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APPROXIMATING EIGENSOLUTIONS OF DISTRIBUTED STRUCTURES USING ADJUSTABLE BASE FUNCTIONS

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ABSTRACT

The prediction of resonance frequencies and mode shapes of distributed structures requires high convergence rate and accuracy. When approximated methods, such as the Rayleigh-Ritz method, are used, these requirements are largely dependent upon the base functions used to express the mode shape functions. This paper shows that while base functions selected to satisfy all the boundary conditions of the structure, their linear independency may be reduced. This will results in ill conditioned mass matrix in the generalised eigenequations (large matrix condition number) and prediction error. On the other hand, the base functions with good linear independent property may not satisfy part of the boundary conditions. This paper propose the use of adjustable base functions to allow the selection of the base functions capable of satisfying the all boundary conditions and keeping the condition number of the mass matrices sufficiently small. As a result, the two requirements can be simultaneously satisfied.

INTRODUCTION

The vibratory characteristics of a distributed structure can be predicted using approximated methods such as Rayleigh-Ritz method. These methods often involves the use of base functions to represent the mode shapes of the structure. The prediction accuracy is related to the property of the selected base functions. Previous study shown that the prediction error and low convergence rate may be found if the base functions only satisfy some of the boundary conditions. Efforts have been made to use a combination of two sets of base functions to achieve a fast convergence rate. One set satisfies the boundary conditions of displacement and another satisfies the boundary conditions of force.

The longitudinal vibration of a beam of L in length and with one end fixed and the other attached to an elastic spring of stiffness K can be used as illustrating example. If the eigenfunctions of the fixed-free beam $\sin[(2i-1)\pi x / 2L]$ ($i = 1, 2, \ldots N$) are used as base functions, the boundary condition of the force at the spring end will never be satisfied. As a result, the convergence rate in calculating the resonance frequencies and mode shapes will be low. If two sets of base functions, such as eigenfunctions of fixed-free and fixed-fixed beams, are used, the boundary condition for the spring force can be satisfied by the linear combination of these functions. Therefore a high convergence rate was achieved in the prediction. High convergence rate indicates the use of small number of base functions and reduced computational effort.

The problem of using two different sets of base functions is that they may be linearly dependent in $0 \leq x \leq L$ as the number of the functions increases. The linearly dependent behaviour of base functions may produce singular mass matrix in the generalised eigenequation obtained form the Rayleigh-Ritz method. As a result of this singularity, the prediction of the system eigensolutions and response may not be accurate. To reduce the singularity, a set of adjustable base functions is used. The adjustable base functions are a set or several sets of base functions with an adjustable parameter. In addition to satisfy all the boundary conditions, a suitable selection of the adjustable parameter may make the mass matrix of the eigenequation less singular. Consequently, a high convergence rate of the prediction can be achieved and sufficient accuracy of the prediction can be obtained with small number of base functions.

DESCRIPTION OF THE SYSTEM MODEL

The idea of adjustable base functions is illustrated by solving the eigenvalue problem of a simple system shown in Fig. 1. However, this idea can be extended to more complicated structures, as the selection of suitable base functions is the common concern of many structural dynamic problems. The differential equation and boundary conditions of the system shown in Fig. 1 are:

$$\frac{d^2u}{dx^2} = -\lambda^2 u, \quad 0 < x < L, \tag{1}$$

$$u\big|_{x=0} = 0, \quad AE\frac{du}{dx}\bigg|_{x=L} = -KEu\big|_{x=L} \tag{2}$$

where u and λ are respectively eigenfunction and eigenvalue of the system. A, E and L are respectively the cross section area, Young's modules and length of the beam. The exact eigenvalues and eigenfunctions of this system can be obtained analytically and used for comparison with that from the approximated methods.

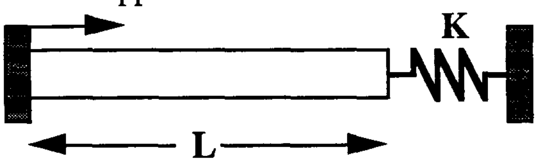

Figure 1. The beam model.

