic motion (∂_t= -iω), the set of linear homogeneous equations governing small amplitude disturbances, includes the following [1] :

$$p = p_a + p_h \tag{3}$$

$$\left(\Delta + k_a^2\right) p_a = 0 \tag{4}$$

$$\left(\Delta + k_h^2\right) p_h = 0 \tag{5}$$

$$T = T_a + T_h = \tau_a\, p_a + \tau_h\, p_h \tag{6}$$

$$\vec{v} = \vec{v}_e + \vec{v}_t \tag{7}$$

$$\left(\Delta + k_v^2\right) \vec{v}_t = 0 \quad ; \quad \operatorname{div} \vec{v}_t = 0 \tag{8}$$

$$\vec{v}_e = \vec{v}_e^{\,a} + \vec{v}_e^{\,h} = \phi_a\, \vec{\operatorname{grad}}\, p_a + \phi_h\, \vec{\operatorname{grad}}\, p_h \tag{9}$$

where p_a is the acoustic pressure, p_h is the entropic pressure, T_a is the acoustic temperature, T_h is the entropic temperature, $\vec{v}_e$ is the irrotational velocity and $\vec{v}_t$ is the rotational velocity.

$$k_a^2 = k^2(1 - i\,k\,\ell_{vh} - k^2\ell_h\ell'_{vh})^{-1}, \qquad k_h^2 = \frac{ik}{\ell_h}(1 + i\,k\,\ell'_{vh})^{-1}, \qquad k_v^2 = \frac{i\,k}{\ell'_v}, \qquad k = \frac{\omega}{c}$$

$$\tau_{a,h} = \frac{\gamma - 1}{\beta\gamma}\left(1 + i\,\frac{c}{\omega}\,\ell_h\,k_{a,h}^2\right)^{-1}, \quad \phi_{a,h} = \frac{-i}{\rho_0\,\omega}\left(1 + i\,\frac{c}{\omega}\,\ell_v\,k_{a,h}^2\right)^{-1}, \quad \ell_h = \frac{\chi}{c}$$

$$\ell_{vh} = \ell_v + (\gamma - 1)\,\ell_h, \qquad \ell'_{vh} = (\gamma - 1)(\ell_h - \ell_v), \quad \ell_v = \frac{\eta + \frac{4}{3}\mu}{\rho_0 c}, \quad \ell'_v = \frac{\mu}{\rho_0 c}$$

β is the increase in pressure per unit increase in temperature at constant density, μ is the dynamic viscosity, η is the bulk viscosity, ρ_0 is the density, χ is the thermal diffusivity, γ is the specific heat ratio and c is the adiabatic speed of sound.

To these equations, we associate the boundary conditions at the edge of fluid domain. We consider that the normal component of rotational velocity is negligible as compared to the irrotational velocity [5] and the temperature fluctuations are nil on the boundaries, so that we have :

Over S_p : $\quad p_a = 0 \quad (10) \qquad p_h = 0 \quad (11)$

Over S_v : $\quad \phi_a \dfrac{\partial p_a}{\partial n} + \phi_h \dfrac{\partial p_h}{\partial n} = 0 \quad (12) \qquad T = \tau_a\, p_a + \tau_h\, p_h = 0 \quad (13)$

Over Γ : $\quad \phi_a \dfrac{\partial p_a}{\partial n} + \phi_h \dfrac{\partial p_h}{\partial n} = -i\omega\, w \quad (14) \qquad T = \tau_a\, p_a + \tau_h\, p_h = 0 \quad (15)$

