FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION

DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

AN ELLIPTIC PERTURBATION METHOD FOR CERTAIN STRONGLY NON-LINEAR ROTORS

Livija Cveticanin

Faculty of Technical Sciences, Trg D. Obradovica 6, 21000 Novi Sad, Yugoslavia

ABSTRACT

An elliptic perturbation method is developed for calculating solutions of strongly nonlinear systems of the form $\ddot{z} + c_1 z + c_3 z^3 = \varepsilon f(z, \dot{z}, cc)$, where z is the complex deflection function. The Jacobian elliptic functions are employed instead of the usual circular functions. The suggested procedure can give also a second approximate solution. The method is applied for the equation which describes cyclic motion. The analytically obtained results are compared with numerical ones. They show a good agreement.

INTRODUCTION

The vibrations of the rotors are usually described with differential equations with complex functions. Due to nonlinear properties of the rotors the differential equations are also nonlinear. These non-linearities are not only weak but very often strong. To solve these equations various asymptotic methods are developed [1-9]. When the non-linearity is of cubic type the generating solution is a complex function of two Jacobian elliptic functions [2,4,7]. If beside the strong cubic nonlinear term also weak non-linearities exist, the perturbation methods based on the Bogolubov-Mitropolski method [4], Krylov-Bogolubov [2] and Elliptic-Krylov-Bogolubov [7,8] method are developed. For all of them it is common that the methods are correct only for small non-linearities.

In this paper a perturbation method developed for the systems with one degree of freedom [9] is extended for systems with two degrees of freedom described with complex function. The method is applicable not only for small but also large values of parameter ε but it is correct only for a strictly defined group of problems. The constraints of the method are discussed in the paper. An example with nonlinear gyroscopic force is discussed. The analytically obtained results are compared with numerical ones.

MATHEMATICAL MODEL

The mathematical model of the strong nonlinear rotor system is assumed as

$$\ddot{z} + c_1 z + c_3 z^3 = \varepsilon f(z, \dot{z}, cc), \qquad (1)$$

where z is the complex deflection function, x,y are the coordinates of rotor center, c_1 is the coefficient of linear and c_3 of nonlinear terms, f is the nonlinear function, cc is the complex conjugate function and ε is the small parameter. The equation (1) is a strong nonlinear differential equation with complex function. If the parameter ε is negligible the differential equation transforms to

$$\ddot{z} + c_1 z + c_3 z^3 = 0, \qquad (2)$$

where $z=x+iy$ is the complex function, x,y are coordinates, $i=(-1)^{1/2}$, c_1, c_3 are coefficients of the linear and cubic term. It represents the generating equation. For $c_1>0$, $c_3>0$ the solution is

$$z = A_0[\mathrm{cn}(\tau, k) + i\,\mathrm{sn}(\tau, k)], \quad \tau = \Omega_0 t \qquad (3)$$

where cn(τ,k) and sn(τ,k) are cosine and sine Jacobian elliptic functions, A_0, Ω_0, k are called the amplitude, the angular frequency and the modulus of the elliptic functions, respectively. The first and the second time derivatives of the function (3) are

$$\dot{z} = \frac{dz}{d\tau}\frac{d\tau}{dt} = -A_0 i\,\mathrm{dn}(\tau, k)[\mathrm{cn}(\tau, k) + i\,\mathrm{sn}(\tau, k)],$$

$$\ddot{z} = \frac{d\dot{z}}{d\tau}\frac{d\tau}{dt} = -\Omega_0^2 A_0 (ik^2 sncn[\mathrm{cn}(\tau, k) + i\,\mathrm{sn}(\tau, k)], \qquad (4)$$

where dn(τ,k) is the Jacobian elliptic function. Substituting equations (3) and (4) into (2), separating the real and imaginary parts and equating the coefficients of cn and cn^3 to zero we obtain

$$\Omega_0^2 = c_1 + c_3 A_0^2, \qquad k^2 = \frac{2c_3 A_0^2}{\Omega_0^2}. \qquad (5)$$

