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AN INTRODUCTORY STUDY OF THE CONVERGENCE OF THE DIRECT BOUNDARY ELEMENT METHOD

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ABSTRACT
Although boundary element methods have been used for three decades for the numerical solution of acoustic problems, the issue of convergence is not well known among acoustic engineers. In this paper the concept of convergence is introduced in an intuitive and empirical style. The convergence of an axisymmetric boundary element formulation is studied using linear, quadratic or superparametric elements. It is demonstrated that the rate of convergence of these formulations is reduced for calculations involving bodies with edges (geometric singularities). Two methods for improving the rate of convergence for these cases are suggested and examined.

1. INTRODUCTION
During the last three decades boundary element methods have evolved to become one of the standard numerical methods for solving acoustic problems. Several products are now commercially available for the calculation of acoustic problems using numerical techniques including the boundary element method. Hence, a typical user of the boundary element method is now an engineer, who is not necessarily acquainted with all details of the boundary element method and code. One of the important topics for the end-user is the question of the accuracy of the resulting solution. It is well accepted that using a finer division of the mesh (more elements) improves the accuracy. Likewise, it is well known that the use of advanced interpolation functions (shape functions) also improves accuracy. However, the underlying theory and assumptions of these results do not seem to be well known. The study of the accuracy of a numerical solution *versus* the computational work in terms of computer time is the study of convergence. The formal analysis of convergence involves advanced functional analysis, and the results of this somewhat recondite theory do not seem to be well known among acoustic engineers.

In convergence analysis the methods of improving accuracy traditionally are divided into two groups. In the h-method higher accuracy is obtained by a finer division of the mesh. The typical element dimension is normally denoted by h, which gives the method its name. The p-method improves accuracy by using increasingly higher order of the interpolation functions. The order of the interpolation function used, is normally denoted by p giving the method its name (in this paper m is used in order to avoid confusion with the pressure). If both the mesh is refined and the order of interpolation functions is increased the method is termed a hp-method. The p-method requires that shape functions of high order are available in the code, which is often not the case at present time. Hence, the method that is usually applied is the h-method, i.e. mesh refinement. The present paper conducts a convergence study of the direct collocation boundary element method using the h-method. Rather than presenting the rigorous mathematical theory described e.g. in reference [1], an alternative development is presented, where focus has been put on intuitive understanding.

2. THE HELMHOLTZ INTEGRAL EQUATION

The direct boundary element method is the numerical solution of the Helmholtz integral equation. The Helmholtz integral equation relates the pressure $p(P)$ outside a vibrating or scattering body to the pressure $p(Q)$ on the surface of the body, the surface velocity $v(Q)$ normal to the body and (if desired) an incoming wave $p^I(P)$. The formulation used in this paper is a restriction of the general three-dimensional integral equation to axisymmetric geometries and boundary conditions [2] using a transformation in to cylindrical coordinates (r, z, θ):

$$C(P)p(P) = \int_L \left(p(Q)\, F^B(P,Q) + ikz_0 v(Q)\, F^A(P,Q)\right) \mathrm{d}L + 4\pi p^I(P), \tag{1a}$$

where

$$F^A(P,Q) = r_Q \int_0^{2\pi} G(R)\ \mathrm{d}\theta_Q, \tag{1b}$$

and

$$F^B(P,Q) = r_Q \int_0^{2\pi} \frac{\partial G(R)}{\partial n} \mathrm{d}\theta_Q. \tag{1c}$$

This equation relates the pressure at the point P, $p(P)$, to the pressure $p(Q)$ and the normal velocity $v(Q)$ on the surface of a closed axisymmetric body described by its generator L. In the medium p satisfies $\nabla^2 p + k^2 p = 0$. The time factor $e^{i\omega t}$ has been suppressed. $R = | P\text{-}Q |$ is the distance between P and Q. $G(R) = e^{-ikR}/R$ is the free-space Green's function; $k = \omega/c$ is the wavenumber, where ω is the circular frequency and c is the speed of sound; i is the imaginary unit, z_0 is the characteristic impedance of the medium and n is the unit normal to the surface at the point Q directed away from the body. The quantity $C(P)$ has the value 0 for P inside B and 4π for P outside B. In the case of P on the surface S, $C(P)$ equals the solid angle measured from the medium ($=2\pi$ for a smooth surface).

