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NONLINEAR BENDING-TORSION MODAL INTERACTION UNDER PARAMETRIC EXCITATION

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ABSTRACT

The purpose of this study is to understand the main differences between deterministic and random response characteristics of a cantilever beam in the neighborhood of combination parametric resonance. The beam orientation with respect to the excitation is made in such a way that the bending and torsion modes are in cross coupling through the excitation. In the absence of excitation the two modes are also coupled due to nonlinear inertia forces. This means that both linear generalized and normal coordinates are the same. For sinusoidal parametric excitation the beam experiences instability in the neighborhood of the combination parametric resonance $\Omega = \omega_u + \omega_\phi$, where Ω is the excitation frequency, ω_u and ω_ϕ are the bending and torsion first mode natural frequencies, respectively. The dependence of the response amplitude on the excitation level reveals three distinct regions which include linear behavior, jump phenomena, and energy transfer. Under random excitation, with center frequency close to the sum of the bending and torsion mode frequencies, the system may experience a single response, two possible responses or nonstationary responses depending on the excitation level. The response may also be Gaussian or non-Gaussian depending on the excitation level as well. Experimentally, it is possible to obtain two different responses for the same excitation level by providing some perturbation to the system.

ANALYTICAL MODELING

Figure 1 shows a schematic diagram of a cantilever beam carries a mass m whose mass moment of inertia about z-axis is I_0. The beam can experience bending and torsion oscillations under a support excitation Y(t). The coupling can be visualized by representing the base excitation as an inertia force on the end -mass (Roberts, 1985). According to Roberts, if the beam is instantaneously bent out of the excitation plane and twisted slightly, then the end inertia force acting through the bending displacement gives a torque on the beam cross-section. Similarly, because of the smaller rotation of the principal planes of the cross-section, the end inertia force contributes a bending moment about a local plane of minimum bending stiffness proportional to the twist angle ϕ. The nonlinear coupling may be traced directly to the presence of v_0, a small but important displacement in the plane of the excitation. A comprehensive analytical modeling (excluding nonlinear curvature) has been given by Cartmell (1990). The displacements of the beam elastic axis in the x, y, and z directions at z=L are described by $u_0(L,t)$, $v_0(L,t)$, and $w_0(L,t)$, respectively. The displacement along y axis $v(z,t)$ can be written in terms of bending and torsion displacement by setting the curvature about x-axis to zero, i.e. $v'' \cong u''\phi$, where a prime denotes differentiation with respect to z. Considering only the first mode in bending and torsion, the following solutions are adopted

$$u(z,t) = u_0(t)\sin\left(\frac{\pi z}{2L}\right), \qquad \phi(z,t) = \phi_0(t)\sin\left(\frac{\pi z}{2L}\right) \tag{1}$$

where $u_0(t)$ and $\phi_0(t)$ are the generalized coordinates at z=L. Neglecting the extension of the beam elastic axis, and taking into account the axial drop $w_0(L,t)$ in terms of u one can write the equations of motion using Lagrange's equation as (Hijawi, 1996)

$$\left[1 + c_1\phi_0^2 + c_2U^2\right]\ddot{U} + 2\zeta_u\dot{U} + U + 2c_1\phi_0\dot{\phi}_0\dot{U} + c_2U\dot{U}^2 + c_1\phi_0\ddot{\phi}_0U + c_3U^3 + B_1\phi_0\ddot{\tilde{Y}}(\tau) = 0 \tag{2}$$

$$\left[1 + c_4\,U^2\right]\ddot{\phi}_0 + 2r\,\zeta_\phi\dot{\phi}_0 + r^2\,\phi_0 + 2\,c_4\,B_1^2\,U\,\dot{U}\,\dot{\phi}_0 + c_4\,U\,\ddot{U}\,\phi_0 + c_5\,U\,\ddot{\tilde{Y}}(\tau) = 0 \tag{3}$$

where $U = u_0/b$, $\tilde{Y}(\tau) = Y(\tau)/b$, b is the beam width, a dot denotes differentiation with respect to the nondimensional time parameter $\tau = \omega_u t$, $\omega_u = \pi^2\sqrt{EI_y/32mL^3}$ is the bending natural frequency, $r = \omega_\phi/\omega_u$, $\omega_\phi = \pi\sqrt{cGJ/8LI_0}$ is the torsion natural frequency, c is a constant which accounts for the non-circular cross-section of the beam, J is polar area moment of inertia of the beam cross-section, E is Young's modulus, G is the modulus of rigidity, c_i and B_1 are constants, ζ_u and ζ_ϕ are damping factors associated with bending and torsion, respectively. The nonlinear differential equations (2) and (3) are coupled through cubic nonlinear inertia and parametric excitation through the other mode. In addition to the inertia nonlinearity, the bending equation (2) also includes cubic curvature nonlinearity.

