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# AXISYMMETRIC TRANSFER FUNCTIONS ALONG A FLUID-FILLED ELASTIC TUBE

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## ABSTRACT

The equations of motion of a fluid-filled tube moving with axisymmetric motion are described. This leads to a dispersion plot describing the frequency dependence of four wavenumbers. One wave is predominantly controlled by the fluid and wall stretching, the others by wall motion. Homogeneous rubber tubes were found to soften with increasing pressure while braided rubber tubes stiffened significantly with increasing pressure.

## 1. INTRODUCTION

Elastomeric flexible tubes and bellows are used to control structure- and fluid-borne vibrational waves in pipes by providing an impedance mismatch, reflecting both fluid and structural waves back towards the source. This theoretical study is intended to investigate these phenomena by considering the general case of wave propagation in a fluid-filled, orthotropically-stiffened and internally pressurised shell. Only axisymmetric waves were considered as these are probably most significant. In [1], the appropriate wave equations were set up. This leads to plots of dispersion curves relating the wavenumbers of the four possible wave types, s = 1, 2, 3, 4, to frequency. These dispersion relationships are used here to derive the impedance matrix for a fluid-filled tube. This matrix relates input fluid or structural motions to transmitted fluid pressures or axial wall stresses.

The tube shown in Figure 1 has radius a, wall thickness h and length $\ell$. The mean density is $\rho$, the elastic moduli in the axial and circumferential directions are $E_x$, $E_\theta$. The associated Poisson's ratios are $\nu_x$, $\nu_\theta$.

## 2. EQUATIONS OF MOTION AND WAVENUMBERS

It is assumed that four waves of the form, $e^{-ik_s x}$, $k_s = 1, 2, 3, 4$, propagate to the right in an infinite tube. Each of these waves has a radial and axial displacement in the shell, $W_s$, $U_s$, and axial displacement in the fluid, $U_s$. The relative sizes of these waves are derived by considering the tube element shown in Figures 2 and 3. Bending moments, M, shear forces, Q, axial tension, $N_x$, and fluid pressure, P, are considered [1, 2].

Axial equilibrium of the shell gives

$$a\,E'_x h\left[\frac{\partial^2 u}{\partial x^2} + \left(\nu_\theta - \frac{\gamma_p}{\gamma_E}\right)\frac{\partial w}{a\partial x}\right] = a\rho h\ddot{u} \tag{1}$$

where: $\gamma_p = \dfrac{P_a}{E'_\theta h}$ ; $\gamma_E = \dfrac{E_x}{E_\theta}$ ; $E'_x = E_x/(1-\nu_\theta\nu_x)$, $E'_\theta = E_\theta/(1-\nu_\theta\nu_x)$.

The equation for radial equilibrium of the shell is found by consideration of Figure 2.

$$a\,N_x\frac{\partial^2 w}{\partial x^2} - E'_\theta h\left[\frac{w}{a} + \nu_x\frac{\partial u}{\partial x}\right] + ap - a\,B\frac{\partial^4 w}{\partial x^4} + a\,P\left[\frac{w}{a} + \frac{\partial u}{\partial x}\right] = a\rho h\ddot{w} \tag{2}$$

Substitution of harmonic solutions for propagating waves,

$$u = U_s\,e^{-ik_s x}\ ,\ w = W_s\,e^{-ik_s x}\ ,\ p = P_s\,e^{-ik_x x} \tag{3}$$

where $U_s$, $W_s$ and $P_s$ are the axial displacement, radial displacement and fluctuating pressure associated with each wavenumber. Equation (1) becomes:

$$\left(\Omega^2/\gamma_E - \alpha_s^2\right) U_s = i\,\alpha_s\left(\nu_\theta - \gamma_p/\gamma_E\right) W_s \tag{4}$$

where $\Omega^2 = \dfrac{\omega^2}{\omega_r^2}$ , $\omega_r = a\sqrt{\dfrac{E'_\theta}{\rho}}$ is the ring frequency in rads/sec, $\alpha_s = k_s a$ is the normalised axial wavenumber. Equation (2) becomes:

$$W_s\left[\Omega^2 - 1 + \gamma_p - \chi\alpha_s^2 - \alpha_s^4 r\right] + \frac{a^2 P_s}{E'_\theta h} = -\,i\alpha_s\left[\nu_x - \gamma_p\right] U_s \tag{5}$$

where $\chi = \dfrac{N_0}{E'_\theta h}$ is the normalised tension, $r = \dfrac{\gamma_E h^2}{12a^2}$ is the normalised bending stiffness term. The pressure amplitude, $P_s$, is given from [3]:

$$P_s = \frac{-2K_f}{1 - \left(\dfrac{\alpha_s}{\alpha_f}\right)^2} \cdot \frac{W_s}{a} \tag{6}$$

where $K_f$ is the fluid bulk modulus and $\alpha_f$, the free normalised wavenumber in a rigid walled cylinder, is $\alpha_f = a\omega\sqrt{\dfrac{\rho_f}{K_f}}$ ; $\rho_f$ is the fluid density.