THE ELLIPTIC PERTURBATION METHOD

The eq.(1) can be written in the form

$$\ddot{x} + c_1 x + c_3(x^3 - 3xy^2) = \varepsilon f_1,$$
$$\ddot{y} + c_1 y + c_3(3x^2 y - y^3) = \varepsilon f_2, \qquad (6)$$

where f_1 and f_2 are the real and imaginary components of non-linearity, respectively. We assume the solution of the equations (6) in the form of a series $x = x_0 + \varepsilon x_1 + \ldots$, $y = y_0 + \varepsilon y_1 + \ldots$. In order to construct the correct solution which must include the parameter ε we introduce a nonlinear transformation

$$x = A\,\mathrm{cn}(\tau, k), \qquad y = A^*\mathrm{sn}(\tau, k), \qquad (7)$$

where

$$A = A_0 + \varepsilon A_1 + \ldots, \qquad A^* = A_0^* + \varepsilon A_1^* + \ldots \qquad (8)$$

The A_i and A_i^* are constants. We assume that the frequency of vibrations has different values in x and y direction (Ω and Ω^* are not equal) and it is

$$\Omega = \frac{d\tau_1}{dt} = \Omega_0 + \varepsilon\Omega_1(\tau) + \ldots, \qquad \Omega^* = \Omega_0 + \varepsilon\Omega_1^*(\tau) + \ldots \tag{9}$$

The frequencies Ω_1, Ω_1^*,... are dependent on τ. Then the first and second time derivatives of (7) are

$$\dot{x} = \frac{dx}{d\tau_1}\frac{d\tau_1}{dt} = x_0'\Omega_0 + \varepsilon(x_1'\Omega_0 + x_0'\Omega_1) + \varepsilon^2(x_0'\Omega_2 + x_1'\Omega_1 + x_2'\Omega_0) + \ldots, \tag{10}$$

$$\dot{y} = \frac{dy}{d\tau_2}\frac{d\tau_2}{dt} = \Omega_0 y_0' + \varepsilon(\Omega_0 y_1' + \Omega_1 y_0') + \varepsilon^2(\Omega_2^* y_0' + \Omega_1^* y_1' + \Omega_0 y_2') + \ldots, \tag{11}$$

$$\ddot{x} = x_0''\Omega_0^2 + \varepsilon(x_1''\Omega_0^2 + 2\Omega_0\Omega_1 x_0'' + x_0'\Omega_0\Omega_1') + \varepsilon^2(x_2''\Omega_0^2 + 2\Omega_0\Omega_1 x_1'' + x_0''\Omega_1^2 + x_1'\Omega_1' + x_0'\Omega_2'), \tag{12}$$

$$\ddot{y} = y_0''\Omega_0^2 + \varepsilon(y_1''\Omega_0^2 + 2\Omega_0\Omega_1^* y_0'' + y_0'\Omega_0\Omega_1^{*'}) + \varepsilon^2(y_2''\Omega_0^2 + 2\Omega_0\Omega_1^* y_1'' + y_0''\Omega_1^{*2} + y_1'\Omega_1^{*'} + y_0'\Omega_2^{*'}), \tag{13}$$

where $x' \equiv \dfrac{dx}{d\tau_1}$, $y_0' \equiv \dfrac{dy_0}{d\tau_2}$.