DETERMINISTIC RESPONSE BACKGROUND

The linearized form of equations (2) and (3) only include cross coupling through the parametric excitation terms $B_1\phi_0\ddot{\tilde{Y}}(\tau)$ and $c_5U\ddot{\tilde{Y}}(\tau)$. Under sinusoidal excitation $\tilde{Y}''(\tau) = z_0r_f^2\cos(r_f\tau)$, where $r_f = \Omega/\omega_u$, the beam may experience parametric instability of summed combination type $r_f = r+1$ (see, e.g., Dugundji and Mukhopadhyay, 1973 and Roberts, 1985). The instability is limited by a boundary curve of bounded non-zero response of V-shaped and centered. Inside the instability boundary the response can be estimated by considering the full nonlinear equations (2) and (3) using the method of multiple scales (Hijawi, 1996). The dynamic response of the system in the neighborhood of the combined resonance $r_f = r+1$ was given in the time and frequency domains. In the time domain the interaction is reflected by periodic signals carrying a frequency component of the other mode.

The dependence of the response amplitudes on the external frequency detuning parameter σ ($r - r_f = -1 - \varepsilon\sigma$, where ε is a small parameter) exhibited nonlinear characteristics with the response amplitudes overhang to the left which is the main characteristics of systems with nonlinear inertia. For any detuning parameter there are two solutions, the larger one is belonging to a stable manifold while the smaller one is unstable. At the point where stable and unstable solutions meet (saddle or turning point) the response collapses to the static equilibrium position. The experimental measurements showed that at relatively high excitation levels the energy transfers to the torsion mode keeping the bending mode at the same level. The influence of the system nonlinearities was reflected in the response spectra since some spikes appeared at frequencies other than the natural frequencies of the first bending and torsion modes.

RANDOM RESPONSE ANALYSIS

Linear Stability Analysis: The response of the beam under random parametric excitation is not a simple task. However, the linear part of the system equations of motion coupled through parametric excitation cab be examined for stochastic stability by estimating the maximum value of Lyapunov exponent. Under the condition that $\sqrt{8cmGL^2/EI_0} = \pi mb^2/I_0$, the linear form of equations (2) and (3) takes the form

$$\ddot{U} + 2\zeta_u\dot{U} + U - 0.3669\phi_0\tilde{Y}(\tau) = 0 \tag{4}$$

$$\ddot{\phi}_0 + 2r\,\zeta_\phi\dot{\phi}_0 + r^2\,\phi_0 - 0.3669r\,U\,\tilde{Y}(\tau) = 0 \tag{5}$$

These equations fall under the class examined by Ariaratnam and Xie (1992) who showed that the largest Lyapunov exponent λ is given by the expression

$$\lambda = -2\zeta_u \pm \frac{1}{8}0.3669^2 S(1+r) \quad \text{if } \zeta_u > r\zeta_\phi \tag{6a}$$

$$\lambda = -\,2r\zeta_\phi \pm \frac{1}{8}0.3669^2 S(1+r) \quad \text{if } \zeta_u < r\zeta_\phi \tag{6b}$$

where S(1+r) denotes the power spectral density of the stochastic excitation $\tilde{Y}(\tau)$ at frequency (1+r). If $\lambda < 0$, then the beam equilibrium position $U = \phi_0 = 0$ is almost-surely asymptotically stable, otherwise it is unstable. Once the equilibrium position becomes unstable, the beam motion will grow and achieve bounded variations due to the system nonlinearities.

Nonlinear Response: In view of the complex form of equations (2) and (3), we will estimate the response statistics by employing Monte Carlo simulation. The excitation can be either white noise or band limited random process. The band limited excitation is generated by a linear filter subjected to white noise random process. The filter equation in the nondimensional form

$$q'' + 2\,\zeta_f\,\omega_f q' + \omega_f^2 q = \tilde{Y}''(\tau) \tag{7}$$

The input bandwidth and center frequency are controlled by ζ_f and ω_f, respectively. Applying the coordinate transformation $\{U, U', \phi_0, \phi_0', q, q'\} \rightarrow \{X_1, X_2, X_3, X_4, X_5, X_6\}$ the equations of motion (1) and (2) and the filter equation (21) take the form

$$\begin{gathered} X_1' = X_2 \\ X_2' = -\,[2\zeta_u X_2 + X_1 + 2c_1X_3X_4X_2 + c_2X_1X_2^2 + c_1X_3X_4'X_1 + c_3X_1^3 + B_1X_3X_5]/\left[1 + c_1X_3^2 + c_2X_1^2\right] \\ X_3' = X_4 \\ X_4' = -[2r\,\zeta_\phi X_4 + r^2\,X_3 + 2\,c_4\,B_1^2\,X_1\,X_2\,X_4 + c_4\,X_1\,X_2'\,X_3 + c_5\,X_1\,X_5]/\left[1 + c_4\,X_1^2\right] \\ X_5' = X_6 \\ X_6' = -\,2\,\zeta_f\,\omega_f X_6 - \omega_f^2 X_5 + \tilde{Y}''(\tau) \end{gathered} \tag{8}$$