The approximate eigensolutions of the system described by Eqs. (1) and (2) may be obtained by rendering the Rayleigh's quotient stationary. The Rayleigh's quotient can be defined as:

$$R(u) = \frac{[u,u]}{(\sqrt{m}u, \sqrt{m}u)} \tag{3}$$

where $[u,u]$ represents the energy inner product. In this case:

$$[u,u] = \int_0^L EA\left(\frac{du}{dx}\right)^2 dx + Ku^2(L). \tag{4}$$

$(\sqrt{m}u, \sqrt{m}u)$ is the inner product of $\sqrt{m}u$ with itself. For this case, $\sqrt{m} = \sqrt{\rho A}$ and

$$(\sqrt{m}u,\sqrt{m}u)=\int_0^L \rho A u^2 dx. \qquad (5)$$

If $\varphi_n(x)$, $n=1,2,\ldots N$ are the base functions, mode shape function u can be expressed as:

$$u^{(N)}(x)=\sum_{n=1}^{N} a_n\varphi_n(x). \qquad (6)$$

The coefficients $a_1,a_2,\ldots a_N$ can be determined from the conditions for the stationarity of Rayleigh's quotient. Following eigenequation is resulted:

$$\mathbf{Ka}=(\varpi^2)^{(N)}\mathbf{Ma} \qquad (7)$$

where $\mathbf{M}=[M_{ij}]$ and $\mathbf{K}=[K_{ij}]$ are $N\times N$ real and symmetric mass and stiffness matrices of the system. Solution of Eq.(7) gives rise to estimated eigenvalues of the system $\lambda_r=\varpi_r/c_L$, $r=1,2,\ldots N$, where c_L is the longitudinal wave speed, and the corresponding eigenvectors $\mathbf{a}_r=[a_{r1},a_{r2},\ldots a_{rN}]^T$. The estimated mode shape functions are:

$$u_r^{(N)}=\mathbf{a}_r^T\Phi, \quad r=1,2,\ldots N \qquad (9)$$

where $\Phi=[\varphi_1,\varphi_2,\ldots\varphi_N]^T$.

EFFECT OF BASE FUNCTIONS

To satisfy the boundary condition described in Eq.(2), eigenfunctions of the fixed-free and fixed-fixed beam of length L are used as base functions:

$$\varphi_n(x)=\begin{cases}\sin\dfrac{n\pi x}{2L} & n=1,3,5,\ldots, \quad (fixed-free)\\ \sin\dfrac{n\pi x}{2L} & n=2,4,6,\ldots, \quad (fixed-fixed)\end{cases}. \qquad (10)$$

For this case, the elements of the $\mathbf{M}$ and $\mathbf{K}$ matrices are:

$$M_{ij}=\rho A\int_0^L \sin k_i x \sin k_j x dx\ , \qquad (11)$$

and $$K_{ij}=EAk_ik_j\int_0^L \cos k_i x\cos k_j x dx + K\sin k_i L \sin k_j L \qquad (12)$$

where $k_i=i\pi/2L$ and $k_j=j\pi/2L$. Listed in Table 1 are the exact eigenvalues of the system shown in Fig. 1 (with $L=1.0m$, $E=6.85\times10^{10}N/m^2$, $\rho=2.7\times10^3 kg/m^3$, $A=0.05\times0.025m^2$ and $K=5\times10^8 N/m^2$) in comparison with the estimated eigenvalues by using the first 8 ($N=8$) combined mode shapes as described in Eq.(10) and that using the first 8 eigenfunctions of the fixed-free beam.