A popular way of bringing equation (1a) to a form suitable for computation is the collocation technique. The integral on the right-hand of equation (1a) is divided into a sum of integrals each concerning a segment of the generator L_j, where the index $j = 1,2,..,N$ denotes the number of a segment of the generator. Within each segment a simple curve (usually linear or quadratic) is used to describe the geometry, and the acoustic variables are likewise described by simple functions. After these two approximations are made, the geometry of a linear segment can be described by the coordinates of the starting point and the end point of each segment - for quadratic interpolation also the midpoint values are needed. The term used for these points are nodes. Likewise the values of the acoustic variables are now defined by their nodal values. The term used for a segment after these approximations have been carried out is 'element', giving the method its name. If we assume that the normal velocities are known (this is the usual situation) we then have a finite number of unknown nodal pressures M.

In the collocation approach the M equations needed to match the M unknowns for finding a unique solution of the problem are then found by placing the point P in turn at M positions (collocation points). Normally these positions are chosen to be at the nodes on the surface (the so-called surface formulation), since the resulting coefficient matrix becomes dominated by the $C(P)$ terms in the diagonal, which is desirable from a computational point of view. This leaves the problem of handling singular integrals, since the Green's function and its normal derivative become singular when R tends to zero, i.e. when the integration point Q passes the calculation point P in the integral on the right-hand side of equation (1a). However, this problem is well known and is easily solvable [2]. Alternatively, the calculation point could be placed inside the body, where $C(P) = 0$ (the so-called interior formulation), but this strategy is less desirable since it does not lead to the desired coefficient matrix with large entries in the diagonal.

Once the problem has been discretized the resulting equations may be written in matrix form,

$$(\mathbf{C}-\mathbf{D})\mathbf{p} = -ikz_0\mathbf{M}\mathbf{v} - 4\pi\mathbf{p}^I \,, \tag{2}$$

where the complex vector $\mathbf{p}$ contains the nodal pressures, $\mathbf{v}$ contains the nodal normal velocities and $\mathbf{p}^I$ contains the pressure of the incident wave at the nodes in the absence of the body. The matrix $\mathbf{C}$ is diagonal; its values c_{ii} are the solid angles at node number i. The elements in the matrices $\mathbf{D}$ and $\mathbf{M}$ are integrals over the elements of the segments, so that the column number is related to the segment number and the row number is the calculation point number. $\mathbf{M}$ contains integrals over the Green's function (monopole terms), and $\mathbf{D}$ contains integrals over the normal derivative of Green's function (dipole terms).

3. CONVERGENCE

In the transition from the integral equation (1) to its numerical counterpart (2) five sources of error may be identified: i) uncertainties in calculating the functions F^A and F^B in equation (1), ii) approximations of the boundary, iii) approximations of the acoustic variables, iv) the numerical integration over each element, v) errors in the solution of the system of equations.

In the following it will be assumed that the error in determining the functions F^A and F^B is negligible. This assumption seems allowable if the singular integrals mentioned in the previous section are correctly handled. Moreover, it is possible to test the validity of this assumption at a later stage.

It will also be assumed that the error due to numerical integration is negligible, and it turns out that this criterion can be used to decide the minimum order of the numerical integration formula [3]. Numerical experiments have confirmed that for quadratic elements a four point Gauss-Legendre quadrature formula gives the same accuracy and convergence rate as all higher order quadrature formulas, whereas a two point Gauss-Legendre quadrature formula destroys the accuracy and convergence rate in agreement with the theory in reference [3] (the three point formula has not been used since one of the integration points then coinsides with the singularity).