3. VARIATIONAL FORMULATION

To resolve this problem, an integral representation of acoustic and entropic pressure is used :

$$p_a(X) = \int_{S_p} G_a \frac{\partial p_a}{\partial n_Y} dS_Y - \int_{S_v U \Gamma} p_{aY} \frac{\partial G_a}{\partial n_Y} dS_Y + \int_{S_v U \Gamma} G_a \frac{\partial p_a}{\partial n_Y} dS_Y \qquad X \in \Omega - (S_p U S_v U \Gamma) \quad (16)$$

$$\begin{aligned} p_h(X) = &\int_{S_p} G_h \frac{\partial p_h}{\partial n_Y} dS_Y + \int_{S_v} \left(\frac{\tau_a}{\tau_h} p_{aY} \frac{\partial G_h}{\partial n_Y} - \frac{\phi_a}{\phi_h} G_h \frac{\partial p_a}{\partial n_Y} \right) dS_Y + \\ &\int_{\Gamma} \frac{\tau_a}{\tau_h} p_{aY} \frac{\partial G_h}{\partial n_Y} dS_Y + \int_{\Gamma} \phi_h^{-1} G_h \left(-i\omega\, w_Y - \phi_a \frac{\partial p_a}{\partial n_Y} \right) dS_Y \end{aligned} \qquad X \in \Omega - (S_p U S_v U \Gamma) \quad (17)$$

where p_{aY} is the acoustic pressure at point Y, $\vec{n}$ is the external normal vector of regions containing fluid; G_a and G_h are the elementary solutions of equations (4) and (5) in the free space.

The application of the boundary conditions (10-15) at integral representations of acoustic and entropic pressures and their normal derivatives, gives taking into account the equation of motion of membrane (1), the system of equations which enables us to determine the unknowns of the problem : $(\frac{\partial p_a}{\partial n}, \frac{\partial p_h}{\partial n})$ over S_p, $(\frac{\partial p_a}{\partial n}, p_a)$ over S_v and $(w, \frac{\partial p_a}{\partial n}, p_a)$ over Γ.

To solve these equations, we have developed a total variational formulation of the fluid-structure system. This formulation is obtained by coupling a variational formulation by integral equations of the fluid with a classical variational formulation of the structure [6]:

$$\begin{aligned} &< A_1 >_{S_p x S_p} + < A_2 >_{S_v x S_p} + < A_3 >_{S_p x S_v} + < A_4 >_{\Gamma x S_p} + < A_5 >_{S_p x \Gamma} + \\ &< A_6 >_{S_v x S_v} + < A_7 >_{\Gamma x S_v} + < A_8 >_{S_v x \Gamma} + < A_9 >_{\Gamma x \Gamma} - \frac{i\omega}{\phi_a} \int_{\Gamma} \frac{w\, p_a' + w' p_a}{2} ds_\Gamma \\ &+ \frac{i\omega\tau_h}{\phi_a(\tau_h - \tau_a)} \int_{\Gamma} \left[T(\vec{grad} w . \vec{grad} w') - \rho_s \omega^2 w w' \right] ds_\Gamma = \frac{i\omega\tau_h}{\phi_a(\tau_h - \tau_a)} \int_{\Gamma} p_e w'\, dS_\Gamma \end{aligned} \qquad (18)$$

where the double integrals $< A_i >$ are presented in the appendix A, w' is the dual of w and p_a' the dual of p_a.

The discretisation by finite boundary elements of the variational formulation (18) for the problems with revolution geometry, permits to obtain after assembly the symmetrical matrix system follows :

$$\left[\begin{array}{c|c} Z_s & C \\ \hline C^T & D \end{array}\right] \left\{\begin{array}{c} w \\ \hline \tilde{P} \end{array}\right\} = \left\{\begin{array}{c} F_w \\ \hline 0 \end{array}\right\} \tag{19}$$

Z_s is the matrix of membrane impedance affected by the visco-thermal fluid, C is the coupling matrix between the membrane displacement w and the acoustic variables $\tilde{P}$ ($\frac{\partial p_a}{\partial n}, p_a$ over S_v or Γ and $\frac{\partial p_a}{\partial n}, \frac{\partial p_h}{\partial n}$ over S_p), D is the matrix of admittance of the fluid, F_w is the second member du to the mechanical charge p_e.