Table 1. Eigenvalues of the beam ($N=8$).

r	λr	**Combined**	**Fixed-Free**
1	2.70744550	2.70744550	2.75901507
2	5.52540977	5.52540977	5.59945224
3	8.45822418	8.45822418	8.53394919
4	11.46659801	11.46659803	11.53761057
5	14.51954791	14.52084347	14.58590674
6	17.59913076	17.94220467	17.66325115
7	20.69536334	25.73305232	20.76187933
8	23.80252122	70.81022066	23.88635338

When the combined mode shapes are used, the high convergence rate is demonstrated by comparing the first 4 estimated eigenvalues with the exact ones. The corresponding eigenfunctions are shown in Fig. 2. The first 4 eigenfunctions using the combined mode shapes overlap with the exact ones, while the estimated eigenfunctions by the fixed-free mode shape functions show a poor convergence in particular near $x = L$. The main cause of the low convergence is that the fixed-free mode shape functions are not capable of satisfying the force boundary condition. However, the error in higher order eigensolutions ($r = 6,7,8$) using the combined mode shapes is more pronounced than that obtained by the fixed-free eigenfunctions.

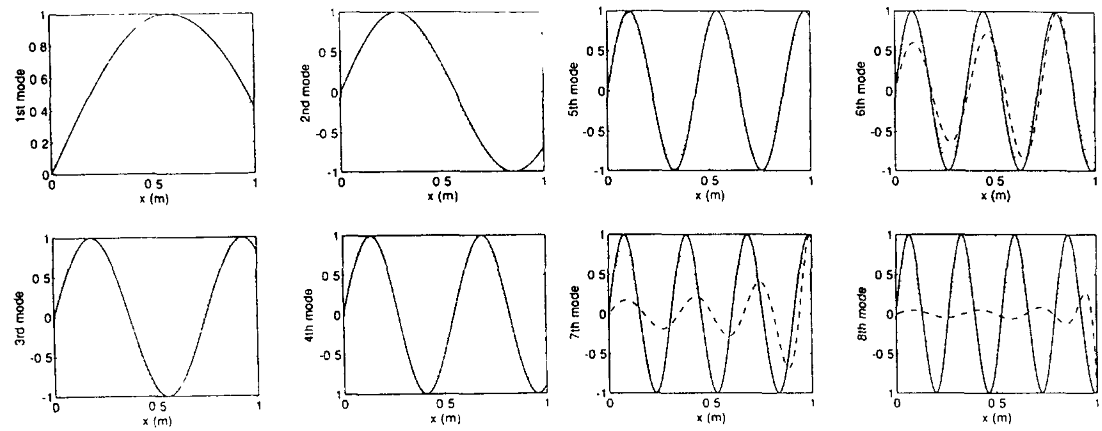

Figure 2. Eigenfunctions of the beam ($N = 8$). Solid lines: exact solution, dash dotted lines: combined base functions, dotted lines: fixed-free base function.

The common approach to obtain the accurate higher order eigensolutions is to increase the number of the base functions. The examination of the eigenfrequency squared ϖ_r^2 (Table 2) obtained using combined base functions ($N = 16$) shows that there is a pair of complex ϖ_r^2 with their real parts (3.5742, converted into the eigenvalue by $\lambda_r = real(\varpi_r) / c_L$) in between the first two eigenvalues from the exact solutions. The complex eigenvalues have no physical significance for the system considered in this study, because the eigenvalues of a generalised eigenequation with the real and symmetric matrices **M** and **K** must be real. The first 8 estimated eigenfunctions using fixed-free mode shape functions ($N = 16$) are examined with comparison to exact eigenfunctions. For this case, all the eigenfunctions using the combined base functions have imaginary parts as shown in Fig. 3. Although the real parts of the eigenfunctions correspond closely to the exact solution, the imaginary parts of the eigenfunctions are not linearly dependent upon their real parts. Figure 4 shows the eigenfunctions corresponding to the complex eigenvalues ($r = 15,16$ in Table 2). The spatial distribution of these eigenfunctions correlates well with the imaginary parts of the eigenfunctions shown in Fig. 3. However, they clearly indicate the error in the prediction.

