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FRENQUENCY ANALYSIS OF COMPOSITE BEAMS USING A HIGHER-ORDER BEAM ELEMENT

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ABSTRACT

A simple and accurate third-order composite beam element is presented in this work. This higher-order element possesses a linear bending strain as opposed to the constant bending strain in existing higher-order composite beam elements. By using the Hamilton's Principle, the variational consistent mass matrix for the third-order theory is derived. The resulting element is more accurate than the existing higher-order elements. The numerical examples show that the mass matrix resulting from the higher-order displacement has a considerable influence on the higher mode frequencies.

INTRODUCTION

The objectives of this paper are two fold, one is to present a simple and accurate third-order composite beam element, and the other is to use this new element for the frequency analysis of composite beams.

In the finite element modeling of composite beams and plates, a higher-order shear deformation theory can lead to finite elements having the same number of nodal variables but giving solutions with different accuracy. By studying the interpolation order of the element bending strain, this paper presents a simple but accurate third-order composite beam element, which possesses a linear bending strain as opposed to the constant bending strain in existing higher-order composite beam elements. In many higher-order elements, the mass matrices are either evaluated by the lumped mass method or based on the first-order theory. In this work, the variational consistent mass matrices for the third-order theory are derived from the Hamilton's Principle. The present new element is used to solve some frequency analysis of composite beams. The numerical examples show that the transverse shear strains play a very important role in the dynamic behavior of composite beams. The numerical examples also illustrate that the present composite beam element is more accurate than the higher-order beam elements which are based on the same higher-order theory and having the same number of nodal variables but using a different bending strain expression.

STRIANS AND VELOCITIES BASED ON THE THIRD-ORDER DISPLACEMENT

The displacement field of a beam in the third-order shear deformation theory [1] is, in general, of the form

$$u(x,y,z,t)=u_0(x,y,t)+\phi(x,y,t)\ z-\alpha(\frac{\partial w_0}{\partial x}+\phi)\ z^3 \qquad (1)$$

$$w(x,y,z,t)=w_0(x,y,t) \qquad (2)$$

where u_0 and w_0 are, respectively, the axial displacement and the deflection of a point on the beam reference plane; ϕ is the rotation of a normal to the reference plane about the y-axis; $\alpha=4/(3h^2)$; and h is the beam thickness. The influence of normal strain in the ydirection on the behavior of composite beams will not be considered here, since the emphasis here is to discuss the efficient finite element modeling.

Equations (1) and (2) lead to the axial normal strain, the transverse shear strain and the transverse normal strain as

$$e_x = e_m + e_b z - \alpha e_{hs} z^3, \quad e_{xz} = (1-\beta z^2) e_s, \quad e_z = 0 \tag{3}$$

where $\beta = 4/h^2$, and e_m, e_b, e_s & e_{hs} are, respectively, the membrane strain, bending strain, transverse shear strain and higher-order transverse shear strain on the reference plane. They take the form

$$e_m = \frac{\partial u_x^0}{\partial x}, \quad e_b = \frac{\partial \phi}{\partial x}, \quad e_s = \frac{1}{2}(\frac{\partial w_0}{\partial x} + \phi), \quad e_{hs} = \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi}{\partial x} \tag{4}$$

The displacement in Eq. (1) can be expressed in a different form. If one defines

$$\gamma = \frac{\partial w_0}{\partial x} + \phi \tag{5}$$

Eq. (1) then can be rewritten as

$$u(x,y,z,t) = u_0(x,y,t) - [\frac{\partial w_0}{\partial x} - \gamma(x,y,t)] \; z - \alpha\gamma(x,y,t) \; z^3 \tag{6}$$

Accordingly, the normal strain, transverse shear strain and the velocities take the form

$$e_x = e_m + \overset{*}{e}_b z - \alpha \overset{*}{e}_{hs} z^3, \quad 2e_{xz} = (1-\beta z^2)\gamma \tag{7}$$

$$v_x = \frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} - \frac{\partial}{\partial t}(\frac{\partial w_0}{\partial x} - \gamma) z - \alpha \frac{\partial \gamma}{\partial t} z^3, \quad v_z = \frac{\partial w}{\partial t} = \frac{\partial w_0}{\partial t} \tag{8}$$

in which $\beta = 4/h^2$ and

$$\overset{*}{e}_b = -\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \gamma}{\partial x}, \quad \overset{*}{e}_{hs} = \frac{\partial \gamma}{\partial x} \tag{9}$$

Equation (9) gives the new expressions for the bending and higher-order transverse shear strains.

