FREQUENCY ANALYSIS OF COMPOSITE BEAMS USING A HIGHER-ORDER BEAM ELEMENT

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ABSTRACT

A simple and accurate third-order composite beam element is presented in this work. This higher-order element possesses a linear bending strain as opposed to the constant bending strain in existing higher-order composite beam elements. By using the Hamilton’s Principle, the variational consistent mass matrix for the third-order theory is derived. The resulting element is more accurate than the existing higher-order elements. The numerical examples show that the mass matrix resulting from the higher-order displacement has a considerable influence on the higher mode frequencies.

INTRODUCTION

The objectives of this paper are two fold, one is to present a simple and accurate third-order composite beam element, and the other is to use this new element for the frequency analysis of composite beams.

In the finite element modeling of composite beams and plates, a higher-order shear deformation theory can lead to finite elements having the same number of nodal variables but giving solutions with different accuracy. By studying the interpolation order of the element bending strain, this paper presents a simple but accurate third-order composite beam element, which possesses a linear bending strain as opposed to the constant bending strain in existing higher-order composite beam elements. In many higher-order elements, the mass matrices are either evaluated by the lumped mass method or based on the first-order theory. In this work, the variational consistent mass matrices for the third-order theory are derived from the Hamilton’s Principle. The present new element is used to solve some frequency analysis of composite beams. The numerical examples show that the transverse shear strains play a very important role in the dynamic behavior of composite beams. The numerical examples also illustrate that the present composite beam element is more accurate than the higher-order beam elements which are based on the same higher-order theory and having the same number of nodal variables but using a different bending strain expression.

STRIANS AND VELOCITIES BASED ON THE THIRD-ORDER DISPLACEMENT

The displacement field of a beam in the third-order shear deformation theory [1] is, in general, of the form

\[ u(x, y, z, t) = u_0(x, y, t) + \phi(x, y, t) z - \alpha \left( \frac{\partial w_0}{\partial x} + \phi \right) z^3 \]  

[1]

\[ w(x, y, z, t) = w_0(x, y, t) \]  

[2]

where \( u_0 \) and \( w_0 \) are, respectively, the axial displacement and the deflection of a point on the beam reference plane; \( \phi \) is the rotation of a normal to the reference plane about the \( y \)-axis; \( \alpha = 4/(3h^2) \); and \( h \) is the beam thickness. The influence of normal strain in the \( y \)-
direction on the behavior of composite beams will not be considered here, since the emphasis here is to discuss the efficient finite element modeling.

Equations (1) and (2) lead to the axial normal strain, the transverse shear strain and the transverse normal strain as

\[ e_x = e_m + e_b z - \alpha e_{hs} z^3, \quad e_{xz} = (1 - \beta z^2) e_x, \quad e_z = 0 \tag{3} \]

where \( \beta = 4/h^2 \), and \( e_m, e_b, e_s \) & \( e_{hs} \) are, respectively, the membrane strain, bending strain, transverse shear strain and higher-order transverse shear strain on the reference plane. They take the form

\[ e_m = \frac{\partial u_0}{\partial x}, \quad e_b = \frac{\partial \phi}{\partial x}, \quad e_s = \frac{1}{2} \left( \frac{\partial w_0}{\partial x} + \phi \right), \quad e_{hs} = \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \phi}{\partial x} \tag{4} \]

The displacement in Eq. (1) can be expressed in a different form. If one defines

\[ \gamma = \frac{\partial w_0}{\partial x} + \phi \tag{5} \]

Eq. (1) then can be rewritten as

\[ u(x, y, z, t) = u_0(x, y, t) - \left[ g - y(L - \gamma) + w(L - \gamma) \right] z + \alpha \gamma(x, y, t) z^3 \tag{6} \]

Accordingly, the normal strain, transverse shear strain and the velocities take the form

\[ e_x = e_m + e_b^* z - \alpha e_{hs}^* z^3, \quad 2e_{xz} = (1 - \beta z^2) \gamma \tag{7} \]

\[ v_x = \frac{\partial u}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{\partial}{\partial t} \left( \frac{\partial w_0}{\partial x} - \gamma \right) z - \alpha \frac{\partial \gamma}{\partial t} z^3, \quad v_z = \frac{\partial w}{\partial t} = \frac{\partial w_0}{\partial t} \tag{8} \]

in which \( \beta = 4/h^2 \) and

\[ e_b^* = -\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \gamma}{\partial x}, \quad e_{hs}^* = \frac{\partial \gamma}{\partial x} \tag{9} \]

Equation (9) gives the new expressions for the bending and higher-order transverse shear strains.