Table 2 The beam eigenfrequency squared ($N = 16$).

r	ϖ_r^2 **Combined** $(\times 10^{10})$	ϖ_r^2 **Fixed-Free** $(\times 10^{10})$
1	0.0185971441244	0.01894542809071
2	0.0774561294208	0.07844720812182
3	0.1815035781042	0.18299919598895
4	0.3335769111352	0.33539526391967
5	0.5348512259744	0.53687866188681
6	0.7857949688217	0.78797133228146
7	1.0866080512718	1.08890428288120
8	1.4373837461642	1.43978889238310
9	1.8381711703520	1.84068433033446
10	2.2921571689099 + 0.0000000000000i	2.29162594298094
11	2.9160708428437 + 0.0000000000000i	2.79263868498729
12	4.4077042619969 + 0.0000000000000i	3.34374516420512
13	10.0549737603058 + 0.0000000000000i	3.94497359712145
14	81.4279112036678	4.59637317082678
15	0.0324114006714 + 0.93986481478l4i	5.29806584883849
16	0.0324114006714 - 0.93986481478l4i	6.05058586964743

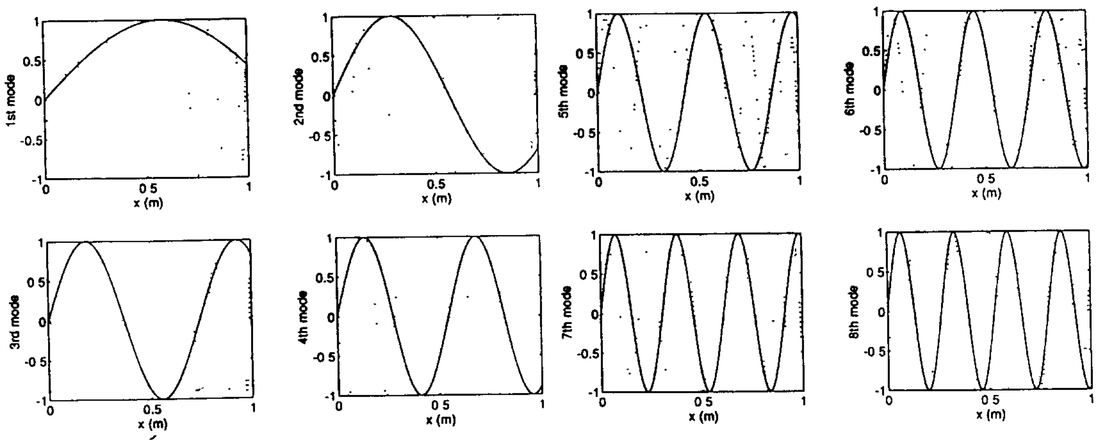

Figure 3. Imaginary part (dash dotted lines) of the beam eigenfunctions ($N = 16$) using combined base functions, compared with the real part (solid lines).

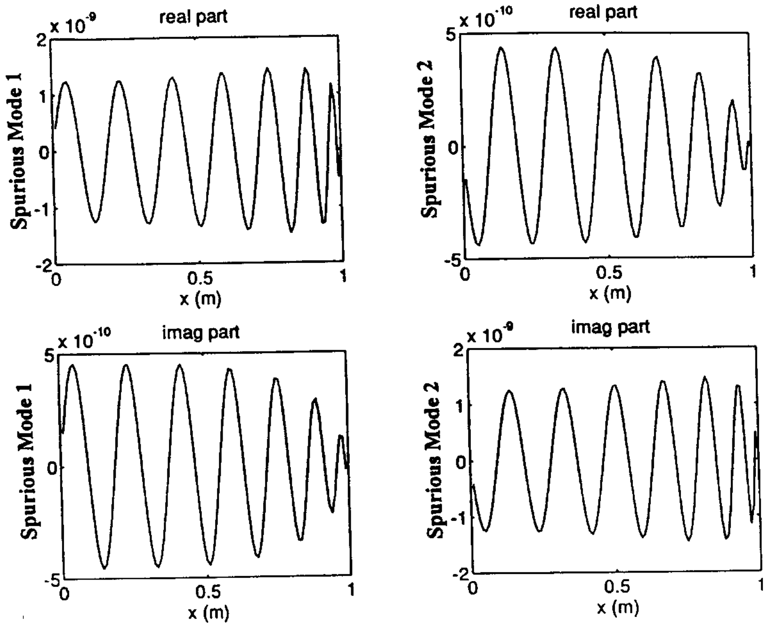

Figure 4. Beam eigenfunctions ($N = 16$) corresponding to the complex eigenvalues when using combined base functions.