Combining equations (4), (5) and (6) leads to an eighth order wave equation [1] in terms of $\alpha_s$, the normalised wavenumber. The solution gives four pairs of wavenumber, $s = 1, 2, 3, 4$.

## 3. DERIVATION OF THE DYNAMIC STIFFNESS MATRIX

For a tube of finite length the axial displacement fields, $u$, $u_f$, and the shell radial displacement, $W$, may both be described as a sum of the four pairs of waves corresponding to the roots, $s = 1, 2, 3, 4$. The axial motion for the shell, for example, is

$$u = \sum_{s=1,4} [L_s] \ \{U_s\} \tag{7}$$

where $[L_s] = \begin{bmatrix} e^{-ik_s x} & 0 \\ 0 & e^{ik_s x} \end{bmatrix}$ and $\{U_s\} = \begin{Bmatrix} U_s^+ \\ U_s^- \end{Bmatrix}$, $\{W_s\} = \begin{Bmatrix} W_s^+ \\ W_s^- \end{Bmatrix}$ are the amplitudes of the right and left travelling waves.

Using equations (1), (2) and (3), the displacements, $U_s$, $U_{sf}$, and forces, $\sigma_{xs}$, $P_s$, can all be expressed in terms of the radial displacement, $W_s$, i.e.

$$\{U_s\} = [C_s]\,\{W_s\},\ \ [U_{sf}] = [B_s]\,\{W_s\},\ \ [\sigma_{xs}] = [E_s]\,\{W_s\},\ [P_s] = [F_s]\,\{W_s\} \tag{8}$$

The dynamic stiffness expressions require that the stresses and pressures are multiplied by the respective cross-sectional areas, such that the axial shell force, $N(x) = -2\pi a h\,\sigma(x)$, and the pressure force, $P(x) = \pi a^2\,p(x)$. The matrices describing these forces at any position, $x$, are the sum of the four roots.

$$\begin{Bmatrix} N(x) \\ P(x) \end{Bmatrix} = [M]\,[L]\,\{W\}\ , \quad \begin{Bmatrix} u(x) \\ u_f(x) \end{Bmatrix} = [N]\,[L]\,\{W\} \tag{9}$$

where $\{W\} = \{W_1, W_2, W_3, W_4\}$, $[L]$ is a diagonal matrix, $L_s$, $[M]$ and $[N]$ are composed from equations (8).

The displacements and the forces have been stated in terms of a common variable $\{W\}$. The forces and displacements at the boundaries, $x = 0, x = \ell$, are substituted.

$$\begin{aligned} N_1 = N(0) \quad &, \quad N_2 = -N(\ell) \quad , \quad P_1 = p(0), \quad P_2 = -p(\ell) \\ U_1 = u(0) \quad &, \quad U_2 = u(\ell) \quad , \quad U_{f1} = u_f(0) \quad , \quad U_{f2} = u_f(\ell) \end{aligned} \tag{10}$$

The $\{W\}$ in equations (9) are then eliminated by division of the forces by the displacements to yield the impedance matrix.

The impedance matrix $[Z] = {}^1/i\omega\,[K]$.

## 4. PREDICTED TRANSFER FUNCTIONS

The first set of tests involved changing the axial tension and internal pressure on a homogeneous rubber tube filled with water. The tube parameters were $a = 25$ mm; $h = 10$ mm; $\ell = 0.2$ m; $E'_x = E'_\theta = 10^6\ N/m^2$; $\nu_\theta$, $\nu_x = 0.5$; $\rho = 10^3\ kg/m^2$, $K_f = 2.7 \times 10^9\ N/m^2$, $\rho_f = 10^3\ kg/m^2$. The wall loss factor was 0.1.