Both of the strain sets given, respectively, in Eqs. (3) & (4) and in Eqs. (5), (7) and (9) can be used for finite element modeling, and both of them result in C^1-continuity elements under the displacement-based formulation. The bending strains in Eq. (4) and Eq. (9) are associated with different displacement functions. However, the one in Eq. (4) is a function of the rotation, and the one in Eq. (9) is the function of the deflection and transverse shear deformation. Because the orders of the approximations for the deflection and rotation are different in the finite element analysis under the given number of nodal variables, these two different bending strain expressions will lead to the finite element solutions with different accuracy. This will be demonstrated latter.

HIGHER-ORDER COMPOSITE BEAM ELEMENT

Let U_e and K_{ke} be the element strain energy and kinetic energy respectively, then the Himilton's principle states that

$$\delta \sum_{elem} \int_{t_0}^{t} (U_e - K_{ke}) dt = 0 \tag{10}$$

in which the work done by external forces is neglected. The Himilton's principle leads to the equilibrium equation of a system as

$$\mathbf{M}\ \ddot{\mathbf{q}} + \mathbf{K}\ \mathbf{q} = \mathbf{0} \tag{11}$$

where **M, K** and **q** are, respectively, the global mass matrix, stiffness matrix and nodal variable vector of the system. Consequently, the frequency ω can be evaluated by

$$(\mathbf{K} - \omega^2 \mathbf{M})\ \mathbf{q} = \mathbf{0} \tag{12}$$

The derivation of the element stiffness matrix and mass matrix based on the third-order theory is presented here.

ELEMENT STIFFNESS MATRIX

Let us consider a straight beam of length l and rectangular cross-section h x b in which h is the beam thickness and b is the beam width. The strain energy of an element, U_e, is of the form

$$U_e = \frac{b}{2}\int_l \int_{-h/2}^{h/2} (e_x Q_{xx} e_x + 4 e_{xz} Q_{xz} e_{xz}) dz dx \tag{13}$$

where Q_{xx} and Q_{xz} are, respectively, the longitudinal Young's modulus and transverse shear modulus, and they are functions of z. Substituting Eqs. (7) and (9) into Eq. (13) leads to

$$\begin{aligned} U_e = \frac{1}{2}\int_l [& e_m A_{xx} e_m + e_b^* D_{xx} e_b^* + \gamma\ S_{xx} \gamma + e_{hs}^* \alpha^2 H_{xx} e_{hs} \\ & + 2 e_m \alpha B_{xx} e_b^* - 2 e_m \alpha E_{xx} e_{hs}^* - 2 e_b^* \alpha F_{xx} e_{hs}^*] dx \end{aligned} \tag{14}$$

in which

$$(A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx}) = b\int_{-h/2}^{h/2} (1, z, z^2, z^3, z^4, z^6) Q_{xx} dz \tag{15}$$

$$S_{xx} = b\int_{-h/2}^{h/2} (1 - \beta z^2)^2 Q_{xz} dz \tag{16}$$

In the finite element modeling of displacement-based formulation, element strains in Eq. (14) can be expressed in terms of the element nodal displacement vector $\mathbf{q}_e$ and the element strain matrices as follows

$$e_m = \mathbf{B}_m \mathbf{q}_e,\ \ e_b^* = \mathbf{B}_b \mathbf{q}_e,\ \ 2e_s = \gamma = \mathbf{B}_s \mathbf{q}_e,\ \ e_{hs}^* = \mathbf{B}_{hs} \mathbf{q}_e \tag{17}$$