Both of the strain sets given, respectively, in Eqs. (3) & (4) and in Eqs. (5), (7) and (9) can be used for finite element modeling, and both of them result in \( C^1 \)-continuity elements under the displacement-based formulation. The bending strains in Eq. (4) and Eq. (9) are associated with different displacement functions. However, the one in Eq. (4) is a function of the rotation, and the one in Eq. (9) is the function of the deflection and transverse shear deformation. Because the orders of the approximations for the deflection and rotation are different in the finite element analysis under the given number of nodal variables, these two different bending strain expressions will lead to the finite element solutions with different accuracy. This will be demonstrated latter.

**HIGHER-ORDER COMPOSITE BEAM ELEMENT**

Let \( U_e \) and \( K_{ke} \) be the element strain energy and kinetic energy respectively, then the Himilton’s principle states that

\[ \delta \sum_{elem} \int_0^l (U_e - K_{ke}) dt = 0 \tag{10} \]

in which the work done by external forces is neglected. The Himilton’s principle leads to the equilibrium equation of a system as
\[ M \ddot{q} + K \ q = 0 \]  
(11)

where \( M, K \) and \( q \) are, respectively, the global mass matrix, stiffness matrix and nodal variable vector of the system. Consequently, the frequency \( \omega \) can be evaluated by

\[ (K - \omega^2 M) \ q = 0 \]  
(12)

The derivation of the element stiffness matrix and mass matrix based on the third-order theory is presented here.

**ELEMENT STIFFNESS MATRIX**

Let us consider a straight beam of length \( l \) and rectangular cross-section \( h \times b \) in which \( h \) is the beam thickness and \( b \) is the beam width. The strain energy of an element, \( U_e \), is of the form

\[ U_e = \frac{b}{2} \int_{-h/2}^{h/2} (e_x Q_{xx} e_x + 4e_{xx} Q_{xx} e_{xx}) dz dx \]  
(13)

where \( Q_{xx} \) and \( Q_{xz} \) are, respectively, the longitudinal Young’s modulus and transverse shear modulus, and they are functions of \( z \). Substituting Eqs. (7) and (9) into Eq. (13) leads to

\[ U_e = \frac{1}{2} \int \left[ e_m A_x e_m + e_b^* D_{xx} e_b^* + \gamma S_{xx} \gamma + e_h^* \alpha^2 H_{xx} e_h^* + 2e_m \alpha B_{xx} e_b^* - 2e_m \alpha E_{xx} e_h^* - 2e_b^* \alpha F_{xx} e_h^* \right] dx \]  
(14)

in which

\[ (A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx}) = b \int_{-h/2}^{h/2} (1, z, z^2, z^3, z^4, z^6) Q_{xx} dz \]  
(15)

\[ S_{xx} = b \int_{-h/2}^{h/2} (1 - \beta z^2)^2 Q_{xz} dz \]  
(16)

In the finite element modeling of displacement-based formulation, element strains in Eq. (14) can be expressed in terms of the element nodal displacement vector \( q_e \) and the element strain matrices as follows

\[ e_m = B_m q_e, \quad e_b^* = B_b q_e, \quad 2e_s = \gamma = B_s q_e, \quad e_h^* = B_h q_e \]  
(17)

Consequently, Eq. (14) becomes

\[ U_e = \frac{1}{2} q_e^T \int \left[ B_b^T D_{xx} B_b + B_m^T A_{xx} B_m + B_s^T S_{xx} B_s + B_h^T \beta^2 H_{xx} B_h + (B_m^T B_{xx} B_b + B_b^T B_{xx} B_m) \right] - \left( B_m^T \alpha E_{xx} B_h + B_h^T \alpha E_{xx} B_m \right) - \left( B_m^T \alpha F_{xx} B_h + B_h^T \alpha F_{xx} B_m \right) \]  
(18)