Finally it will be assumed that the error in solving the system of equations is negligible. This is more disputable since it is a well-known fact [4] that the system of equations is ill-conditioned at or near the so-called characteristic frequencies, and that the solution then may be wrong. However, for the geometries considered here the characteristic frequencies are known in advance and have been avoided, and it has been found that for all cases the resulting system of equations is well conditioned. A favourable feature of the surface formulation considered here is that the condition number of the resulting coefficient matrix tend to a constant as the number of elements is increased [3], so the set of equations does not become more ill-conditioned in the limit of $h \rightarrow 0$ as is the case in some other numerical schemes. Furthermore, in reference [3] it was found that the convergence rate was not destroyed even when very close to the characteristic frequencies (although the level of error was dramatically increased). Only exactly at a characteristic frequency (up to six significant digits) the formulation failed to converge. A similar experiment (not shown here) carried out on the present formulation showed the same behaviour.

Hence, the present work deals with the errors due to the discretization of the geometry and of the acoustic variables. Initially we consider the error due to the approximation of the acoustic variables only. The effect of discretizing the geometry is considered later.

Consider the solution of a scattering problem where the body is infinitely hard (i.e. $\mathbf{v}=\mathbf{0}$) and smooth (i.e., $C(P)=2\pi$ for all collocation points P). Hence, it is assumed that the dipole matrix is the sum of an exact part $\bar{\mathbf{D}}$ and a part $\tilde{\mathbf{D}}$ that contains the error due to the discretization of the pressure. Equation (2) may now be written as

$$\left(\mathbf{C}-\bar{\mathbf{D}}-\tilde{\mathbf{D}}\right)\cdot\left(\bar{\mathbf{p}}+\tilde{\mathbf{p}}\right) = \mathbf{p}^{I}, \tag{3}$$

where the tilde denotes the error made by the approximation and the bar denotes the true vectors or matrices. Likewise, it is assumed that the pressure may be written as a sum of an exact part and a resulting error. Note that the right-hand side is given with no errors. Working out the left-hand side of equation (3) and using $(\mathbf{C}-\bar{\mathbf{D}})\bar{\mathbf{p}} = \mathbf{p}^{I}$, the equation

$$(\mathbf{C}-\bar{\mathbf{D}})\tilde{\mathbf{p}} = \tilde{\mathbf{D}}(\bar{\mathbf{p}}+\tilde{\mathbf{p}}) \tag{4}$$

is found. If we assume that making the mesh finer reduces the error (i.e., if we assume that the method converges, which is proven in reference [1]), then $\tilde{\mathbf{p}}$ will become much smaller than $\bar{\mathbf{p}}$ as $M\to\infty$, and equation (4) then reduces to

$$(\mathbf{C}-\bar{\mathbf{D}})\tilde{\mathbf{p}} = \tilde{\mathbf{D}}\bar{\mathbf{p}}. \tag{5}$$

Note that this analysis concerns the effect of discretization of the acoustic variables only - the discretization with respect to the geometry is treated later in this paper (and theoretically in references [1,3]). Suppose that the object is divided into linear or quadratic elements of the same size. For convenience of the following analysis each element is transformed into a standard interval $[0;h]$. (In an actual implementation the parent element interval is normally $[-1;1]$, but if this interval is used the analysis below would then be a bit more tedious and give the same result.) The elements of $\mathbf{D}=\bar{\mathbf{D}}+\tilde{\mathbf{D}}$ can then be written in the following form:

$$\int_0^h N_\alpha(x)\, F^{B}(P,Q(x))\, J(x)\ \mathrm{d}x, \tag{6}$$

where $J(x)$ is the Jacobean of the transformation, and the N_α's are the shape functions. If it is assumed that the pressure and its derivatives are sufficiently smooth, the exact value of the pressure p in an element will be given as

$$p(x) = \sum_{\alpha=1}^{m} p_\alpha N_\alpha(x) + h^{m}\frac{\partial^{m}p(x_0)}{\partial x^{m}}\frac{1}{m!}, \tag{7}$$

according to Taylor's formula. Here x_0 is an unknown point between 0 and h. Hence the elements in $\tilde{\mathbf{D}}$ are of the form

$$\int_0^h h^{m}\frac{\partial^{m}p(x_0)}{\partial x^{m}}\frac{1}{m!}\, F^{B}(P,Q(x))\, J(x)\ \mathrm{d}x. \tag{8}$$