Substituting (7)-(13) into (6) and separating the terms with the same values of parameter ε a system of differential equations is obtained:

ε^0:

$$x_0''\Omega_0^2 + c_1 x_0 + c_3(x_0^3 - 3x_0 y_0^2) = 0, \tag{14}$$

$$y_0''\Omega_0^2 + c_1 y_0 + c_3(-y_0^3 + 3x_0^2 y_0) = 0, \tag{15}$$

ε^1:

$$x_1''\Omega_0^2 + 2\Omega_0\Omega_1 x_0'' + \Omega_0\Omega_1' x_0' + c_1 x_1 + c_3(3x_0^2 x_1 - 3x_1 y_0^2 - 6x_0 y_0 y_1) = f_1(x_0, y_0, x_0'\Omega_0, y_0'\Omega_0), \tag{16}$$

$$y_1''\Omega_0^2 + 2\Omega_0\Omega_1^{*} y_0'' + \Omega_0\Omega_1^{*'} y_0' + c_1 y_1 + c_3(-3y_0^2 y_1 + 3y_1 x_0^2 + 6x_0 y_0 x_1) = f_2(x_0, y_0, x_0'\Omega_0, y_0'\Omega_0), \tag{17}$$

ε^2:

$$x_2''\Omega_0^2 + 2\Omega_0\Omega_1 x_1'' + x_0''\Omega_1^2 + \Omega_0\Omega_1' x_1' + \Omega_0\Omega_2' x_0' + c_1 x_2 + c_3[3x_0 x_1^2 + 3x_2 x_0^2 - 3(x_2 y_0^2 + 2x_1 y_0 y_1 + x_0 y_1^2 + 2x_0 y_2 y_0)]$$
$$\frac{\partial f_1}{\partial x}x_1 + \frac{\partial f_1}{\partial y}y_1 + \frac{\partial f_1}{\partial x'}(x_1'\Omega_0 + x_0'\Omega_1) + \frac{\partial f_1}{\partial y'}(y_1'\Omega_0 + y_0'\Omega_1^*), \tag{18}$$

$$y_2''\Omega_0^2 + 2\Omega_0\Omega_1 y_1'' + y_0''\Omega_1^{*2} + \Omega_0\Omega_1^{*'} y_1' + \Omega_0\Omega_2^{*'} y_0' + c_1 y_2 - c_3[3y_0 y_1^2 + 3y_2 y_0^2 - 3(y_2 x_0^2 + 2y_1 x_0 x_1 + y_0 x_1^2 + 2y_0 x_2 x_0)] =$$
$$\frac{\partial f_2}{\partial x}x_1 + \frac{\partial f_2}{\partial y}y_1 + \frac{\partial f_2}{\partial x'}(x_1'\Omega_0 + x_0'\Omega_1) + \frac{\partial f_2}{\partial y'}(y_1'\Omega_0 + y_0'\Omega_1^*), \tag{19}$$

where $f_1 \equiv f_1(x_0, y_0, \Omega_0 x_0', \Omega_0 y_0')$, and $f_2 \equiv f_2(x_0, y_0, \Omega_0 x_0', \Omega_0 y_0')$.

The solutions of (14) and (15) are as for the generating solution

$$x_0 = A_0 cn(\tau, k), \qquad y_0 = A_0 sn(\tau, k),$$

where $\tau = \Omega_0 t$.

To transform the equations (16) and (17) they have to be multiplied with x_0' and y_0', respectively. The transformed equations are

$$\Omega_0\Omega_1 x_0'^2\Big|_0^t = -\Omega_0^2\frac{A_1}{A_0}x_0'^2\Big|_0^t - \frac{A_1}{A_0}(c_1x_0^2+c_3x_0^4-3x_0^2y_0^2c_3\Big|_0^t - 3\frac{A_1^*}{A_0}x_0^2y_0^2c_3\Big|_0^t - 3c_3\int_0^t x_0[2y_0'y_0\frac{A_1^*}{A_0}x_0 + 2\frac{A_1}{A_0}y_0y_0'x_0]d\tau + \int_0^t f_1x_0'd\tau,$$

$$\Omega_0\Omega_1^* y_0'^2\Big|_0^t = -\Omega_0^2\frac{A_1^*}{A_0}y_0'^2\Big|_0^t - \frac{A_1^*}{A_0}(c_1y_0^2+c_3y_0^4-3x_0^2y_0^2c_3\Big|_0^t - 3\frac{A_1}{A_0}x_0^2y_0^2c_3\Big|_0^t + 3c_3\int_0^t y_0[2x_0'x_0\frac{A_1^*}{A_0}y_0 + 2\frac{A_1}{A_0}y_0x_0'x_0]d\tau + \int_0^t f_2y_0'd\tau. \tag{20}$$