REMARKS ON MATRIX SINGULARITY

The behaviour of the linear independence of the selected base functions $\varphi_n(x)$ can be determined by examining:

$$c_1\varphi_1(x) + c_2\varphi_2(x)+\ldots+c_N\varphi_N(x) = 0 . \tag{13}$$

If non-zero coefficients exist, the functions are referred as linearly dependent. Multiplying Eq.(13) by $\varphi_s(x)$, and integrating the resultant equations in the range of $0 \le x \le L$, we than have:

$$\Lambda \mathbf{c} = 0 \tag{14}$$

where $\Lambda_{sn} = \int_0^L \varphi_s(x)\varphi_n(x)dx$. If $\varphi_n(x)$ are orthogonal functions in $0 \le x \le L$ then $\mathbf{c} = 0$. Therefore orthogonal functions in $0 \le x \le L$ are linearly independent. However, if $\varphi_n(x)$ are selected from two separate sets of base functions, the orthogonality and the linear independence of the functions are not guaranteed. For this case, the degree of the singularity of matrix Λ can be used to measure the degree of the linear independence of the base functions. If the base function is selected as in Eq. (10), then the mass matrix in Eq.(7) is identical to Λ. This indicates that degree of the singularity of mass matrix $\mathbf{M}$ is directly related to the degree of the linear independence of the base functions. The degree of the matrix singularity may be measured by the matrix condition number. If the real and symmetric matrix $\mathbf{M}$ is not singular, its condition number is defined as:

$$cond(\mathbf{M}) = \frac{\max\{|\lambda_1|,\ldots,|\lambda_N|\}}{\min\{|\lambda_1|,\ldots,|\lambda_N|\}} \tag{15}$$

where $\lambda_1,\ldots,\lambda_N$ are eigenvalues of $\mathbf{M}$. If $cond(\mathbf{M})$ approaches to 1, the matrix $\mathbf{M}$ is well conditioned. If $cond(\mathbf{M})$ reaches infinity, $\mathbf{M}$ becomes singular. If $\mathbf{M}$ approaches to singular, the corresponding eigensolution of Eq.(7) even using the QZ method will be inaccurate due to inherited division of very small values. As a result, some eigenvalues will be in error and so the eigenfunctions.

ADJUSTABLE BASE FUNCTIONS

Considering a set of base functions $\varphi_n(x,\xi)$, where ξ is an adjustable parameter, they are used to express the mode shape function of the beam vibration using Eq. (6). The selection of ξ should allow $u^{(N)}$ capable of satisfying all the boundary conditions and matrix **M** in Eq. (7) being less singular. For example, we may select the mode shape functions of fixed-free beam with an adjustable length of $L_e = L + \xi$. In this case, the base functions are:

$$\varphi_n(x,\xi) = \sin[(n-\frac{1}{2})\frac{\pi x}{L_e}], \quad 0 \le x \le L, \text{ for } n = 1,2,\ldots,N. \qquad (16)$$

The Rayleigh-Ritz method gives rise to the mass and stiffness matrices of the eigenequation (Eq.(7)) with the elements having the similar expressions to Eqs. (11) and (12) except: $k_i = (i-\frac{1}{2})\frac{\pi}{L_e}$ and $k_j = (j-\frac{1}{2})\frac{\pi}{L_e}$.

The condition number of matrix **M** is shown as a function of L_e in Fig. 5 ($N = 32$). When $L_e \le 1m$, **M** is well conditioned, while when $L_e \ge 1.25m$ **M** becomes very ill conditioned. In particular for these L_e where the condition numbers of **M** are not shown in the curve, their condition numbers approach to 'infinity'. There is a transition region ($1 < L_e < 1.25m$) where **M** is sufficiently well conditioned. It will be shown that within this region the eigensolutions have high convergent rate and accuracy.