Consequently, Eq. (14) becomes

$$\begin{aligned} U_e = \frac{1}{2}\mathbf{q}_e^T \int_l [& \mathbf{B}_b^T D_{xx} \mathbf{B}_b + \mathbf{B}_m^T A_{xx} \mathbf{B}_m + \mathbf{B}_s^T S_{xx} \mathbf{B}_s + \mathbf{B}_{hs}^T \beta^2 H_{xx} \mathbf{B}_{hs} + (\mathbf{B}_m^T B_{xx} \mathbf{B}_b + \mathbf{B}_b^T B_{xx} \mathbf{B}_m) \\ & - (\mathbf{B}_m^T \alpha E_{xx} \mathbf{B}_{hs} + \mathbf{B}_{hs}^T \alpha E_{xx} \mathbf{B}_m) - (\mathbf{B}_b^T \alpha F_{xx} \mathbf{B}_{hs} + \mathbf{B}_{hs}^T \alpha F_{xx} \mathbf{B}_b)] dx\ \mathbf{q}_e \end{aligned} \tag{18}$$

Then the Himilton's principle leads to the element stiffness matrix $\mathbf{K}_e$ as

$$\mathbf{K}_e = \mathbf{K}_b + \mathbf{K}_m + \mathbf{K}_s + \mathbf{K}_{hs} + \mathbf{K}_c \tag{19}$$

with

$$\begin{aligned} & \mathbf{K}_b = \int_l \mathbf{B}_b^T D_{xx} \mathbf{B}_b dx,\ \ \mathbf{K}_m = \int_l \mathbf{B}_m^T A_{xx} \mathbf{B}_m dx, \\ & \mathbf{K}_s = \int_l \mathbf{B}_s^T S_{xx} \mathbf{B}_s dx,\ \ \ \mathbf{K}_{hs} = \int_l \mathbf{B}_{hs}^T \beta^2 H_{xx} \mathbf{B}_{hs} dx \\ & \mathbf{K}_c = \int_l [(\mathbf{B}_m^T B_{xx} \mathbf{B}_b + \mathbf{B}_b^T B_{xx} \mathbf{B}_m) - (\mathbf{B}_m^T \alpha E_{xx} \mathbf{B}_{hs} + \mathbf{B}_{hs}^T \alpha E_{xx} \mathbf{B}_m) \\ & \qquad - (\mathbf{B}_b^T \alpha F_{xx} \mathbf{B}_{hs} + \mathbf{B}_{hs}^T \alpha F_{xx} \mathbf{B}_b)] dx \end{aligned} \tag{20}$$

where $\mathbf{K}_b$, $\mathbf{K}_m$, $\mathbf{K}_s$, $\mathbf{K}_{hs}$ and $\mathbf{K}_c$ are the element bending, membrane, transverse shear, higher-order shear and coupling stiffness matrix respectively.

CONSISTENT MASS MATRIX FOR HIGHER-ORDER BEAM THEORY

The element kinetic energy K_{ke} corresponding to the higher-order theory takes the form

$$K_{ke}=\frac{b}{2}\int_l\int_{-h/2}^{h/2}(v_z^2+v_x^2)\rho dzdx=\frac{b}{2}\int_l\int_{-h/2}^{h/2}[(\frac{\partial w}{\partial t})^2+(\frac{\partial u}{\partial t})^2)\rho dzdx$$
$$=\frac{b}{2}\int_l\int_{-h/2}^{h/2}[(\frac{\partial w_0}{\partial t})^2+(\frac{\partial u_0}{\partial t})^2+z^2(\frac{\partial^2 w_0}{\partial t\partial x})^2+\alpha^2 z^6(\frac{\partial \gamma}{\partial t})^2$$
$$+2z\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial t\partial x}-2z^3\frac{\partial u_0}{\partial t}\frac{\partial \gamma}{\partial t}-2\alpha z^4\frac{\partial^2 w_0}{\partial t\partial x}\frac{\partial \gamma}{\partial t}]\rho dzdx \quad (21)$$