By the use of partial integration it can now be seen that in the limit of small element length h the elements in $\tilde{\mathbf{D}}$ are proportional to h^{m+1}. Note that the dimension M of $\tilde{\mathbf{D}}$ is inversely proportional to h. Hence, each element in the vector $\tilde{\mathbf{D}}\bar{\mathbf{p}}$ is the sum of $M\propto 1/h$ terms of magnitude h^{m+1}, and therefore of the magnitude h^m, since the true solution $\bar{\mathbf{p}}$ is slowly varying in the limit of small h. Now, consider the left-hand side of equation (5). The elements in a single row of the matrix $\bar{\mathbf{D}}$ are related to a surface integral over the object, since each element in $\bar{\mathbf{D}}\,\bar{\mathbf{p}} = C(P)\,p_{true}(P)$ is the value of the surface integral of the true pressure and the derivative of Green's function with respect to the collocation point P. This integral is in the limit a constant (the solid angle 2π times the true pressure at P), and therefore each element in the matrix $\bar{\mathbf{D}}$ is of magnitude h (i.e., the magnitude of each element decreases proportionally to the length of the integration interval, which seems reasonable). Now, *suppose* that the elements in $\tilde{\mathbf{p}}$ are of order m_1 (i.e., of magnitude h^{m1}). Then each element of $\bar{\mathbf{D}}\,\tilde{\mathbf{p}}$ must also be of magnitude m_1 since each element of $\bar{\mathbf{D}}\,\tilde{\mathbf{p}}$ is a sum of $M \propto 1/h$ terms of magnitude $hh^{m1}=h^{m1+1}$. Likewise $\mathbf{C}\,\tilde{\mathbf{p}}=2\pi\,\tilde{\mathbf{p}}$ is of magnitude m_1. Hence, the rigth-hand side of equation (9) is of the same magnitude as $\tilde{\mathbf{p}}$, m_1, and since the left-hand side of equation (5) is of magnitude m, $m_1=m$ is found.

For linear shape functions the expected convergence rate $m=2$ is found, and for quadratic shapefunctions the expected convergence rate is $m=3$, which is consistent with the findings of others [1,3].

3.1. First test case: Scattering by a rigid sphere

The scattering of a plane wave by a rigid sphere at $ka=1$ serves as the first test case. For this problem an analytical solution is known, so the error made by the BEM formulations may easily be obtained. Figure 1 shows the errors measured as the ratio between the length of the residual and the length of the analytical vector as functions of the number of nodes, M, which is a good measure of the computational work. The results of three different formulations are shown. It is evident that the curves are straight lines in a double logarithmic coordinate system, verifying the result of the previous paragraph that the error is of the form M^{-m}. For the isoparametric linear formulation which uses a piecewise linear approximation for both the geometry and the pressure, a slope of -2 is found. This corresponds to an order of two and is in agreement with the findings of the previous section. For the isoparametric quadratic formulation a slope of about -4 is found. The order of four is higher than expected, and is due to symmetry of the formulation cancelling third order terms. A similar effect has been reported for constant elements [3]. The last curve shown represents a superparametric formulation using quadratic approximation for the geometry but a linear approximation for the pressure. Evidently, this curve is parallel to the curve representing the isoparametric formulation, but the level of error is smaller. Hence, it may be concluded that the error due to the discretization of the geometry is of the same order as the error due to the discretization of the pressure since the curves otherwise would not be parallel. This observation is consistent with the theoretical findings [1,3]. More frequencies have been investigated showing the same qualitative behaviour (and