Then the Hamilton’s principle leads to the element stiffness matrix \( K_e \) as

\[ K_e = K_b + K_m + K_s + K_h + K_c \]  
(19)

with

\[ K_b = \int B_b^T D_{xx} B_b dx, \quad K_m = \int B_m^T A_{xx} B_m dx, \]
\[ K_s = \int B_s^T S_{xx} B_s dx, \quad K_h = \int B_h^T \beta^2 H_{xx} B_h dx, \]
\[ K_c = \int \left[ (B_m^T B_{xx} B_b + B_b^T B_{xx} B_m) - (B_m^T \alpha E_{xx} B_h + B_h^T \alpha E_{xx} B_m) \right] - \left( B_m^T \alpha F_{xx} B_h + B_h^T \alpha F_{xx} B_m \right) \]  
(20)

where \( K_b, K_m, K_s, K_h \) and \( K_c \) are the element bending, membrane, transverse shear,
higher-order shear and coupling stiffness matrix respectively.

**CONSISTENT MASS MATRIX FOR HIGHER-ORDER BEAM THEORY**

The element kinetic energy $K_{ke}$ corresponding to the higher-order theory takes the form

$$K_{ke} = \frac{b}{2} \int_{-h/2}^{h/2} (v_z^2 + v_s^2) \rho \, dz \, dx = \frac{b}{2} \int_{-h/2}^{h/2} [\left(\frac{\partial w}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2] \rho \, dz \, dx$$

$$= \frac{b}{2} \int_{-h/2}^{h/2} \left[\left(\frac{\partial w_0}{\partial t}\right)^2 + \left(\frac{\partial u_0}{\partial t}\right)^2 + z^2 \left(\frac{\partial^2 w_0}{\partial t \partial x}\right)^2 + \alpha^2 z^6 \left(\frac{\partial^2 w_0}{\partial t \partial x}\right)^2 + 2z \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial t \partial x} - 2z^3 \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial t \partial x} - 2\alpha^4 \frac{\partial^2 w_0}{\partial t \partial x} \frac{\partial^2 w_0}{\partial t \partial x} \right] \rho \, dz \, dx$$

(21)

By defining

$$(J_A, J_B, J_D, J_E, J_F, J_H) = b \int_{-h/2}^{h/2} (1, z, z^2, z^3, z^4, z^6) \rho \, dz$$

(22)

the element kinetic energy $K_{ke}$ can be written as

$$K_{ke} = \frac{1}{2} \int [J_A \left(\frac{\partial w_0}{\partial t}\right)^2 + J_A \left(\frac{\partial u_0}{\partial t}\right)^2 + J_D \left(\frac{\partial^2 w_0}{\partial t \partial x}\right)^2 + \alpha^2 J_H \left(\frac{\partial^2 w_0}{\partial t \partial x}\right)^2$$

$$+ 2J_B \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial t \partial x} - 2\alpha J_E \frac{\partial u_0}{\partial t} \frac{\partial^2 w_0}{\partial t \partial x} - 2\alpha J_F \frac{\partial^2 w_0}{\partial t \partial x} \frac{\partial^2 w_0}{\partial t \partial x}] \rho \, dz \, dx$$

(23)

The element displacements can be interpolated in terms of the element nodal displacement vector $q_e$ as

$$u = N_u q_e, \quad w_0 = N_w q_e, \quad \frac{\partial w_0}{\partial x} = \frac{\partial}{\partial x} N_w q_e = N_{wx} q_e, \quad \gamma = N_\gamma q_e \quad \text{and} \quad \ddot{q}_e = \frac{\partial^2}{\partial t^2} q_e$$

(24)

where $N_j$ ($j = u, w, wx$ and $\gamma$) are the interpolation matrices. The equation above and the Hamilton’s principle give the consistent element mass matrix $M_e$ as

$$M_e = M_w + M_{u_0} + M_{w_0} + M_\gamma + M_{u_{ow}} + M_{u_{ow}} + M_{w_{xy}}$$

(25)

with

$$M_w = \int_i N_w^T J_A N_w \, dx, \quad M_{u_0} = \int_i N_{u_0}^T J_A N_{u_0} \, dx, \quad M_{w_0} = \int_i N_{w_0}^T J_A N_{w_0} \, dx$$

$$M_\gamma = \alpha^2 \int_i N_\gamma^T J_H N_\gamma \, dx, \quad M_{u_{ow}} = \int_i (N_{u_0}^T J_B N_w + N_{w}^T J_B N_{u_0}) \, dx$$

$$M_{u_{ow}} = -\alpha \int_i (N_{u_0}^T J_E N_\gamma + N_\gamma^T J_E N_{u_0}) \, dx, \quad M_{w_{xy}} = -\alpha \int_i (N_{w_0}^T J_F N_\gamma + N_\gamma^T J_F N_{w_0}) \, dx$$

(26)

$M_w$, $M_{u_0}$ and $M_{w_0}$ are, respectively, the consistent transverse, axial and rotary mass matrices; $M_\gamma$ is the mass matrix resulting from the higher-order terms; and $M_{u_{ow}}$, $M_{u_{ow}}$ and $M_{w_{xy}}$ are the coupling terms.