By defining

$$(J_A,J_B,J_D,J_E,J_F,J_H)=b\int_{-h/2}^{h/2}(1,z,z^2,z^3,z^4,z^6)\rho dz \quad (22)$$

the element kinetic energy K_{ke} can be written as

$$K_{ke}=\frac{1}{2}\int_l[J_A(\frac{\partial w_0}{\partial t})^2+J_A(\frac{\partial u_0}{\partial t})^2+J_D(\frac{\partial^2 w_0}{\partial t\partial x})^2+\alpha^2 J_H(\frac{\partial \gamma}{\partial t})^2$$
$$+2J_B\frac{\partial u_0}{\partial t}\frac{\partial^2 w_0}{\partial t\partial x}-2\alpha J_E\frac{\partial u_0}{\partial t}\frac{\partial \gamma}{\partial t}-2\alpha J_F\frac{\partial^2 w_0}{\partial t\partial x}\frac{\partial \gamma}{\partial t}]dx \quad (23)$$

The element displacements can be interpolated in terms of the element nodal displacement vector $\mathbf{q}_e$ as

$$u=\mathbf{N}_u\mathbf{q}_e,\ w_0=\mathbf{N}_w\mathbf{q}_e,\ \frac{\partial w_0}{\partial x}=\frac{\partial}{\partial x}\mathbf{N}_w\mathbf{q}_e=\mathbf{N}_{wx}\mathbf{q}_e,\ \gamma=\mathbf{N}_\gamma\mathbf{q}_e \text{ and } \ddot{\mathbf{q}}_e=\frac{\partial^2}{\partial t^2}\mathbf{q}_e \quad (24)$$

where $\mathbf{N}_j$ (j = u, w, wx and γ) are the interpolation matrices. The equation above and the Himilton's principle give the consistent element mass matrix $\mathbf{M}_e$ as

$$\mathbf{M}_e=\mathbf{M}_w+\mathbf{M}_{uo}+\mathbf{M}_{wx}+\mathbf{M}_\gamma+\mathbf{M}_{uow}+\mathbf{M}_{uo\gamma}+\mathbf{M}_{wx\gamma} \quad (25)$$

with

$$\mathbf{M}_w=\int_l\mathbf{N}_w^T J_A\mathbf{N}_w dx,\quad \mathbf{M}_{uo}=\int_l\mathbf{N}_{uo}^T J_A\mathbf{N}_{uo}dx,\ \mathbf{M}_{wx}=\int_l\mathbf{N}_{wx}^T J_A\mathbf{N}_{wx}dx$$
$$\mathbf{M}_\gamma=\alpha^2\int_l\mathbf{N}_\gamma^T J_H\mathbf{N}_\gamma dx,\quad \mathbf{M}_{uow}=\int_l(\mathbf{N}_{uo}^T J_B\mathbf{N}_w+\mathbf{N}_w^T J_B\mathbf{N}_{uo})dx \quad (26)$$
$$\mathbf{M}_{uo\gamma}=-\alpha\int_l(\mathbf{N}_{uo}^{T\,2}J_E\mathbf{N}_\gamma+\mathbf{N}_\gamma^T J_E\mathbf{N}_{uo})dx,\ \mathbf{M}_{wx\gamma}=-\alpha\int_l(\mathbf{N}_{wx}^T J_F\mathbf{N}_\gamma+\mathbf{N}_\gamma^T J_F\mathbf{N}_{wx})dx$$

$\mathbf{M}_w$, $\mathbf{M}_{uo}$ and $\mathbf{M}_{wx}$ are, respectively, the consistent transverse, axial and rotary mass matrices; $\mathbf{M}_\gamma$ is the mass matrix resulting from the higher-order terms; and $\mathbf{M}_{uow}$, $\mathbf{M}_{uo\gamma}$ and $\mathbf{M}_{wx\gamma}$ are the coupling terms.

The consistent mass matrix here has two meanings, one is opposed to the lumped mass method, and the other refers that the contribution of the higher-order term to the mass matrix is also taken into account.

