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OUTPUT PROBABILITY OF AN ENVIRONMENTAL VIBRATORY SYSTEM WITH A NON-LINEAR FEEDBACK ELEMENT

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Abstract

A statistical treatment for the output probability is proposed by introducing a statistical Lagrange series expansion method, where a general random process of arbitrary distribution type is passed through a time-variant linear environmental vibratory system with an arbitrary non-linear feedback element. A typical example is seen in an environmental vibratory system described by Duffing's non-linear differential equation. In order to find the effect of non-linear feedback element reflecting an environmental criterion, the explicit expression of the output probability distribution is derived in the general form of non-orthogonal expansion series, reflecting the effects of the forward linear element of the system into the first term. In view of the arbitrariness of the input characteristics, non-linear elements and fluctuation forms of system parameters, the validity of theoretical expression is experimentally confirmed by the method of digital simulation.

1. INTRODUCTION

As is well-known, the non-linear feedback operation in various types of actual vibratory systems may be conveniently divided into two categories: unaviodabley present and intentionally inserted non-linearities. As an example of the former the most prevalent forms of unavoidable non-linearity are the saturation and the third power of elasticity characteristics as seen in soft and hard spring forces of Duffing's non-linear vibratory system[1]. As an example of the latter, it can be taken up to add some artificial constraint to an elastic term as a positive control in the meaning of an environmental counter-measure, such as the use of a stopper in vibration control of vehicles and power turbo-governors. It is obvious that the above non-linear operation in the environmental vibratory systems is characterized as a non-linear feedback operation if the total vibratory system is illustrated by a block diagram of closed-loop type as shown in Fig.1.

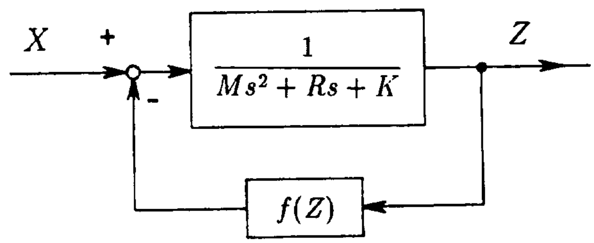


Fig.1 A vibratory system given by Duffing's non-linear equation.

On the other hand, when a statistical analysis for the prediction and control of environmental vibratory systems is especially sought, the following two approaches are basically important:

(1) Generating approach; where main attention is paid to the internal deterministic mechanism based on mathematical relationships or physical laws. The possible variety of probabilistic behavior of the output fluctuation is then considered by using the probabilistic measure-preserving law associated with the above *a priori* information of the system transformation.

(2) Integrated approach; where a universal framework for the output probability expression is introduced in advance or realized automatically. The specific features of internal mechanism and of input statistics are reflected successively into the characterizing parameters of the probability expression. This approach is, for example, seen in Wiener's classical theory of non-linear systems, where the combined networks by Hermite type non-linear elements and Laguerre filters are introduced in advance[2].

On the basis of the latter point of view, in this paper, a statistical treatment of output probability is proposed in the form of a universalized theory for acutual cases when an arbitrarily distributed random signal passes through a time-variant linear vibratory system with an arbitrary non-linear feedback element, with special reference to an environmental non-linear vibration. Namely,

(1) If a statistical method which is more effective for the close-looped system is especially sought, it is necessary to focus attention on how an output probability distribution of the system is affected by the existence of a feedback element. For this purpose, the explicit expression of the output probability distribution is newly derived in the general form of non-orthogonal expansion series, *i.e.*, the statistical Lagrange series expansion.

(2) On the other hand, if we especially attend to the temporal characteristics *i.e.*, memory effect of the forward linear element, a different approach is possible by taking instantaneous response probability without memory as the first expansion term[3].

(3) In view of the arbitrariness of possible input statistics, the possible variety of non-linear elements and fluctuation pattern of system parameters, and the complexity of the statistical treatment involved, the method of digital simulation must necessarily be utilized for experimental confirmation. The experimental simulation results are in good agreement with the theoretically calculated values for typical model cases.