If the equations (20) are integrated in the period 4K where K is the first order Jacobian intergral, we obtain

$$-\frac{6c_3}{A_0}\int_0^{4K}x_0^2y_0y_0'(A_1+A_1^*)d\tau + \int_0^{4K}f_1x_0'd\tau = 0, \qquad \frac{6c_3}{A_0}\int_0^{4K}y_0^2x_0x_0'(A_1+A_1^*)d\tau + \int_0^{4K}f_2y_0'd\tau = 0. \tag{21}$$

If

$$\int_0^{4K}f_1x_0'd\tau = \int_0^{4K}f_2y_0'd\tau = 0, \tag{22}$$

the identities (21) are satisfied for any A_1 and A_1^*. This relation is the constraint condition for the developed method. As an example, it is $f = \pm iF\bar{z}_0$, i.e., $f_1 = \pm Fy_0'$, and $f_2 = \mp Fx_0$, where F is a real function of x_0, y_0, x_0', y_0'. This type of functions describes the nonlinear gyroscopic force.
Let us transform the eqs.(18) and (19) by multiplying it with x_0' and y_0', respectively. If we integrate the transformed equations in the period 4K it is

$$\int_0^{4K}[\frac{\partial f_1}{\partial x}x_1 + \frac{\partial f_1}{\partial y}y_1 + \frac{\partial f_1}{\partial x'}(x_1'\Omega_0 + x_0'\Omega_1) + \frac{\partial f_1}{\partial y'}(y_1'\Omega_0 + y_0'\Omega_1^*)]x_0'd\tau = 0, \tag{23}$$

$$\int_0^{4K}[\frac{\partial f_2}{\partial x}x_1 + \frac{\partial f_2}{\partial y}y_1 + \frac{\partial f_2}{\partial x'}(x_1'\Omega_0 + x_0'\Omega_1) + \frac{\partial f_2}{\partial y'}(y_1'\Omega_0 + y_0'\Omega_1^*)]y_0'd\tau = 0. \tag{24}$$

The values of A_1, A_1^*, Ω_1, Ω_1^*, are determined from the equations (20), (23) and (24). For computational reasons it is convenient to calculate the previous values according to the following relations

$$\Omega_1 = A_1W_1 + A_1^*W_1^* + W_0, \qquad \Omega_1^* = A_1B_1 + A_1^*B_1^* + B_0,$$
$$A_1 = \frac{P_1I_{22} - P_2I_{22}}{I_{11}I_{22} - I_{21}I_{12}}, \qquad A_1^* = \frac{P_1I_{21} - P_2I_{11}}{I_{12}I_{21} - I_{22}I_{11}}, \tag{25}$$

where

$$W_0 = \frac{1}{\Omega_0 x_0'^2}\int_0^{\tau} f_1(x_0, y_0, \Omega_0x_0', \Omega_0y_0')x_0'd\tau,$$

$$W_1 = -\frac{1}{A_0\Omega_0 x_0'^2}[\Omega_0^2 x_0'^2 + c_1 x_0^2 + c_3 x_0^4 - 3c_3 x_0^2 y_0^2 + 6c_3 \int_0^\tau x_0^2 y_0 y_0' d\tau],$$

$$W_1^* = -\frac{1}{A_0\Omega_0 x_0'^2}[3c_3 x_0^2 y_0^2 + 6c_3 \int_0^\tau x_0^2 y_0 y_0' d\tau],$$

$$B_0 = \frac{1}{\Omega_0 y_0'^2}\int_0^\tau f_2(x_0, y_0, \Omega_0 x_0', \Omega_0 y_0') y_0' d\tau,$$