The dispersion curve for the first test with zero pressure and tension is shown in Figure 4. The $s = 2$ wave is the axial wave in the rubber and is responsible for the structural transfer functions $Z_{12}$, $Z_{11}$ in Figure 5. This wave is not affected by internal pressure or wall tension. The $s = 1$ fluid wave is much slower and is below about 100 Hz, where it is called the Korteweg wave involving wall stretching. The fluid transfer function, $Z_{34}$, is shown in Figure 5 to be much lower, with a higher modal density than the structural transfer function, $Z_{12}$. At about 300 Hz the tube ring frequency occurs and bending in the wall thickness is dominant. The other two waves $s = 3, 4$ are non-propagating waves responsible for local mass effects.

Internal pressurization was found to be relatively unimportant for homogeneous rubber except at low frequencies. In Figure 6 it can be seen that when $\gamma_p = 1$ the $s = 1$ wave becomes almost non-propagating. The ring frequency drops to 20 Hz as seen in Figure 7. Further increase would move the ring frequency to zero causing catastrophic expansion.

Tests were performed on a water-filled braided rubber tube with properties as follows: $a = 100$ mm; $h = 10$ mm; $\ell = 1$ m; $E'_x$ $E'_\theta = 5.1 \times 10^8\ N/m^2$; $\nu_\theta, \nu_x = 1$; $\rho = 1.07 \times 10^3\ kg/m^3$; loss factor = 0.1. When there was no pressurization the fluid input and transfer impedances are given in Figure 8. Under pressurization these impedances increased dramatically as seen in Figure 8. The reason for this is that the braiding causes the high Poisson's ratio values given, therefore pressurization causes axial tension, which stiffens the tube to act as a membrane.

## 5. REFERENCE


[1] R.J. Pinnington. The axisymmetric wave transmission properties of pressurized flexibles. Journal of Sound and Vibration, 204(2), 1997, 271-289.

[2] R.J. Pinnington. Axisymmetric wave transfer functions of flexible tubes. Journal of Sound and Vibration, 204(2), 1997, 290-310.

[3] R.J. Pinnington and A.R. Briscoe. Externally applied sensor for axisymmetric waves in a fluid-filled pipe. Journal of Sound and Vibration, 173, 1994, 503-516.


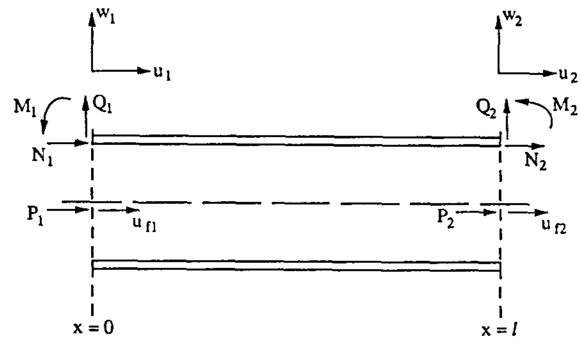


Figure 1. Section through a fluid-filled pipe showing the sign convention.

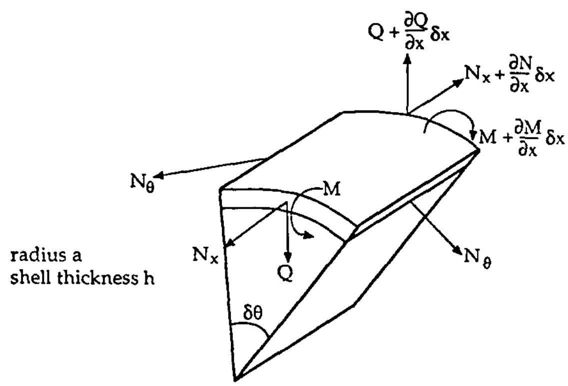


Figure 2. Forces on an element.

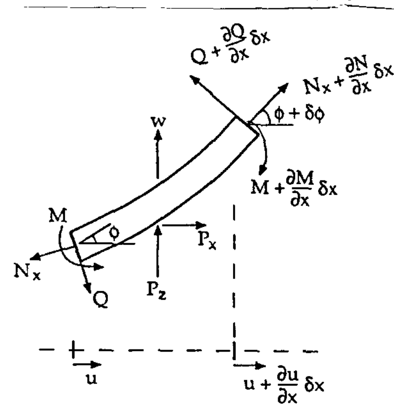


Figure 3. Forces on a section of an element.