2. MULTI-VARIATE LAGRANGE SERIES EXPANSION

The single-variate Lagrange expansion is known and proved from the viewpoint of complex function theory[4]. We introduce the multi-variate form to treat the memory effect of the system; the result is as follows:

[*Theorem*]

Consider two functions, $f(Z_i)$ and $F(\boldsymbol{Z})(\boldsymbol{Z} = \{Z_1, Z_2, \ldots, Z_K\})$, which are infinitely differentiable over $(-\infty, \infty)$. If $Z_i = X_i + a_i f(Z_i)$ (a_i is an arbitrary constant for $i = 1, 2, \ldots, K$), $F(\boldsymbol{Z})$ can be expressed in the form of expansion series:

$$F(\boldsymbol{Z}) = F(\boldsymbol{X}) + \sum_{n=1}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{1}{n_1! n_2! \cdots n_K!}$$

$$\frac{\partial^{n-K}}{\partial X_1^{n_1-1} X_2^{n_2-1} \cdots X_K^{n_K-1}} \left[\prod_{i=1}^{K} \{a_i f(X_i)\}^{n_i} \frac{\partial^K}{\partial X_1 \partial X_2 \cdots \partial X_K} F(\boldsymbol{X}) \right], \tag{1}$$

where $\boldsymbol{X} = \{X_1, X_2, \ldots, X_K\}$. When n_i is equal to zero, $(\partial/\partial X_i)^{n_i-1}$ represents an intergral over $(-\infty, X_i)$.

3. FEEDBACK EFFECT ON THE OUTPUT PROBABILITY

Consider a time-variant feedback system with finite memory shown in Fig. 2. When an arbitrary input sequence, $X_i (i = 1, 2, \ldots)$, is passed through the system, the output sequence, $Z_j (j = 1, 2, \ldots)$, at an arbitrarily sampled time point, $t = t_j$, is expressed as

$$Z_j = \sum_{i=1}^{j} \alpha_i (X_{j-i+1} - f(Z_{j-i+1}; \boldsymbol{\beta})), \tag{2}$$

where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$: descrete impulse response function in the forward linear element and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \ldots, \beta_r\}$: parameters expressing the non-linear feedback operation.

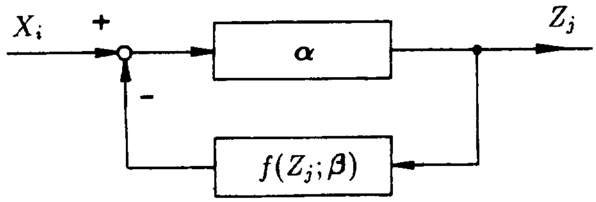


Fig.2 A linear vibratory system with a non-linear feedback element.

3.1 Basic formulation

From a structural viewpoint of the non-linear feedback system under consideration, the input-output relationship, Eq.(2), can be rewritten as

$$Z_j = \zeta_j - h(Z_j; \boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{3}$$

where

$$\zeta_j = \sum_{i=1}^{j} \alpha_i X_{j-i+1}$$

and

$$h(Z_j; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^{j} \alpha_i f(Z_{j-i+1}; \boldsymbol{\beta})).$$

3.2 Muti-variate probability expression

Let us now introduce an arbitrary function, $\varphi(\boldsymbol{Z})$, of multi-variate type, as a kind of catarytic function for the purpose of reorganizing the probability expression. $\varphi()$ and its successive derivatives satisfy the following condition:

$$\left.\frac{\partial^{r_1+r_2+\cdots+r_K}}{\partial Z_1^{r_1} \partial Z_2^{r_2} \cdots \partial Z_K^{r_K}} \varphi(\boldsymbol{Z}) \right|_{Z_j \to \pm\infty} \longrightarrow 0, \tag{4}$$

where $r_j \geq 0$. The above aribitrary functions, $\varphi()$, are sometimes called a set of Baire functions. Our present purpose is to derive a specific probability expression which is matched to investigating the feedback effect to output response on the basis of the statistics of random input signal, $\boldsymbol{X}$, and system parameters, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Here, the probabilistic stability for the output is assumed[5].