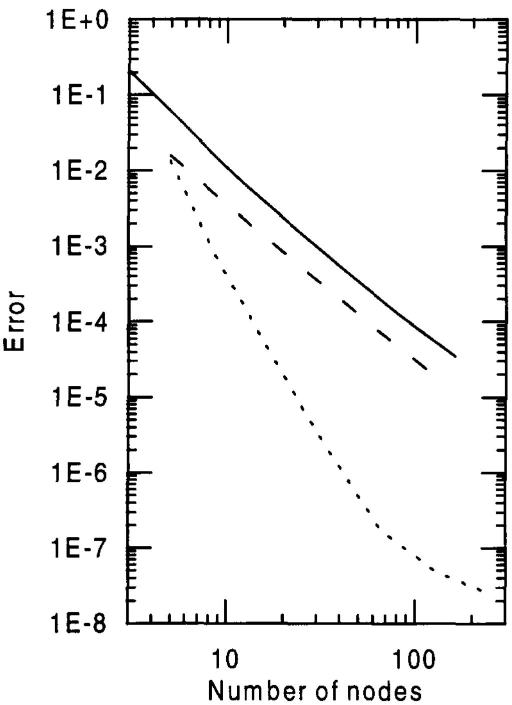

Figure 1. Error as a function of the number of nodes M for different boundary element formulations applied to the problem of scattering by a rigid sphere at ka=1. ———, Isoparametric linear formulation; - - - - - - -, superparametric linear/quadratic formulation; · · · · · · · ·, isoparametric quadratic formulation

hence the results are not shown) - the influence of the frequency is a vertical displacement of the curves in Figure 1.

3.2. Geometric singularities

One of the assumptions in paragraph 3 is that the sound pressure can be described by its Taylor expansion so that the order of the formulation is determined by the first term of the expansion that is not included in the shape functions. If the sound pressure has not a valid Taylor expansion, this assumption is flawed. Consider the sound field near the 270-degree edge of a cylinder. The asymptotic behaviour of the pressure near an edge may be found by the following loose study (for a more rigid development the reader should refer to Pierce [5 p. 187]). Consider the sound field near the edge of the cylinder. For the diffraction problem, the term of interest is *kr*, where *r* is the distance from the edge. Now, for any finite frequency *kr* tends to zero when *r* tends to zero, but since the diffraction problem is governed by *kr* it is mathematically legitimate to keep *r* constant and let *k* tend to zero instead, and still draw the same conclusions from the approximative study. Hence, in the limit of $r \ll 1/k$ Laplace's equation may be used. For Laplace's equation, traditionally used in the limit of small *k*, it is well known that an edge of angle $3\pi/2$ produces an $r^{-1/3}$ behaviour of the flow velocity at the edge - the general rule is that an angle α produces an $r^{\pi/\alpha-1}$ behaviour of the flow velocity [6, p.69]. Hence, the particle velocity tends to infinity as *r* tends to zero. The well-known $r^{-1/2}$ behaviour of the particle velocity near the edge of a thin screen [5 p.505,7-8] may also be explained in this way. Hence the pressure near the edge of the cylinder shows an $r^{2/3}$ behaviour that is taken into account neither by the linear nor the quadratic formulation and therefore becomes the limiting factor on the rate of convergence. This is illustrated in Figure 2, which shows the errors of the isoparametric linear and quadratic formulations as functions of the number of nodes. In the lack of an analytical solution a very fine meshed BEM solution has been used as the true solution, for which it has been ensured that the error is far below the level of errors shown in Figure 2. It can be seen that both formulations now have the reduced slope of about -1.2, which is far from the values found for the sphere. The accuracy and the rate of convergence are clearly reduced due to the presence of the singularity.