4. RESULTS AND DISCUSSION

In this section we consider a vibro-acoustic interaction problem between a vibrating membrane and a cylindrical air layer considered visco-thermal fluid (Figure 1). The membrane was made out of titanium with a thickness of 10^{-5} m and a density ρ_m of 4500 Kg/m³, the tension applied at the periphery of the membrane is T=387.5 N/m. The physical properties of air are: $\rho_0 = 1.1614$ Kg/m³, $c = 340$ m/s, $\mu = 184.6\ 10^{-7}$ Ns/m², $\eta = 110\ 10^{-7}$ Ns/m², $\beta = 458$ N/(m² °K), $\chi = 22.5\ 10^{-6}$ m²/s, $\gamma = 1.403$. The temperature fluctuations are assumed to be null at all boundaries. The pressure is zero at the boundary (r = R). The speed is null at the boundary (z = - L/2) and at the boundary (z = L/2), we have imposed a uniform driving force per unit area $p_e = 1$ N/m². All results presented in this paper are nondimensionalized by referring them to the following parameters : ρ_0 (the density), c (the adiabatic speed of sound), L (the height of the cavity) and $T_0 = 293°$K (the ambient temperature).

The figures 2 and 3 show respectively the modus of acoustic pressure at point A1 and the displacement of the membrane center versus to the frequency. The obtained peaks correspond to the four first natural frequencies of the coupled membrane-cavity system. the three first ones are associated to the membrane modes 01, 02 and 03 and the fourth one is associated to the cavity mode 010. The Numerical and analytical results of visco-thermal fluid model, which taking into account of acoustic and entropic waves coupling, are in good agreement and are clearly different from those of perfect fluid model. One clearly sees a a small damping of the three first modes and a great damping of the fourth mode and a decrease of its natural frequency about 17 %.

By taking into account of the normal component of rotational velocity, the acoustic, entropic and shear waves are coupled at the boundary conditions level. This complete model was investigated by Bruneau [4]. The analytic results obtained by this model show the importance of the effects of the shear wave on the damping of coupled membrane-cavity system modes.

5. CONCLUSION

To expect by the calcul the vibro-acoustic behaviour of miniaturised transducers, A total variational formulation of vibro-acoustic interaction between vibrating membrane and a visco-thermal fluid layer was investigated. This formulation combines a new variational formulation by integral equations of the fluid, which taking into account of acoustic and entropic waves coupling, with a classical variational formulation of the membrane. The numerical results predicted by this formulation for the problems with revolution geometry are in good agreement with analytic ones and show the effects of acoustic and entropic waves coupling on the cavity-membrane system modes. However, the coupling of the shear wave with acoustic and entropic waves remains to be integrated in this numerical work. Indeed the analytic results show the importance of the shear wave effects on the damping of the cavity-membrane system modes.

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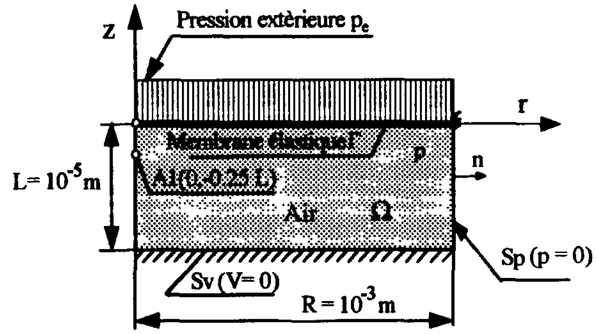