Consider an expectation with respect to $\varphi(\boldsymbol{Z})$:

$$I = <\varphi(\boldsymbol{Z})> = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi(\boldsymbol{Z}) P(\boldsymbol{Z}) d\boldsymbol{Z}$$

$$= \left\langle \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi(\zeta_1 - h(Z_1; \boldsymbol{\alpha}, \boldsymbol{\beta}), \zeta_2 - h(Z_2; \boldsymbol{\alpha}, \boldsymbol{\beta}), \ldots, \zeta_K - h(Z_K; \boldsymbol{\alpha}, \boldsymbol{\beta})) \right.$$

$$\left. P_\zeta(\boldsymbol{\zeta}|\boldsymbol{\alpha}, \boldsymbol{\beta}) d\boldsymbol{\zeta} \right\rangle_{\alpha,\beta}, \quad (\boldsymbol{\zeta} = \{\zeta_1, \zeta_2, \ldots, \zeta_K\}). \tag{5}$$

Let J be the definite integral inside $< . >_{\alpha,\beta}$. Using the multi-variate Lagrange series expansion, we can obtain the following relationship:

$$J = \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{(-1)^n}{n_1! n_2! \cdots n_K!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\partial^{n-K}}{\partial \zeta_1^{n_1-1} \partial \zeta_2^{n_2-1} \cdots \partial \zeta_K^{n_K-1}}$$

$$\left[\prod_{i=1}^{K} \{h(\zeta_i; \boldsymbol{\alpha}, \boldsymbol{\beta})\}^{n_i} \frac{\partial^K}{\partial \zeta_1 \partial \zeta_2 \cdots \partial \zeta_K} \varphi(\boldsymbol{\zeta}) \right] P_\zeta(\boldsymbol{\zeta}|\boldsymbol{\alpha}, \boldsymbol{\beta}) d\boldsymbol{\zeta}. \tag{6}$$

Carrying out an integration by parts $(n_j - 1)$ times for each variable, ζ_j, respectively and considering the limiting condition, Eq.(4), Eq.(6) becomes

$$J = \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{(-1)^K}{n_1!n_2!\cdots n_K!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\partial^K}{\partial\zeta_1\partial\zeta_2\cdots\partial\zeta_K}\varphi(\boldsymbol{\zeta})$$

$$\prod_{i=1}^{K}\{h(\zeta_i;\boldsymbol{\alpha},\boldsymbol{\beta})\}^{n_i} \frac{\partial^{n-K}}{\partial\zeta_1^{n_1-1}\partial\zeta_2^{n_2-1}\cdots\partial\zeta_K^{n_K-1}} P_\zeta(\boldsymbol{\zeta}|\boldsymbol{\alpha},\boldsymbol{\beta})d\boldsymbol{\zeta}. \tag{7}$$

By using the same procedure(integration by parts one time for each ζ_j), we obtain the following relationship:

$$J = \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{1}{n_1!n_2!\cdots n_K!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi(\boldsymbol{Z})$$

$$\frac{\partial^K}{\partial Z_1\partial Z_2\cdots\partial Z_K}\left[\prod_{i=1}^{K}\{h(Z_i;\boldsymbol{\alpha},\boldsymbol{\beta})\}^{n_i} \frac{\partial^{n-K}}{\partial Z_1^{n_1-1}\partial Z_2^{n_2-1}\cdots\partial Z_K^{n_K-1}} P_\zeta(\boldsymbol{Z}|\boldsymbol{\alpha},\boldsymbol{\beta})\right]d\boldsymbol{Z}, \tag{8}$$

where we replace the variables $\boldsymbol{\zeta}$ with $\boldsymbol{Z}$ because of the definite integral operation in Eq.(8). Substituting Eq.(8) into Eq.(5) and exchanging the integral for averaging operation, we consequently have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \varphi(\boldsymbol{Z})\Bigg[P(\boldsymbol{Z}) - \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{1}{n_1!n_2!\ldots n_K!}$$

$$\frac{\partial^K}{\partial Z_1\partial Z_2\cdots\partial Z_K}\left\langle\prod_{i=1}^{K}\{h(Z_i;\boldsymbol{\alpha},\boldsymbol{\beta})\}^{n_i} \frac{\partial^{n-K}}{\partial Z_1^{n_1-1}\partial Z_2^{n_2-1}\cdots\partial Z_K^{n_K-1}} P_\zeta(\boldsymbol{Z}|\boldsymbol{\alpha},\boldsymbol{\beta})\right\rangle_{\alpha,\beta}\Bigg]d\boldsymbol{Z} = 0. \tag{9}$$