FORMULATION OF ELEMENT STIFFNESS MATRIX

Let us consider a simple two-node beam element. The element displacement vector $\mathbf{q}_e$ of a beam with nodes 1 and 2 is of the form

$$\mathbf{q}_e = \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix} \tag{27}$$

Corresponding to the strains defined in Eq. (4), the simplest nodal degrees of freedom at node *i*, $\mathbf{q}_i$, can be chosen as

$$\mathbf{q}_i = [u_i, w_i, w_{,xi}, \phi_i]^T, \qquad i = 1,2 \tag{28}$$

Consequently, a cubic interpolation for deflection w and a linear interpolation for rotation ϕ can be achieved. It follows from Eq. (4) that the resulting bending strain approximation is constant over the element domain.

Corresponding to the strains defined in Eqs. (9) and (5), the simplest nodal degrees of freedom at node i, $\mathbf{q}_i$, can be chosen as

$$\mathbf{q}_i = [u_i, w_i, w_{,xi}, \gamma_i]^T, \qquad i = 1,2 \tag{29}$$

The nodal variable vector above has the same number of degrees of freedom at each node as that in Eq. (28). The nodal variables in Eq. (29) result in a cubic approximation for deflection w and a linear transverse shear strain γ. However, it follows from Eq. (9) that the cubic interpolation of w gives a linear element bending strain.

Since the bending strain is the dominant term in bending problems, we can predict that the strain expressions derived from the displacement filed defined in Eq. (6) would lead to a more accurate solution than those given by the displacement of Eq. (1) in finite element analysis, even though they have the same number of degrees of freedom at each node. This prediction will be verified later by numerical examples.

The element strain matrices in Eq. (17) can be evaluated by the standard displacement-based formulation. However, these matrices will be evaluated by the Quasi-conforming element technique [2] in this work.

BEAM ELEMENT BASED ON EQ. (9): HQCB-8A

In the quasi-conforming element formulation [2], the element strain field is interpolated directly over the element domain, and the compatibility in an element domain is satisfied in a weak form. When the continuity along inter-element boundaries is satisfied *a priori*, the element strain energy in Eq. (14) is modified as

$$U_e^* = U_e + \int_l \tilde{M}(e_b^* - e_b')dx + \int_l \tilde{N}(e_m - e_m')dx$$
$$+ \int_l \tilde{Q}(e_s - e_s')dx + \int_l \tilde{P}(e_{hs}^* - e_{hs}')dx \tag{29}$$

where the *prime* signifies the assumed element strain field and $\tilde{M}$, $\tilde{N}$, $\tilde{Q}$ and $\tilde{P}$ are the test functions corresponding to their relevant strains.

A cubic transverse displacement w and a linear rotation ϕ can be interpolated over the element from the element nodal variables. Then a suitable element strain field for the strains defined in Eqs. (9) & (5) and tensile strain in Eq. (4) can be approximated as

$$e_b^* = \frac{d^2 w_0}{dx^2} - \frac{d\gamma}{dx} \approx e_b' = \alpha_{b1} + x\alpha_{b2}, \quad e_m = \frac{du_0}{dx} \approx e_m' = \alpha_m,$$

$$\gamma \approx 2e_s' = \alpha_s, \quad e_{hs}^* = \frac{d\gamma}{dx} \approx e_{hs}' = \alpha_{hs} \tag{30}$$

Where α_{b_i} ($i = 1, 2$), α_m, α_s and α_{hs} are the assumed element strain parameters which can be determined from the weak form of compatibility given in Eq. (29) at element level. The influence of the y-axis is neglected in Eq. (30) as mentioned earlier.