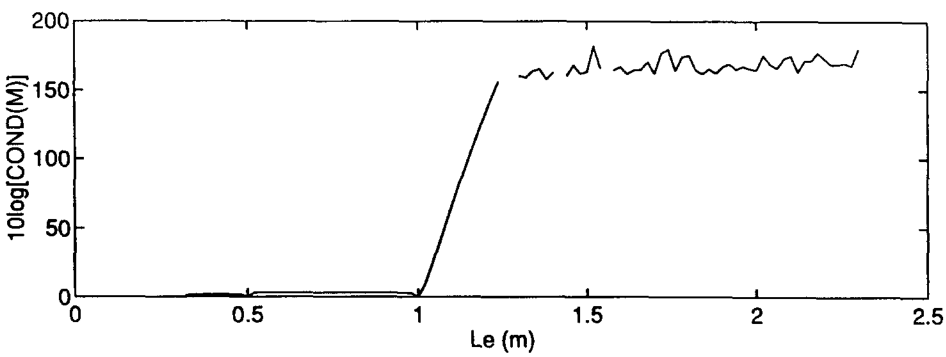

Figure 5. Mass matrix condition number as a function of adjustable parameter ($N = 32$).

Tables 3 lists the first 3 estimated eigenvalues using the adjustable base functions (Eq.(16) for $N = 32$) as a function of L_e. It is shown that when $L_e \le L = 1m$, the convergence rate of the eigenvalues is slow, although the matrix condition numbers are small. As soon as $L_e > L$, the estimated eigenvalues start to approach the system eigenvalues. It is worth noting in Table 3 that the convergence rates of the eigenvalues are higher for those L_e giving relatively larger *cond*(M). There is a range of L_e ($1.05m \le L_e < 1.25m$) where convergence rate of the eigenvalues is high, and condition number of the matrix **M** is not extremely large. Within this range, we haven't found any complex eigenvalues and eigenfunctions. Above the limit of this range however some spurious eigensolutions can be resulted, because the corresponding condition number of matrix **M** becomes extremely large.

Table 3. The beam eigenvalues as functions of $cond(\mathbf{M})$ and adjustable parameter ($N = 32$).

L_e (m)	λ_1	λ_2	λ_3	$cond(\mathbf{M})$
0.8	3.191871481538	5.719611254141	10.13650509789	2.00000000000001
0.85	3.206562886396	5.525909493333	9.140701180114	2.00000000000000
0.9	3.096795788143	5.622539018884	8.549197493233	1.99999999592705
0.95	2.915355321629	5.706405543836	8.491972785434	1.99962858061440
1.0	2.719935276498	5.542662401730	8.475056002434	1.00000000000001
1.05	2.707446996034	5.525412541664	8.458228353277	1.044831850006807e+03
1.1	2.707445509913	5.525409777444	8.458224184164	4.500829942024973e+06
1.15	2.707445509647	5.525409776871	8.458224183188	1.171132061853340e+10
1.2	2.707445509647	5.525409776871	8.458224183187	1.821138592816997e+13
1.25	2.707445509647	5.525409776871	8.458224183187	3.629081306428040e+16

CONCLUSIONS

The convergence rate of the approximated eigensolutions of distributed structures can be greatly increased when the adjustable base functions are used. In many cases, the conveniently selected orthogonal base functions may not satisfy all the boundary conditions of the system. A transformation of the functions using an adjustable parameter ξ allows the possible satisfaction of the boundary condition through the superposition of the transferred functions. However, the orthogonality of the new base functions in $0 \le x \le L$ may be reduced. It is found that the high convergence rate of the eigenvalues often corresponds to the relatively large condition number of the mass matrix. There is also a limit for increase of the condition number of the matrix **M**. Beyond this limit, where $cond(\mathbf{M})$ is extremely large, spurious eigensolutions can be resulted. Prediction of the system characteristics will be in error.