Because of the arbitrariness of $\varphi(\boldsymbol{Z})$, $P(\boldsymbol{Z})$ is expressed by the series expansion of non-orthogonal type, as follows:

$$P(\boldsymbol{Z}) = \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{1}{n_1!n_2!\ldots n_K!}$$

$$\frac{\partial^K}{\partial Z_1\partial Z_2\cdots\partial Z_K}\left\langle\prod_{i=1}^{K}\{h(Z_i;\boldsymbol{\alpha},\boldsymbol{\beta})\}^{n_i} \frac{\partial^{n-K}}{\partial Z_1^{n_1-1}\partial Z_2^{n_2-1}\cdots\partial Z_K^{n_K-1}} P_\zeta(\boldsymbol{Z}|\boldsymbol{\alpha},\boldsymbol{\beta})\right\rangle_{\alpha,\beta}. \tag{10}$$

In practical problem, $P_X()$ with respect to the random input sequence, X_i, may be usually given instead of P_ζ with respect to the artificially introduced variable, ζ_i. Using the measure-preserving law: $P_\zeta((\boldsymbol{\zeta}|\boldsymbol{\alpha},\boldsymbol{\beta}) = P_X(\boldsymbol{X}|\boldsymbol{\alpha},\boldsymbol{\beta})|\partial\boldsymbol{X}/\partial\boldsymbol{\zeta}|\ |_{X_j\to\zeta_j}$. Thus, Eq.(10) can be rewritten as

$$P(\boldsymbol{Z}) = \left\langle \frac{1}{|\alpha_1^K|} P_X(\boldsymbol{X}|\alpha,\beta)\Big|_{X_j \Rightarrow Z_j} \right\rangle_{\alpha,\beta} \Bigg|_{Z_i' \to Z_i} + \sum_{n=1}^{\infty} \sum_{n_1+n_2+\cdots+n_K=n} \frac{1}{n_1! n_2! \cdots n_K!}$$
$$\frac{\partial^K}{\partial Z_1 \partial Z_2 \cdots \partial Z_K} \left\langle \frac{1}{|\alpha_1^K|} \prod_{i=1}^{K} \Big\{\alpha_1 f(Z_j;\boldsymbol{\beta}) + \sum_{i=2}^{j} \alpha_i f(Z'_{j-i+1};\boldsymbol{\beta})\Big\}^{n_j} \right.$$
$$\left. \frac{\partial^{n-K}}{\partial Z_1^{n_1-1} \partial Z_2^{n_2-1} \cdots \partial Z_K^{n_K-1}} P_X(\boldsymbol{X}|\boldsymbol{\alpha},\boldsymbol{\beta})\Big|_{X_j \Rightarrow Z_j} \right\rangle_{\alpha,\beta} \Bigg|_{Z_i' \to Z_i}. \tag{11}$$

Hereupon,

(1)partially differential operations with respect to Z_j denote differentiation along each sample path and the notation $X_j \Rightarrow Z_j$ expresses the transformation: $X_j = (Z_j - \sum_{i=2}^{j} \alpha_i X'_{j-i+1})/\alpha_1$. That is, past values of the input sequence, $\{X_1, X_2, \ldots, X_{j-1}\}$, and the output sequence, $\{Z_1, Z_2, \ldots, Z_{j-1}\}$, are specially distinguished by putting primes at a certain time point, j. Accordingly, all differential operations in Eq.(11) are carried out with respect to the present value of the output, $Z_j (j = 1, 2, \ldots, K)$, where all past values of Z''s are considered to keep constant. After completing all differntial operations, replacing operations are carried out as shown by $Z_i' \to Z_i$.