Let the integrals for the weak form of strain compatibility in Eq. (29) be satisfied individually, and let the test functions in Eq. (29) be the same as the trial functions. Then the last four integrals in Eq. (29) lead to the strain matrices defined in Eq. (17) as

$$\mathbf{B}_m = [-1/l \quad 0 \quad 0 \quad 0 \quad 1/l \quad 0 \quad 0 \quad 0] \tag{31}$$

$$\mathbf{B}_b = \{1 \quad x\} \begin{bmatrix} 1/l & 0 \\ 0 & 12/l^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & l/2 & 0 & 0 & -1 & l/2 & 0 \end{bmatrix} \tag{32}$$

$$\mathbf{B}_s = [0 \quad 0 \quad 0 \quad 1/2 \quad 0 \quad 0 \quad 0 \quad 1/2] \tag{33}$$

$$\mathbf{B}_{sh} = [0 \quad 0 \quad 0 \quad -1/l \quad 0 \quad 0 \quad 0 \quad 1/l] \tag{34}$$

Only $\mathbf{K}_b$ involves a simple polynomial integration and it can be carried out easily. Therefore, the resulting element stiffness matrix can be evaluated explicitly, which makes the resulting beam element very computationally efficient. This beam element is designated as HQCB-8A (8-dof Higher-order Quasi-Conforming Beam element).

The detailed evaluation of the element stiffness matrix based on the quasi-conforming element technique can be found in Ref. 3.

BEAM ELEMENT BASED ON EQ. (1): HQCB-8B

For comparison, the element based on the strains defined in Eq.(4), and the nodal degrees of freedom defined in Eq. (28) is also given here.

As in HQCB-8A, a cubic deflection w and a linear rotation ϕ can be interpolated from the element nodal displacements. However, the cubic w contributes only to the transverse shear strain, but not to the bending strain which is the dominant term in beam analysis. For the given nodal variables, a suitable element strain field can be assumed as

$$e_b = \frac{d\phi}{dx} \approx e_b' = \alpha_b, \quad e_m = \frac{du_0}{dx} \approx e_m' = \alpha_m,$$

$$2e_s = \frac{dw_0}{dx} + \phi \approx 2e_s' = \alpha_{s1} + \alpha_{s2}x, \quad e_{hs} = \frac{d^2w_0}{dx^2} + \frac{d\phi}{dx} \approx e_{hs}' = \alpha_{hs} \tag{35}$$

Where α_b, α_m, α_{si} (i = 1,2) and α_{hs} are the assumed element strain parameters. Similar to Eqs. (32), (33) and (34), one can obtain

$$\mathbf{B}_b^B = [0 \quad 0 \quad 0 \quad -1/l \quad 0 \quad 0 \quad 0 \quad 1/l] \tag{36}$$

$$\mathbf{B}_s^B = \{1 \quad x\} \begin{bmatrix} 0 & -1/l & 0 & 1/2 & 0 & 1/l & 0 & 1/2 \\ 0 & 0 & -1/l & -1/l & 0 & 0 & l/1 & 1/l \end{bmatrix} \tag{37}$$

$$\mathbf{B}_{sh}^B = [0 \quad 0 \quad -1/l \quad -1/l \quad 0 \quad 0 \quad 1/l \quad 1/l] \tag{38}$$

The superscript B here is used to distinguish the strain matrices above from those given in Eqs. (32), (33) and (34). $\mathbf{B}_m$ is the same as that in Eq. (31). This beam element is designated as HQCB-8B.

The mass matrices in Eqs. (26) can be calculated easily by the cubic interpolation for w and linear interpolation for u_0 and γ.

NUMERICAL EXAMPLES

The efficiency and accuracy of the element based on displacement defined in Eq. (9) over that based on Eq. (1) is demonstrated by three examples in this section

EXAMPLE 1. Deflections of Isotropic Cantilevered Beam under Point Load at the Free End

To compare the performance of HQCB-8A and HQCB-8B, a cantilevered beam of length L subjected to a concentrated tip load P is considered. The material of the beam is isotropic with $\nu = 0.3$. The non-dimensional tip deflections for various aspect rations are given in Table I. The non-dimensional deflection is defined as

$$\overline{w} = w\frac{D_{xx}}{PL^3}$$

The numerical results in the table clearly show that the performance of HQCB-8A is much better than that of HQCB-8B as predicted, even though they have the same number of nodal degrees of freedom. This is because HQCB-8A makes the best use of the nodal variables to interpolate a linear bending strain, but HQCB-8B only has a constant bending strain.