$$B_1 = -\frac{1}{A_0\Omega_0 y_0'^2}[3c_3 x_0^2 y_0^2 - 6c_3 \int_0^\tau y_0^2 x_0 x_0' d\tau],$$

$$B_1^* = -\frac{1}{A_0\Omega_0 y_0'^2}[\Omega_0^2 y_0'^2 + c_1 y_0^2 - c_3 y_0^4 + 3c_3 x_0^2 y_0^2 - 6c_3 \int_0^\tau y_0^2 x_0 x_0' d\tau],$$

$$P_1 = -\int_0^{4K} \frac{\partial f_1}{\partial x'} W_0 x_0'^2 - \int_0^{4K} \frac{\partial f_1}{\partial y'} B_0 x_0' y_0', \qquad P_2 = -\int_0^{4K} \frac{\partial f_2}{\partial x'} W_0 x_0' y_0' - \int_0^{4K} \frac{\partial f_2}{\partial y'} B_0 y_0'^2,$$

$$I_{11} = \int_0^{4K} \frac{\partial f_1}{\partial x} \frac{x_0 x_0'}{A_0} + \int_0^{4K} \frac{\partial f_1}{\partial x'} \Omega_0 x_0'^2 + \int_0^{4K} \frac{\partial f_1}{\partial x'} W_1 x_0'^2 + \int_0^{4K} \frac{\partial f_1}{\partial y'} B_1 x_0' y_0',$$

$$I_{12} = \int_0^{4K} \frac{\partial f_1}{\partial y} \frac{y_0 x_0'}{A_0} + \int_0^{4K} \frac{\partial f_1}{\partial y'} \Omega_0 x_0' y_0' + \int_0^{4K} \frac{\partial f_1}{\partial x'} W_1^* x_0'^2 + \int_0^{4K} \frac{\partial f_1}{\partial y'} B_1^* x_0' y_0',$$

$$I_{21} = \int_0^{4K} \frac{\partial f_2}{\partial x} \frac{x_0 y_0'}{A_0} + \int_0^{4K} \frac{\partial f_2}{\partial x'} \Omega_0 x_0' y_0' + \int_0^{4K} \frac{\partial f_2}{\partial x'} W_1 x_0' y_0' + \int_0^{4K} \frac{\partial f_2}{\partial y'} B_1 y_0'^2,$$

$$I_{22} = \int_0^{4K} \frac{\partial f_2}{\partial y} \frac{y_0 y_0'}{A_0} + \int_0^{4K} \frac{\partial f_2}{\partial y'} \Omega_0 y_0'^2 + \int_0^{4K} \frac{\partial f_2}{\partial x'} W_1^* x_0' y_0' + \int_0^{4K} \frac{\partial f_2}{\partial y'} B_1^* y_0'^2.$$

In the first approximation the solution is $x = x_0 + \varepsilon x_1$, and $y = y_0 + \varepsilon y_1$, i.e.

$$x = (A_0 + \varepsilon A_1) cn(\tau, k),$$

$$\dot{x} = -(A_0 + \varepsilon\Omega_1 A_0 + \varepsilon\Omega_0 A_1) sn(\tau, k) dn(\tau, k), \tag{26}$$

and

$$y = (A_0 + \varepsilon A_1^*) sn(\tau, k),$$

$$\dot{y} = (A_0 + \varepsilon\Omega_1^* A_0 + \varepsilon\Omega_0 A_1^*) cn(\tau, k) dn(\tau, k), \tag{27}$$