(2)It must be noticed that the first term of Eq.(11) exhibits the output probability based on the forward linear element and the second and higher expansion terms reflect the non-linear operational effects in the feedback element in terms of the probability form.

3.3 Special cases

(a)Single-variate probability expression

In a special case when the vibratory system is a zero-memory (memoryless) type($K = 1$ in Eq.(11)), the output probability density $P(Z)$ and cumulative distribution $Q(Y)(Y = (Z - \mu)/\sigma)$ functions are reduced respectively, as follows:

$$P(Z) = \Big\langle \frac{1}{|\alpha|} P_X\left(\frac{Z}{\alpha}|\alpha,\boldsymbol{\beta}\right)\Big\rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d}{dZ} \Big\langle \frac{\alpha^n}{|\alpha|} \{f(Z;\boldsymbol{\beta})\}^n \frac{d^{n-1}}{dZ^{n-1}} P_X\left(\frac{Z}{\alpha}|\alpha,\boldsymbol{\beta}\right)\Big\rangle. \tag{12}$$

$$Q(Y) = |\sigma| \int_{-\infty}^{Y} \Big\langle \frac{1}{|\alpha|} P_X\left(\frac{\sigma Y + \mu}{\alpha}|\alpha,\boldsymbol{\beta}\right)\Big\rangle dY$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|\sigma|}{\sigma^n} \Big\langle \frac{\alpha^n}{|\alpha|} \{f(\sigma Y + \mu;\boldsymbol{\beta})\}^n \frac{d^{n-1}}{dZ^{n-1}} P_X\left(\frac{\sigma Y + \mu}{\alpha}|\alpha,\boldsymbol{\beta}\right)\Big\rangle. \tag{13}$$

(b)Input/time dependency of parameters

In particular, when the random input, X_i, is statistically independent of the parameters, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the condition on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is eliminated in the conditional probability expression of Eqs.(10)-(13). Also, when the system is time-invariant, the averaging operation $< \ldots >$ on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is eliminated too.

4. EXPERIMENTAL CONSIDERATION

The objectives of the experimental study is to see how the probability distribution of the output fluctuation is influenced by the non-linear operation in the feedback element and memory effects of the forward linear element and/or the fluctuation form of the system parameters. For this purpose it is a good plan to classify all possible types of actual random input and non-linear feedback operation into some idealized special cases, where the above aribitrariness for the input and the system characteristics may correspond to reflection of actual situations of the environmental management problems. In the present experimental consideration, all random inputs consists of random numbers of specified distribution and normalized with mean 0 and variance 1. Thus, we will confirm the theoretical results by means of digital simulation.

4.1 Experimental models

(a)Random input models

1)Gaussian distribution, 2)Uniform distribution

3)logarithmic exponential distribution

4)Gram-Charlier series type

$P_X(X) = n(X) + An^{(3)}(X) + Bn^{(4)}(X) + \cdots;$

$n(X)$: normalized Gaussian distribution, $A = -0.2459, B = 0.0968$.

(b)Non-linear element models in the feedback path

1) $f(Z;\beta) = \beta_1 Z + \beta_2 Z^3$(soft spring model); $\beta_1 = 0.25, \beta_2 = 0.125 \sim 0.25$.

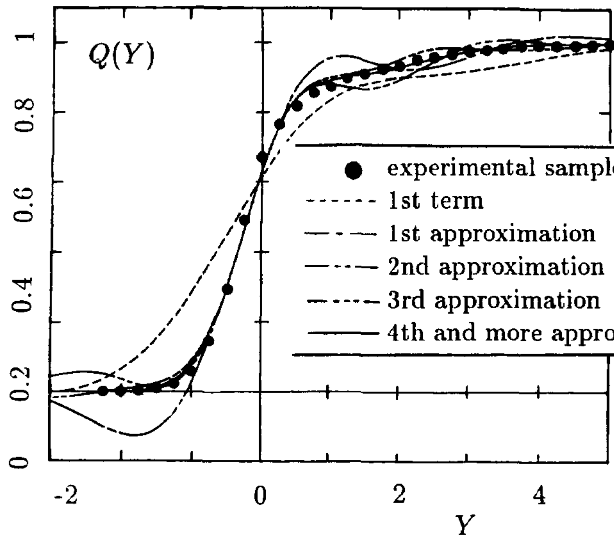
2) $f(Z;\beta) = \tanh \beta Z$(hard spring model); $\beta = 0.5 \sim 2.0$.