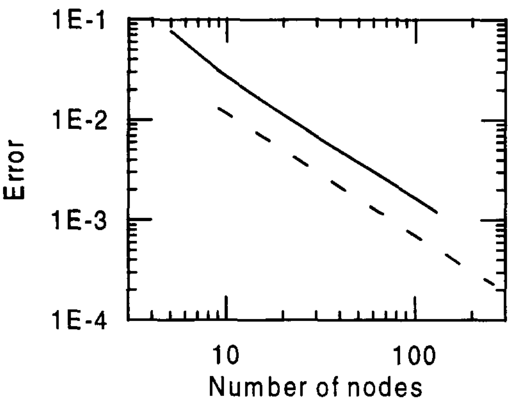

Figure 2. Error as a function of number of nodes M for different boundary element formulation applied to the problem of scattering by a rigid cylinder. ———, Isoparametric linear formulation; - - - - - - - -, isoparametric quadratic formulation.

3.3. Singular shape functions

The most obvious way of handling the singularity found for the problem of scattering by a rigid cylinder is to use a mesh which is graded towards the singularity. However, this leads to additional computational work. An alternative is to model the singularity by the shape functions, i.e. to use 'singular' shape functions near the singularity and regular shape function where the solution is smooth. Regular linear shape functions are developed from the set $\{1,x\}$ so the 'singular' shape functions designed to handle the $r^{2/3}$ singularity must be developed from the set $\{1,x^{2/3}\}$. Hence the approximation of the pressure in $x\in[-1,1]$ using singular shape functions aiming at a singularity at x=+1 is:

$$p(x) = 2^{-2/3}(1-x)^{2/3}p_{-1} + (1-2^{-2/3}(1-x)^{2/3})p_{+1} \tag{9}$$

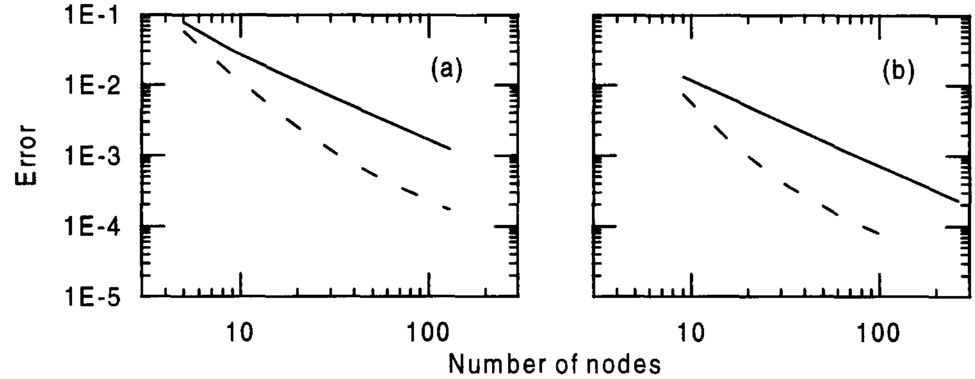

Figure 3. Error as a function of number of nodes M for different boundary element formulation applied to the problem of scattering by a rigid cylinder. (a) ———, Isoparametric linear formulation; - - - - - - - -, specialized linear formulation. (b) ———, Isoparametric quadratic formulation; - - - - - - - -, isoparametric quadratic formulation with the generalized quarter point formulation.

Analogous shape functions can by found that models the singularity in x=-1.

Figure 3(a) shows the error of the original linear formulation and the improved formulation as a function of the number of nodes. For rough meshes (i.e. meshes using few nodes) the original order of two of the linear formulation is restored, but for fine meshes the slope is gradually reduced to -1.2. This reduction of the slope for finer meshes is due to the fact that as the mesh is refined the element next to the singular element gets close to the singularity and exhibits singular behaviour. It should be noted that the accuracy at the stage when the improved formulation reduces its slope is sufficient for most practical purposes

3.4 GENERALIZED QUARTER POINT TECHNIQUE

In mechanics singular elements have been used to model the $r^{1/2}$ behaviour of the stress field near a crack. For quadratic elements it was found that there was no need to design special shape functions in order to model this singularity. If the mid-element node of a quadratic element was displaced to its quarter point position, the element exactly modelled the desired $r^{1/2}$ behaviour.