EXAMPLE

Let us consider the case when the force has the form

$$f = (y_0 + i x_0) F, \tag{28}$$

i.e.,

$$f_1 = y_0 F, \qquad f_2 = x_0 F,$$

where

$$F = x_0 y_0 + (x_0^2 + y_0^2) x_0' y_0'. \tag{29}$$

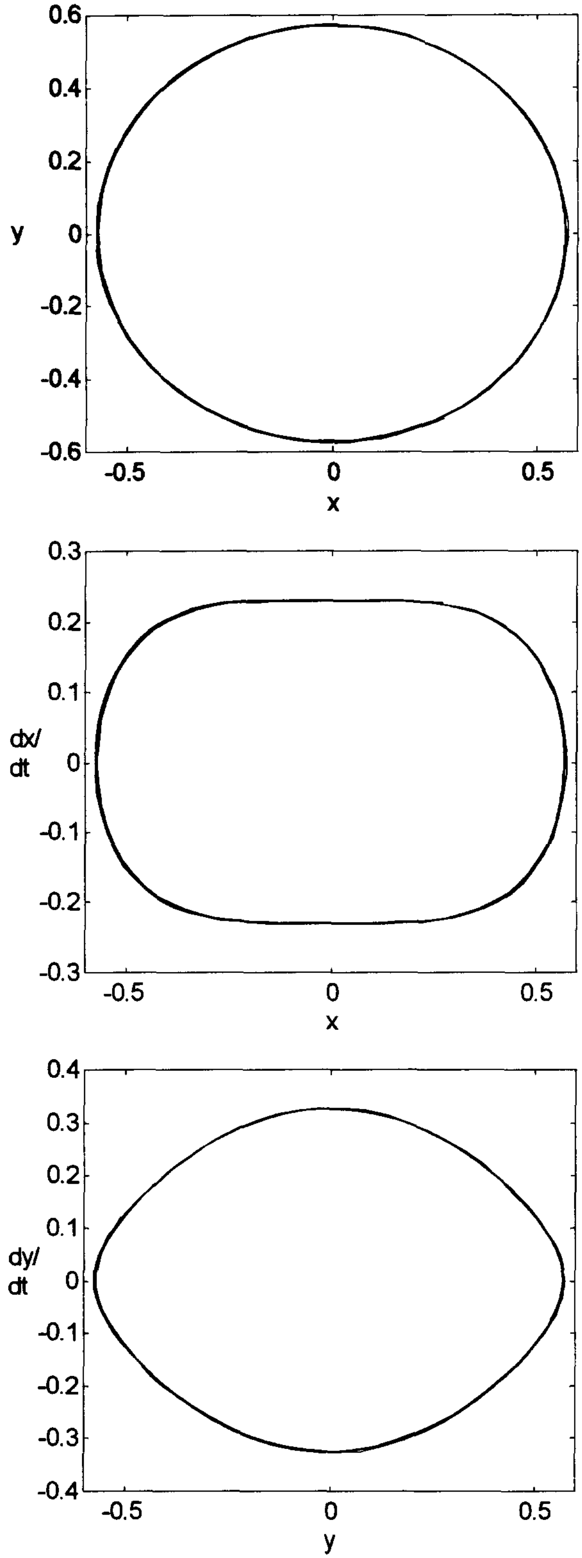

Fig.1. Limit cycles: ___ numerical solution, ---- analytical solution for ε=0.5.

Substituting (28) into (22) it is

$$A_0^2 = \frac{7K(1-k^2)(k^2-2)+14E(k^4+1-k^2)}{[4(1+k^2)(8+k^2)-5(9k^2+1)][K(2+k^2)-2E(1+k^2)]}. \qquad (30)$$

Due to the relations (25) it is evident that A_1 and A_1^* are zero and the frequencies are $\Omega_1 = W_0$, and $\Omega_1^* = B_0$, where

$$W_0 = -\frac{A_0^2}{\Omega_0 sn^2 dn^2}\{\frac{1}{15k^4}[(1-k^2)(k^2-2)\tau + 2(k^4+1-k^2)\frac{E}{K}\tau + 2(k^4+1-k^2)Z(\tau) +$$