Fig. 1 membrane coupled to a cavity filled with a viscothermal fluid

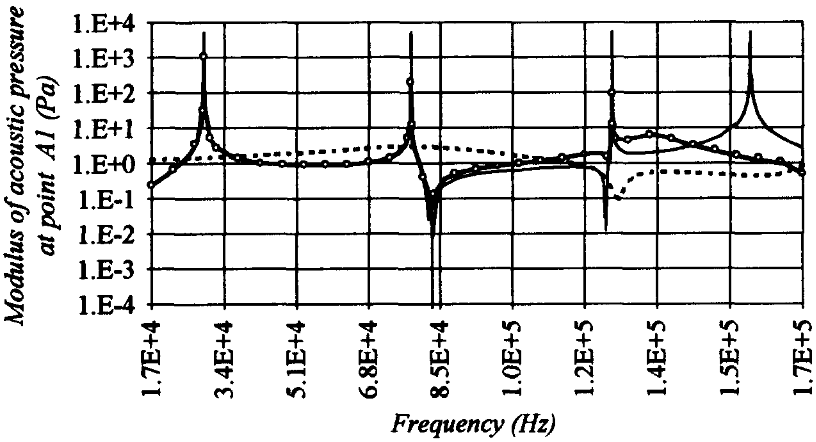

Fig. 2 Modulus of acoustic pressure at point A1 versus frequency. (—) analytical perfect fluid, (—) analytical viscothermal fluid (coupling acoustic and entropic waves), (◦) numerical viscothermal fluid (coupling acoustic and entropic waves), model of Bruneau 1990 (coupling acoustic, entropic and shear waves).

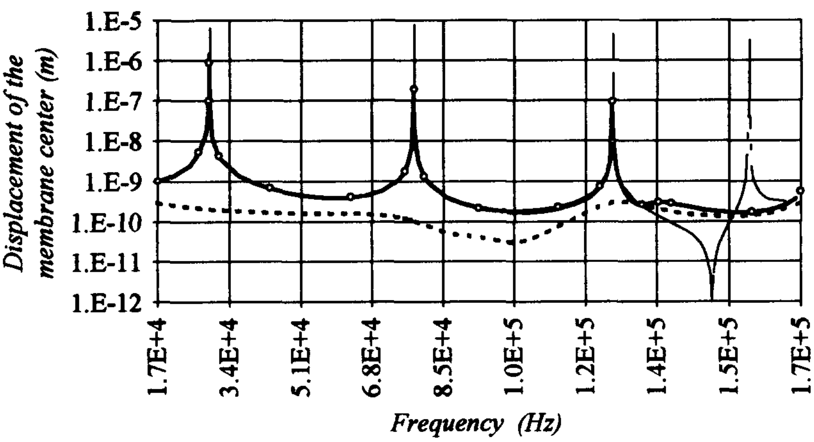

Fig. 3 Modulus of displacement of membrane center versus frequency. (—) analytical perfect fluid, (—) analytical viscothermal fluid (coupling acoustic and entropic waves), (◦) numerical viscothermal fluid (coupling acoustic and entropic waves), model of Bruneau 1990 (coupling acoustic, entropic and shear waves).

APPENDIX A

The integrals $< A_i >$ presented in the first member of variational formulation (18) are given by :

$$< A_1 >_{S_p x S_p} = < G_a \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' - \frac{\tau_h}{\tau_a}\frac{\phi_h}{\phi_a} G_h \frac{\partial p_h}{\partial n_Y} (\frac{\partial p_h}{\partial n_X})' >_{S_p x S_p},$$

$$< A_2 >_{S_v x S_p} = < - \frac{\partial G_a}{\partial n_Y} p_{aY} (\frac{\partial p_a}{\partial n_X})' >_{S_v x S_p} + < - \frac{\phi_h}{\phi_a} \frac{\partial G_h}{\partial n_Y} p_{aY} (\frac{\partial p_h}{\partial n_X})' >_{S_v x S_p} +$$
$$< G_a \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{S_v x S_p} + < \frac{\tau_h}{\tau_a} G_h \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_h}{\partial n_X})' >_{S_v x S_p},$$

$$< A_3 >_{S_p x S_v} = < - \frac{\partial G_a}{\partial n_X} \frac{\partial p_a}{\partial n_Y} p'_{aX} >_{S_p x S_v} + < - \frac{\phi_h}{\phi_a} \frac{\partial G_h}{\partial n_X} \frac{\partial p_h}{\partial n_Y} p'_{aX} >_{S_p x S_v} +$$
$$< G_a \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{S_p x S_v} + < \frac{\tau_h}{\tau_a} G_h \frac{\partial p_h}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{S_p x S_v},$$