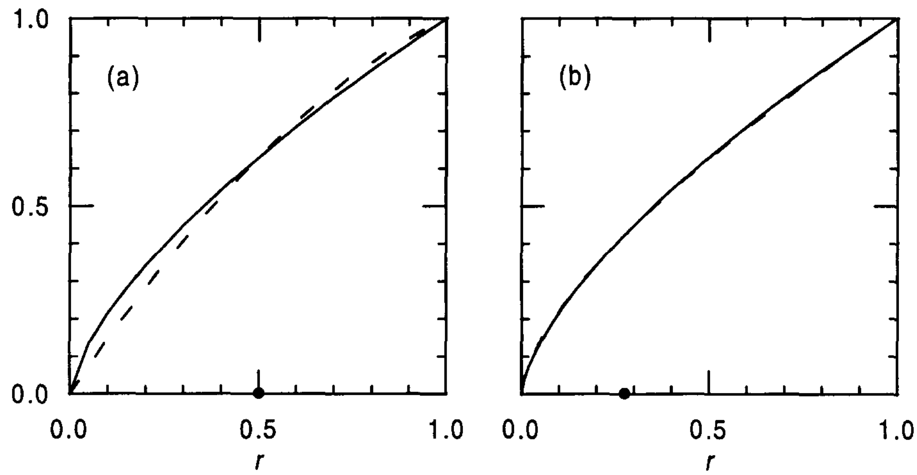

Figure 4. Modelling of the $r^{2/3}$ function (a) using normal isoparametric quadratic elements; (b) using the generalized quarter point formulation. The placement of the mid-element node is indicated with a bold dot on the r-axis.

The acoustic equivalent of the crack is the thin screen, and Wu and Wan [8] have adopted the quarter point technique to these problems. They found that the accuracy was considerably improved when using the quarter point technique.

The idea that now arises is a generalization of the quarter point technique. By trial and error the mid-element node of an isoparametric quadratic element has been moved towards the position of the singularity of the $r^{2/3}$ function, and it was found that at a distance of 0.275 times the element length from the singularity the $r^{2/3}$ function was well modelled. Figure 4(a) shows how the $r^{2/3}$ function is modelled with a normal quadratic element, and Figure 4(b) shows how the function is modelled using the generalized quarter point technique. It is evident that the function is not modelled exactly using this approach as it is the case for the square root singularity and the quarter point technique, but the modelling is still very good. The main benefit of this formulation is that the coding of special singular elements into a BEM program is avoided - the technique can readily be used with existing codes. Figure 3(b) shows the error of the original quadratic formulation compared to the error of the generalized quarter point formulation as functions of the number of nodes. Initially, a slope of about -3 is found, but for finer meshes the slope decreases to the value -1.2 of the original formulation. Once again the problem is that the elements next to the element containing the singularity exhibit nearly singular behaviour for fine meshes.

4. CONCLUSIONS

This paper examines convergence of boundary element formulations in an intuitive and empirical manner. Convergence results are found for an axisymmetric formulation using linear and quadratic shape functions, and test cases confirm the findings.

It is found that bodies with edges give rise to geometric singularities that reduce the rate of convergence of the formulations. Two methods are presented to overcome this difficulty. One of these applies specialized shape functions to deal with the geometric singularity, whereas the other models the singularity by a displacement of the mid-element node in a quadratic element. The latter approach may readily be used with existing codes, and has been named the generalized quarter point technique. These two specialized formulations improve the accuracy considerably and restore a high order of convergence for the rough meshes that are usually applied for engineering purposes.

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