The consistent mass matrix here has two meanings, one is opposed to the lumped mass method, and the other refers that the contribution of the higher-order term to the mass matrix is also taken into account.

**FORMULATION OF ELEMENT STIFFNESS MATRIX**

Let us consider a simple two-node beam element. The element displacement vector $q_e$ of a beam with nodes 1 and 2 is of the form
Corresponding to the strains defined in Eq. (4), the simplest nodal degrees of freedom at node \( i \), \( q_i \), can be chosen as

\[
q_i = \begin{bmatrix} q_{i1} \\ q_{i2} \end{bmatrix}, \quad i = 1, 2
\]

Consequently, a cubic interpolation for deflection \( w \) and a linear interpolation for rotation \( \phi \) can be achieved. It follows from Eq. (4) that the resulting bending strain approximation is constant over the element domain.

Corresponding to the strains defined in Eqs. (9) and (5), the simplest nodal degrees of freedom at node \( i \), \( q_i \), can be chosen as

\[
q_i = \begin{bmatrix} u_i, w_i, w_{xi}, \gamma_i \end{bmatrix}^T, \quad i = 1, 2
\]

The nodal variable vector above has the same number of degrees of freedom at each node as that in Eq. (28). The nodal variables in Eq. (29) result in a cubic approximation for deflection \( w \) and a linear transverse shear strain \( \gamma \). However, it follows from Eq. (9) that the cubic interpolation of \( w \) gives a linear element bending strain.

Since the bending strain is the dominant term in bending problems, we can predict that the strain expressions derived from the displacement field defined in Eq. (6) would lead to a more accurate solution than those given by the displacement of Eq. (1) in finite element analysis, even though they have the same number of degrees of freedom at each node. This prediction will be verified later by numerical examples.

The element strain matrices in Eq. (17) can be evaluated by the standard displacement-based formulation. However, these matrices will be evaluated by the Quasi-conforming element technique [2] in this work.

**BEAM ELEMENT BASED ON EQ. (9): HQCB-8A**

In the quasi-conforming element formulation [2], the element strain field is interpolated directly over the element domain, and the compatibility in an element domain is satisfied in a weak form. When the continuity along inter-element boundaries is satisfied \emph{a priori}, the element strain energy in Eq. (14) is modified as

\[
U_e^* = U_e + \int_1^2 \tilde{M}(e_b^* - e_b)dx + \int_j \tilde{N}(e_m - e_m')dx \\
+ \int_j \tilde{Q}(e_s - e_s')dx + \int_j \tilde{P}(e_{hs} - e_{hs}')dx
\]

where the \emph{prime} signifies the assumed element strain field and \( \tilde{M} \), \( \tilde{N} \), \( \tilde{Q} \) and \( \tilde{P} \) are the test functions corresponding to their relevant strains.

A cubic transverse displacement \( w \) and a linear rotation \( \phi \) can be interpolated over the element from the element nodal variables. Then a suitable element strain field for the strains defined in Eqs. (9) & (5) and tensile strain in Eq. (4) can be approximated as

\[
e_b^* = \frac{d^2w_0}{dx^2} - \frac{dy}{dx} \approx e_b = \alpha_{b1} + x\alpha_{b2}, \quad e_m = \frac{du_0}{dx} \approx e_m' = \alpha_m,
\]

\[
\gamma \approx 2e_s' = \alpha_s, \quad e_{hs}' = \frac{dy}{dx} \approx e_{hs}' = \alpha_{hs}
\]

Where \( \alpha_{bi} (i = 1, 2) \), \( \alpha_m \), \( \alpha_s \) and \( \alpha_{hs} \) are the assumed element strain parameters which can
be determined from the weak form of compatibility given in Eq. (29) at element level. The
influence of the y-axis is neglected in Eq. (30) as mentioned earlier.