EXAMPLE 2. Natural Frequencies of Simply Supported Composite Beams

A simply supported [0/90/90/0] composite beam is considered. The cross-ply beam has four equal thickness laminae and aspect rations of $L/h = 10$. The material properties are

$$E_1 / E_2 = 25,\quad G_{12} = G_{13} = 0.5E_2,\quad G_{23} = 0.2E_2,\quad \nu_{12} = 0.3$$

The non-dimensional frequencies and central deflection under uniform pressure are tabulated in Table II. All results are given by eight HQCB-8A elements. The non-dimensional deflection frequency are defined as

$$\overline{w} = w_{\max}\frac{bh^3E_2}{p_0L^4}.10^2 \quad \text{and} \quad \varpi_i = \omega_i L^2\sqrt{J_A / E_2h^3}\ ,\ i = 1,2,3,4$$

in which p_0 is the density of the uniform load. Because of the symmetric lamination, $\mathbf{M}_{uow}$ and $\mathbf{M}_{wx\gamma}$ are null. The influence of the mass matrix resulting from the higher-order displacement $\mathbf{M}_{\gamma}$ and the mass matrix from the coupling of rotation and higher-order displacement $\mathbf{M}_{wx\gamma}$ are also given in the table. The finite element solutions given by MSC/Nastran are listed in the table too, and eight 8-noded plate elements are used to model the beam.

Table II demonstrates that $\mathbf{M}_{\gamma}$ and $\mathbf{M}_{wx\gamma}$ have no or little influence on the fundamental and second mode frequencies, but they have a significant contribution on the frequencies of higher modes. This conclusion is also true even in the case of isotropic beams as indicated in Table III. The 3-D element results were obtained by 20-noded brick elements with a 4x8 mesh, and the 2-D solutions are given by eight 8-noded plate elements in MSC/Nastran.

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Table I. Non-dimensional tip deflections of isotropic cantilevered beam

Elements	aspect ratio *L/h*		
	5	10	1000
HQCB-8A(1 elem.)	0.3451	0.3363	0.3333
HQCB-8B (1 elem.)	0.3332	0.2543	0.2325
HQCB-8B (2 elem.)	0.3342	0.3179	0.3008
HQCB-8B (4 elem.)	0.3671	0.3379	0.3286
Analytical Solution	0.3436	0.3359	0.3333

Table II. Non-dimensional frequencies of simply supported [0/90/90/0] beam (*L/h*=10)

	consistent mass	no $\mathbf{M}_\gamma$	no $\mathbf{M}_{wx\gamma}$	no $\mathbf{M}_\gamma$ & $\mathbf{M}_{wx\gamma}$	MSC/Nastran 2-D Model	analytical
ϖ_1	10.4784	10.4702	10.4611	10.4529	9.7513	10.4880
ϖ_2	28.2852	27.9731	27.9443	27.6404	24.7609	
ϖ_3	60.2143	56.2691	56.8160	53.4422	39.9887	
ϖ_4	82.1853	70.2370	72.1778	62.6374	54.0770	
$\overline{w}$	1.1364				1.3570	1.1375

Table III. Non-dimensional frequencies of simply supported isotropic beam (*L/h*=10)

	consistent mass	no $\mathbf{M}_\gamma$	no $\mathbf{M}_{wx\gamma}$	no $\mathbf{M}_\gamma$ & $\mathbf{M}_{wx\gamma}$	MSC/Nastran 2-D	MSC/Nastran 3-D
ϖ_1	2.8016	2.8015	2.8013	2.8013	2.8161	2.8093
ϖ_2	10.6699	10.6680	10.6534	10.6615	10.6035	10.6635
ϖ_3	22.1816	22.1422	22.0230	21.9822	23.0221	22.3620
ϖ_4	35.2983	34.9908	34.5972	31.6181	37.4710	36.7079