$$< A_4 >_{\Gamma x S_p} = < - \frac{\partial G_a}{\partial n_Y} p_{aY} (\frac{\partial p_a}{\partial n_X})' >_{\Gamma x S_p} - < \frac{\phi_h}{\phi_a} \frac{\partial G_h}{\partial n_Y} p_{aY} (\frac{\partial p_h}{\partial n_X})' >_{\Gamma x S_p} +$$
$$< G_a \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{\Gamma x S_p} + < \frac{\tau_h}{\tau_a} G_h \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_h}{\partial n_X})' >_{\Gamma x S_p} +$$
$$\frac{i\omega\tau_h}{\tau_a\phi_a} < G_h w_Y (\frac{\partial p_h}{\partial n_X})' >_{\Gamma x S_p},$$

$$< A_5 >_{S_p x \Gamma} = < - \frac{\partial G_a}{\partial n_X} \frac{\partial p_a}{\partial n_Y} p'_{aX} >_{S_p x \Gamma} - < \frac{\phi_h}{\phi_a} \frac{\partial G_h}{\partial n_X} \frac{\partial p_h}{\partial n_Y} p'_{aX} >_{S_p x \Gamma} +$$
$$< G_a \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{S_p x \Gamma} + < \frac{\tau_h}{\tau_a} G_h \frac{\partial p_h}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{S_p x \Gamma} +$$
$$\frac{i\omega\tau_h}{\tau_a\phi_a} < G_h \frac{\partial p_h}{\partial n_Y} w'_X >_{S_p x \Gamma},$$

$$< A_6 >_{S_v x S_v} = < \left(\frac{\partial^2 G_a}{\partial n_X \partial n_Y} - \frac{\phi_h}{\phi_a}\frac{\tau_a}{\tau_h} \frac{\partial^2 G_h}{\partial n_X \partial n_Y} \right) p_{aY}\, p'_{aX} >_{S_v x S_v} +$$
$$< \left(\frac{\partial G_h}{\partial n_X} - \frac{\partial G_a}{\partial n_X} \right) \frac{\partial p_a}{\partial n_Y} p'_{aX} >_{S_v x S_v} + < \left(\frac{\partial G_h}{\partial n_Y} - \frac{\partial G_a}{\partial n_Y} \right) p_{aY} (\frac{\partial p_a}{\partial n_X})' >_{S_v x S_v}$$
$$< \left(G_a - \frac{\tau_h}{\tau_a}\frac{\phi_a}{\phi_h} G_h \right) \frac{\partial p_a}{\partial n_Y} (\frac{\partial p_a}{\partial n_X})' >_{S_v x S_v},$$

$$
\begin{aligned}
< A_7 >_{\Gamma x S_v} = {} & < \left(\frac{\partial^2 G_a}{\partial n_X \partial n_Y} - \frac{\phi_h}{\phi_a} \frac{\tau_a}{\tau_h} \frac{\partial^2 G_h}{\partial n_X \partial n_Y} \right) p_{aY}\, p'_{aX} >_{\Gamma x S_v} + \\
& < \left(\frac{\partial G_h}{\partial n_X} - \frac{\partial G_a}{\partial n_X} \right) \frac{\partial p_a}{\partial n_Y} p'_{aX} >_{\Gamma x S_v} + < \left(\frac{\partial G_h}{\partial n_Y} - \frac{\partial G_a}{\partial n_Y} \right) p_{aY} \left(\frac{\partial p_a}{\partial n_X}\right)' >_{\Gamma x S_v} \\
& < \left(G_a - \frac{\tau_h}{\tau_a} \frac{\phi_a}{\phi_h} G_h \right) \frac{\partial p_a}{\partial n_Y} \left(\frac{\partial p_a}{\partial n_X}\right)' >_{\Gamma x S_v} + \\
& \frac{i\omega}{\phi_a} < \frac{\partial G_h}{\partial n_X} w_Y p'_{aX} >_{\Gamma x S_v} - \frac{i\omega\tau_h}{\tau_a \phi_h} < G_h w_Y \left(\frac{\partial p_a}{\partial n_X}\right)' >_{\Gamma x S_v},
\end{aligned}
$$