Let the integrals for the weak form of strain compatibility in Eq. (29) be satisfied
individually, and let the test functions in Eq. (29) be the same as the trial functions. Then the
last four integrals in Eq. (29) lead to the strain matrices defined in Eq. (17) as

\begin{align}
B_m &= \begin{bmatrix}
-1/l & 0 & 0 & 0 & 1/l & 0 & 0 & 0
\end{bmatrix} \\
B_b &= \begin{bmatrix}
1/l & 0 & 0 & 0 & 0 & 0 \\
0 & 12/l^3 & 0 & 0 & 1/l & 0 & 0 & -1 & l/2 & 0
\end{bmatrix} \\
B_s &= \begin{bmatrix}
0 & 0 & 1/2 & 0 & 0 & 0 & 1/2
\end{bmatrix} \\
B_{sh} &= \begin{bmatrix}
0 & 0 & -1/l & 0 & 0 & 0 & 1/l
\end{bmatrix}
\end{align}

Only \( K_b \) involves a simple polynomial integration and it can be carried out easily. Therefore,
the resulting element stiffness matrix can be evaluated explicitly, which makes the resulting
beam element very computationally efficient. This beam element is designated as HQCB-8A
(8-dof Higher-order Quasi-Conforming Beam element).

The detailed evaluation of the element stiffness matrix based on the quasi-conforming
element technique can be found in Ref. 3.

**BEAM ELEMENT BASED ON EQ. (1): HQCB-8B**

For comparison, the element based on the strains defined in Eq.(4), and the nodal degrees
of freedom defined in Eq. (28) is also given here.

As in HQCB-8A, a cubic deflection \( w \) and a linear rotation \( \phi \) can be interpolated from
the element nodal displacements. However, the cubic \( w \) contributes only to the transverse
shear strain, but not to the bending strain which is the dominant term in beam analysis. For
the given nodal variables, a suitable element strain field can be assumed as

\begin{align}
e_b &= \frac{d\phi}{dx} = e_b = \alpha_b, \quad e_m = \frac{du_0}{dx} = e_m = \alpha_m,
2e_s &= \frac{dw_0}{dx} + \phi = 2\epsilon_s = \alpha_{s1} + \alpha_{s2}x, \quad e_{hs} = \frac{d^2w_0}{dx^2} + \frac{d\phi}{dx} = e_{hs} = \alpha_{hs}
\end{align}

Where \( \alpha_b, \alpha_m, \alpha_{si} \) (i = 1,2) and \( \alpha_{hs} \) are the assumed element strain parameters. Similar to
Eqs. (32), (33) and (34), one can obtain

\begin{align}
B_b^B &= \begin{bmatrix}
0 & 0 & 0 & -1/l & 0 & 0 & 0 & 1/l
\end{bmatrix} \\
B_s^B &= \begin{bmatrix}
0 & -1/l & 0 & 1/2 & 0 & 1/l & 0 & 1/2
\end{bmatrix} \\
B_{sh}^B &= \begin{bmatrix}
0 & 0 & -1/l & -1/l & 0 & 0 & l/1 & 1/l
\end{bmatrix}
\end{align}

The superscript \( B \) here is used to distinguish the strain matrices above from those given in
Eqs. (32), (33) and (34). \( B_m \) is the same as that in Eq. (31). This beam element is designated
as HQCB-8B.

The mass matrices in Eqs. (26) can be calculated easily by the cubic interpolation for \( w \)
and linear interpolation for \( u_0 \) and \( \gamma \).
NUMERICAL EXAMPLES

The efficiency and accuracy of the element based on displacement defined in Eq. (9) over that based on Eq. (1) is demonstrated by three examples in this section.

EXAMPLE 1. Deflections of Isotropic Cantilevered Beam under Point Load at the Free End

To compare the performance of HQCB-8A and HQCB-8B, a cantilevered beam of length $L$ subjected to a concentrated tip load $P$ is considered. The material of the beam is isotropic with $v = 0.3$. The non-dimensional tip deflections for various aspect ratios are given in Table I. The non-dimensional deflection is defined as

$$\bar{w} = \frac{w}{PL^3}$$

The numerical results in the table clearly show that the performance of HQCB-8A is much better than that of HQCB-8B as predicted, even though they have the same number of nodal degrees of freedom. This is because HQCB-8A makes the best use of the nodal variables to interpolate a linear bending strain, but HQCB-8B only has a constant bending strain.