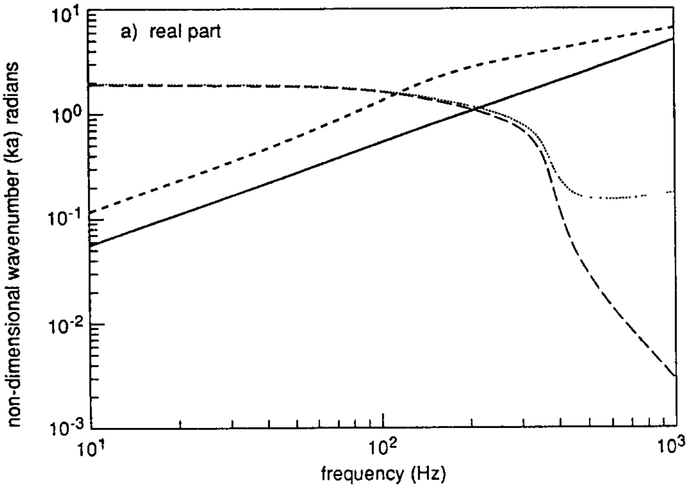


Figure 4. Non-dimensional wavenumber versus frequency.
Test 1: $\gamma_\rho = 0, \chi = 0$, - - - s = 1, — s = 2, — — — s = 3, ····· s = 4.

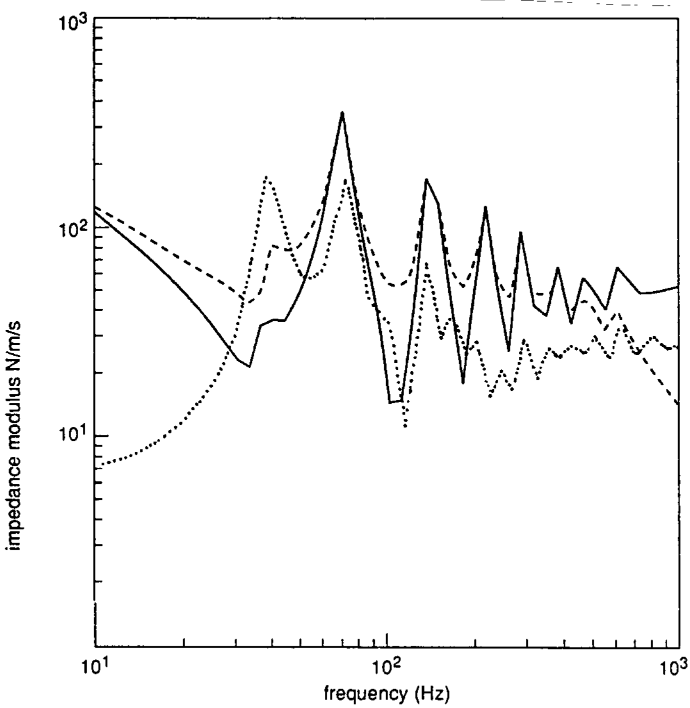


Figure 5. Test 1, $\gamma_\rho = 0, \chi = 0$, structural impedance modulus, — input impedance $Z_{11}$, — — — transfer impedance $Z_{12}$, ····· fluid transfer impedance $Z_{13}$.

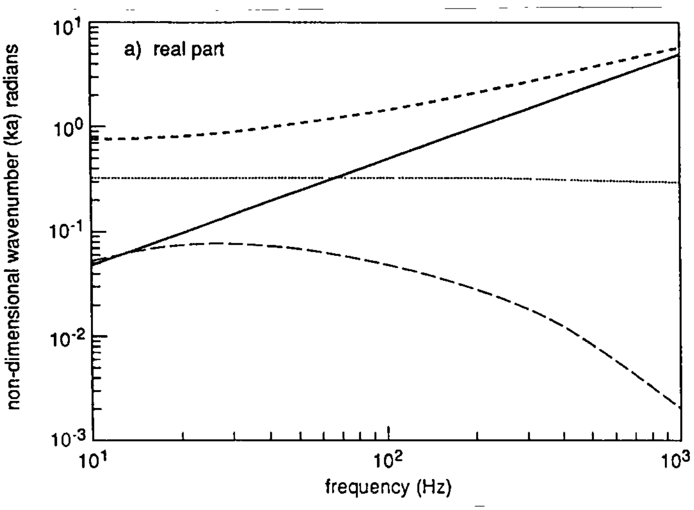


Figure 6. Non-dimensional wavenumber versus frequency.
Test 2, $\gamma_\rho = 1$, $\chi = 0.5$, - - - - s = 1, — s = 2, – – – s = 3, ····· s = 4.