$$
\begin{aligned}
< A_8 >_{S_v x \Gamma} = {} & < \left(\frac{\partial^2 G_a}{\partial n_X \partial n_Y} - \frac{\phi_h}{\phi_a} \frac{\tau_a}{\tau_h} \frac{\partial^2 G_h}{\partial n_X \partial n_Y} \right) p_{aY} p'_{aX} >_{S_v x \Gamma} + < \left(\frac{\partial G_h}{\partial n_X} - \frac{\partial G_a}{\partial n_X} \right) \frac{\partial p_a}{\partial n_Y} p'_{aX} >_{S_v x \Gamma} \\
& + < \left(\frac{\partial G_h}{\partial n_Y} - \frac{\partial G_a}{\partial n_Y} \right) p_{aY} \left(\frac{\partial p_a}{\partial n_X}\right)' >_{S_v x \Gamma} + < \left(G_a - \frac{\tau_h}{\tau_a} \frac{\phi_a}{\phi_h} G_h \right) \frac{\partial p_a}{\partial n_Y} \left(\frac{\partial p_a}{\partial n_X}\right)' >_{S_v x \Gamma} + \\
& \frac{i\omega}{\phi_a} < \frac{\partial G_h}{\partial n_Y} p_{aY} w'_X >_{S_v x \Gamma} - \frac{i\omega\tau_h}{\tau_a \phi_h} < G_h \frac{\partial p_a}{\partial n_Y} w'_X >_{S_v x \Gamma},
\end{aligned}
$$

$$
\begin{aligned}
< A_9 >_{\Gamma x \Gamma} = {} & < \left(\frac{\partial^2 G_a}{\partial n_X \partial n_Y} - \frac{\phi_h}{\phi_a} \frac{\tau_a}{\tau_h} \frac{\partial^2 G_h}{\partial n_X \partial n_Y} \right) p_{aY} p'_{aX} >_{\Gamma x \Gamma} + \\
& < \left(\frac{\partial G_h}{\partial n_X} - \frac{\partial G_a}{\partial n_X} \right) \frac{\partial p_a}{\partial n_Y} p'_{aX} >_{\Gamma x \Gamma} + < \left(\frac{\partial G_h}{\partial n_Y} - \frac{\partial G_a}{\partial n_Y} \right) p_{aY} \left(\frac{\partial p_a}{\partial n_X}\right)' >_{\Gamma x \Gamma} + \\
& < \left(G_a - \frac{\tau_h}{\tau_a} \frac{\phi_a}{\phi_h} G_h \right) \frac{\partial p_a}{\partial n_Y} \left(\frac{\partial p_a}{\partial n_X}\right)' >_{\Gamma x \Gamma} + \frac{i\omega}{\phi_a} < \frac{\partial G_h}{\partial n_X} w_Y p'_{aX} >_{\Gamma x \Gamma} + \\
& \frac{i\omega}{\phi_a} < \frac{\partial G_h}{\partial n_Y} p_{aY} w'_X >_{\Gamma x \Gamma} - \frac{i\omega\tau_h}{\tau_a \phi_h} < G_h w_Y \left(\frac{\partial p_a}{\partial n_X}\right)' >_{\Gamma x \Gamma} - \frac{i\omega\tau_h}{\tau_a \phi_h} < G_h \frac{\partial p_a}{\partial n_Y} w'_X >_{\Gamma x \Gamma} + \\
& \frac{\omega^2 \tau_h}{\tau_a \phi_a \phi_h} < G_h w_Y w'_X >_{\Gamma x \Gamma},
\end{aligned}
$$

where for example $< \bullet >_{S_p x S_v} = \int_{S_v} \left(\int_{S_p} \bullet \; dS_Y \right) dS_X$.