EXAMPLE 2. Natural Frequencies of Simply Supported Composite Beams

A simply supported $[0/90/90/0]$ composite beam is considered. The cross-ply beam has four equal thickness laminae and aspect ratios of $L/h = 10$. The material properties are

$$E_1 / E_2 = 25, \quad G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \quad \nu_{12} = 0.3$$

The non-dimensional frequencies and central deflection under uniform pressure are tabulated in Table II. All results are given by eight HQCB-8A elements. The non-dimensional deflection frequency are defined as

$$\bar{w} = \frac{w_{max} bh^3 E_2}{p_0 L^4} \cdot 10^2 \quad \text{and} \quad \sigma_i = \omega_i L^2 \sqrt{J_A / E_2 h^3}, \quad i = 1, 2, 3, 4$$

in which $p_0$ is the density of the uniform load. Because of the symmetric lamination, $\mathbf{M}_{w_0w}$ and $\mathbf{M}_{wxy}$ are null. The influence of the mass matrix resulting from the higher-order displacement $\mathbf{M}_y$ and the mass matrix from the coupling of rotation and higher-order displacement $\mathbf{M}_{wxy}$ are also given in the table. The finite element solutions given by MSC/Nastran are listed in the table too, and eight 8-noded plate elements are used to model the beam.

Table II demonstrates that $\mathbf{M}_y$ and $\mathbf{M}_{wxy}$ have no or little influence on the fundamental and second mode frequencies, but they have a significant contribution on the frequencies of higher modes. This conclusion is also true even in the case of isotropic beams as indicated in Table III. The 3-D element results were obtained by 20-noded brick elements with a 4x8 mesh, and the 2-D solutions are given by eight 8-noded plate elements in MSC/Nastran.

REFERENCES


Table I. Non-dimensional tip deflections of isotropic cantilevered beam

<table>
<thead>
<tr>
<th>Elements</th>
<th>aspect ratio $L/h$</th>
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<tbody>
<tr>
<td></td>
<td>5</td>
<td>10</td>
<td>1000</td>
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<tr>
<td>HQCB-8A (1 elem.)</td>
<td>0.3451</td>
<td>0.3363</td>
<td>0.3333</td>
</tr>
<tr>
<td>HQCB-8B (1 elem.)</td>
<td>0.3332</td>
<td>0.2543</td>
<td>0.2325</td>
</tr>
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<td>HQCB-8B (2 elem.)</td>
<td>0.3342</td>
<td>0.3179</td>
<td>0.3008</td>
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<tr>
<td>HQCB-8B (4 elem.)</td>
<td>0.3671</td>
<td>0.3379</td>
<td>0.3286</td>
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<tr>
<td>Analytical Solution</td>
<td>0.3436</td>
<td>0.3359</td>
<td>0.3333</td>
</tr>
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</table>

Table II. Non-dimensional frequencies of simply supported [0/90/90/0] beam ($L/h=10$)

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>10.4784</th>
<th>10.4702</th>
<th>10.4611</th>
<th>10.4529</th>
<th>9.7513</th>
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<tr>
<td>$\omega_2$</td>
<td>28.2852</td>
<td>27.9731</td>
<td>27.9443</td>
<td>27.6404</td>
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<tr>
<td>$\omega_3$</td>
<td>60.2143</td>
<td>56.2691</td>
<td>53.4422</td>
<td>39.9887</td>
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<tr>
<td>$\omega_4$</td>
<td>82.1853</td>
<td>70.2370</td>
<td>62.6374</td>
<td>54.0770</td>
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<tr>
<td>$\bar{w}$</td>
<td>1.1364</td>
<td></td>
<td></td>
<td></td>
<td>1.3570</td>
<td>1.1375</td>
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Table III. Non-dimensional frequencies of simply supported isotropic beam ($L/h=10$)

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>2.8016</th>
<th>2.8015</th>
<th>2.8013</th>
<th>2.8013</th>
<th>2.8161</th>
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<td>$\omega_2$</td>
<td>10.6699</td>
<td>10.6680</td>
<td>10.6534</td>
<td>10.6615</td>
<td>10.6035</td>
<td>10.6635</td>
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<tr>
<td>$\omega_3$</td>
<td>22.1816</td>
<td>22.1422</td>
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