(c)Linear element models in the forward path

1)Zero-memory type, gain $< \alpha >= 0.5 \sim 2.0$.

2)Memory type, $\alpha_1 = 0.5, \alpha_2 = 0.05$.

3)Memory type, $\alpha_j = 0.5 \exp\{-0.8(j-1)\}(j = 1,2,3,4)$.



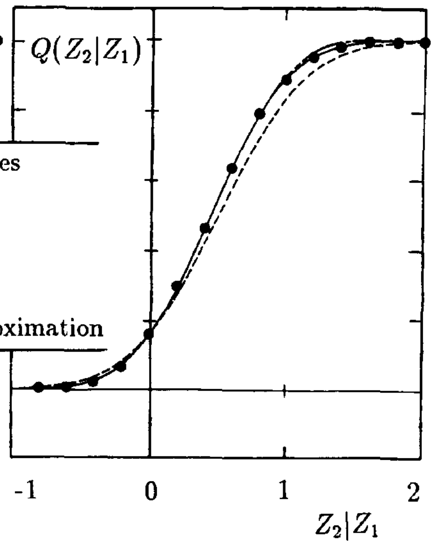


Fig.3 A comparison between theory and experiment for $Q(Y)$ (Eq.(13)) in the case that $f(Z;\beta) = \tanh(2.0Z)$ with a zero-memory forward element ($\alpha = 0.5$) and Gram-Charlier distribution type input random sequence.

Fig.4 A comparison between theory and experiment for conditional probability $Q(Z_2|Z_1)$ (Eq.(14)) in the case that $f(Z;\beta) = 0.25Z + 0.125Z^3$ with memory: $\alpha_1 = 0.5, \alpha_2 = 0.05$ and Gaussian random input sequence.

(d)Fluctuatin models of the parameters, α, β

1)Aperiodic(uniform distribution) model, $\xi[-0.25, 0.25]$.

2)Periodic(sinusoidal) model, $0.125\sin(2\pi t/\sqrt{2})$.

4.2 Experimental results

Figure 3 gives a comparison between theoretical curves and experimental sample values in a case $\alpha = 1.0$(zero-memory) and $f(Z;\beta) = \tanh(2.0Z)$ with Gram-Charlier distribution input model. Experimental confirmations of the multi-variate output probability expression, Eq.(11) for the vibratory system with memory are carried out in terms of a conditional probability distribution,$Q(Z_2|Z_1)$, experimentally defined by

$$Q(Z_2|u \leq Z_1 \leq v) = \int_{-\infty}^{Z_2} \int_u^v P(Z_1, Z_2) dZ_1 dZ_2 \Big/ \int_u^v P(Z_1) dZ_1. \tag{14}$$

Here, values of u and v are set to $\mu - 0.25\sigma$ and$\mu + 0.25\sigma$, respectively(μ and σ^2 are mean and variance values of Z_1). An experimental result in this case is shown in Fig. 4.

Hence, it has been experimentally confirmed that the present theory is useful for fairly wide class of input and system characteristics of the environmental vibratory systems. Many other examples which are not shown here also give a good agreement between theory and experiment.

5. CONCLUSIONS

In this paper, a new approach to the unified statistical treatment of the output probability distribution has been proposed, where random signals of arbitrary distribution type are passed through a class of time-variant vibratory systems with a non-linear element in the feedback path into which we can reflect environmental control and management policy. Experimental confirmations of the present theory are carried out by means of digital simulation where various models are taken into consideration for non-linear elements and fluctuation of system parameter.

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