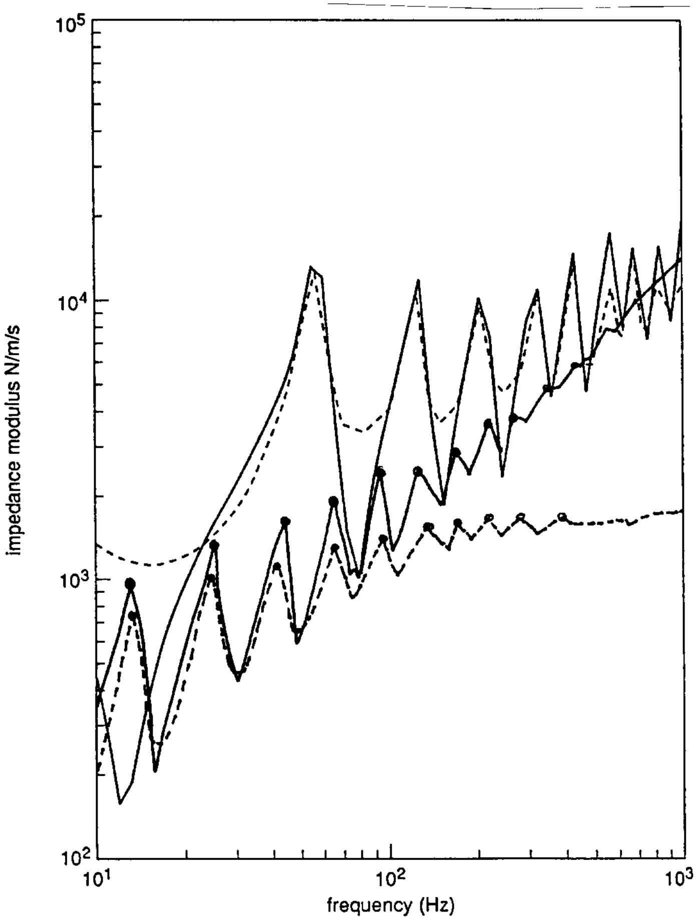


Figure 7. Real part of input fluid impedance $Z_{33}$, — $\gamma_p = 0$, $\chi = 0$,
– – – $\gamma_p = 1$, $\chi = 0.5$.

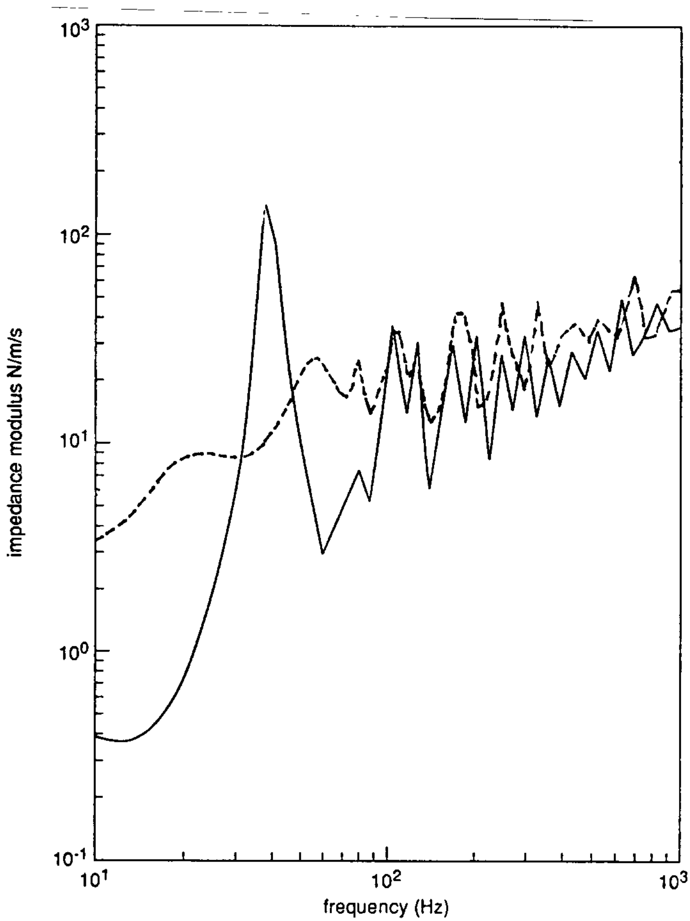


Figure 8. $\gamma_\rho = 0.5$, $\chi = 0.25$, —— fluid input impedance $Z_{33}$, - - - fluid transfer impedance $Z_{34}$. $\gamma_\rho = 0$, $\chi = 0$, —o— $Z_{33}$, - -o- - $Z_{34}$.