$$k^2 sn\, cn\, dn(3k^2 sn^2 - 1 - k^2)]\} + \frac{A_0^4}{\Omega_0 sn^2 dn^2}\{-\frac{k^5}{7} sn^5 cn\, dn + \frac{8k^2+k^4}{7} sn^3 cn\, dn + \frac{11-3k^2}{70k^2}$$

$$[kcn\, dn + (1+k^2)\ln\frac{dn - kcn}{1-k} - k] + \frac{4(1+k^2)(8+k^2)-5(9k^2+1)}{105k^4}[(2+k^2)\tau - 2(1+k^2)E(\tau) + k^2 sn\, cn\, dn]\},$$

$$B_0 = -\frac{A_0^2}{\Omega_0 cn^2 dn^2}\{\frac{1}{15k^4}[(1-k^2)(k^2-2)\tau + 2(k^4+1-k^2)\frac{E}{K}\tau + 2(k^4+1-k^2)Z(\tau) +$$

$$k^2 sn\, cn\, dn(3k^2 sn^2 - 1 - k^2)]\} + \frac{A_0^4}{\Omega_0 cn^2 dn^2}\{-\frac{k^2}{7} sn^5 cn\, dn + \frac{8k^2+k^4}{7} sn^3 cn\, dn + \frac{11-3k^2}{70k^2}$$

$$[kcn\, dn + (1+k^2)\ln\frac{dn - kcn}{1-k} - k] + \frac{4(1+k^2)(8+k^2)-5(9k^2+1)}{105k^4}[(2+k^2)\tau - 2(1+k^2)E(\tau) + k^2 sn\, cn\, dn]\},$$

For the case when c_1=0.243384 and c_3= 0.25, the modulus of Jacobian function is k^2=1/2, and the initial amplitude and frequency are $A_0=\Omega_0=0.56966$ and the solution in the first approximation is $z = 0.56966[cn(0.56966t, \sqrt{0.5}) + isn(0.56966t, \sqrt{0.5})]$. In Fig.1. the limit cycles in $x-y$, $x-\dot{x}$ and $y-\dot{y}$ frames for ε=0.5 are plotted. The solutions obtained numerically by Runge-Kutta method and analytically by the presented method are compared. They are in good agreement.

CONCLUSION

It can be concluded that the elliptic perturbation method suggested in this paper is applicable for solving strong differential equations with periodic solutions. The method gives solutions which are in good agreement with those obtained numerically, even for high values of non-linearity.

REFERENCES

1. Cveticanin, L., Approximate analytical solutions to a class of non-linear equations with complex functions, Journal of Sound and Vibration, Vol.157, No.2, 289-302, 1992.
2. Cveticanin, L., An approximate solution for a system of two coupled differential equations, Journal of Sound and Vibration, Vol.152, No.2, pp.375-380, 1992.
3. Cveticanin, L., An asymptotic solution for weak nonlinear vibrations of the rotor, Mechanism and Machine Theory, Vol.28, No.4, pp.495-505, 1993.
4. Cveticanin, L., An approximate solution of a coupled differential equation with variable parameter, Trans. ASME, Journal of Applied Mechanics, Vol.60, March, pp.214-217, 1993.
5. Cveticanin, L., Dynamic behavior of a rotor with time-dependent parameters, JSME, Int. Journal, Ser.C, Vol.37, No.1, pp.41-48, 1994.

6. Cveticanin L., Approximate solution of a time-dependent differential equation, Meccanica, Vol.30, pp.665-671, 1995.
7. Cveticanin, L., Vibration of strongly nonlinear rotors with time variable parameters, Machine Vibration, Vol.4, pp.40-45, 1995.
8. Yuste, S.B., Bejarano, J.D., Construction of approximate analytical solutions to a new class of non-linear oscillator equation, Journal of Sound and Vibration, Vol.110, pp.347-350, 1986.
9. Chen, S.H., Cheung, Y.K., An elliptic perturbation method for certain strongly non-linear oscillators, Journal of Sound and Vibration, Vol.192, No.2, pp.435-464, 1996.