Time history records of the filtered excitation $E[q^2]$ from the white noise random process $\tilde{Y}''$, and response mean squares, for different filter bandwidths are found to achieve stationarity in the steady state regime. The response stationarity is preserved for excitation levels which are less than a critical level above which the response becomes nonstationary. If the excitation is purely white noise, the response coordinates experience quasi-nonstationarity random process. The white noise excitation level has to be higher than the filtered white noise because the power spectrum of the white noise is uniformly spread over a wide band frequency range for which the bending and torsion modes will pick a small portion of this power corresponding to their natural frequencies. This is not the case for filtered white noise where its power spectrum is concentrated around the sum of the two natural frequencies of the beam.

The dependence of the response mean square on the excitation level is shown in Figure 2 for filter damping ratios ζ_f=0.001 and 0.01 shown by solid and empty small circles, respectively. The filtered excitation bandwidth, given by the filter damping ratio, affects the bifurcation point of response. For example, for small bandwidth, ζ_f=0.001, the bifurcation point occurs at a very small excitation level 0.025, while for relatively larger bandwidth ζ_f= 0.01 the bifurcation point occurs at excitation level 0.4. Figures 3 shows the dependence of the response mean square on the white noise excitation level. It is obvious that the system has zero response for excitation range greater than the filtered white noise case. The bifurcation point occurs at excitation level 2.5. Contrary to direct parametric excitation for each mode, the well known "on-off intermittency" close to the bifurcation point does not take place in the present case.

EXPERIMENTAL RESULTS

The model beam was excited by a limited bandwidth random excitation of 2 Hz and a central frequency of 45 Hz which is very close to the summation of the bending and torsion natural frequencies ($\Omega = \omega_t + \omega_b = 44.6$ Hz).

It was found experimentally that the system has a single response for excitation mean square less than 2.25 and two possible responses for higher excitation mean square. The response mean square time history records are essentially stationary. The probability density functions of both the excitation and the torsion response have Gaussian distribution while the bending response shows some periodicity as reflected by its bimodal distribution which was also revealed from Monte Carlo simulation. Increasing the excitation mean square to 3.0, shows stronger nonlinear interaction between the response modes. The response is found to have more than one attractor depending on the initial conditions or perturbations triggered by hand. When the beam is perturbed by hand the response it is observed that more energy is transferred to the bending mode.

The dependence of the response mean square on the excitation level is shown in Figure 4. It is seen that at low excitation levels the bending mode has zero response because the structural damping forces overcome the input energy. However, the torsion mode has oscillations even for low excitation levels. It was noticed experimentally that for different configurations of the beam (for example vertical beam under parametric excitation) the region of zero response for both bending and torsion is more distinguishable. Figure 4 shows that as the excitation level increases, the bending mode starts to bifurcate to a non-zero mean square value. When the excitation level is higher than 2.25 the system starts to have two different possible responses. Experimentally, it is possible to obtain two different responses for the same excitation level by providing some perturbation to the beam. It should be pointed out that the data was taken after the system reaches a stationary response as shown in the mean square time histories.

When the bandwidth of the excitation is increased to 10 Hz for the same central frequency (45 Hz), new features of the system response are detected. The system has a single response for excitation mean square less than 4.0 after which the system possesses many possible non-stationary responses for the same excitation level. For excitation mean square of 4.64, the mean square time history records of the bending and torsion response show that the response is not stationary and different tests can produce different results. Figures 5a through 5b show the mean square time histories of two tests conducted at the same excitation level. It is obvious that even after a long time (1000 sec) the response does not reach stationarity. The figures also reflect the energy exchange between the bending and torsion modes. The dependence of the response mean square on the excitation level mean square is displayed in Figure 6. The response for excitation mean square greater than 4.0, however, should not be taken into account since it is not stationary.

CONCLUSIONS

Deterministic and stochastic excitations of a cantilever beam with an end mass have revealed different characteristics. Under sinusoidal parametric excitation the beam equilibrium position can be stable or unstable close to combination parametric resonance depending on the excitation frequency and amplitude. In the unstable zone the beam possesses one steady stable response with energy exchange between bending and torsion modes. Under random excitation the response stochastic stability is obtained in terms of the largest Lyapunov exponent. When the Largest Lyapunov exponent takes positive values the beam response statistics are estimated using Monte Carlo simulation. The experimental results showed that at relatively high excitation levels the system has two possible responses for narrow band excitation while many non-stationary responses are evident for wide band excitation. This was verified for different tests at the same excitation level. Both analytical and experimental results showed that the main features of the system dynamical characteristic are in good agreement. For both deterministic and random excitation the investigations perceived that the system nonlinearities play a significant role.

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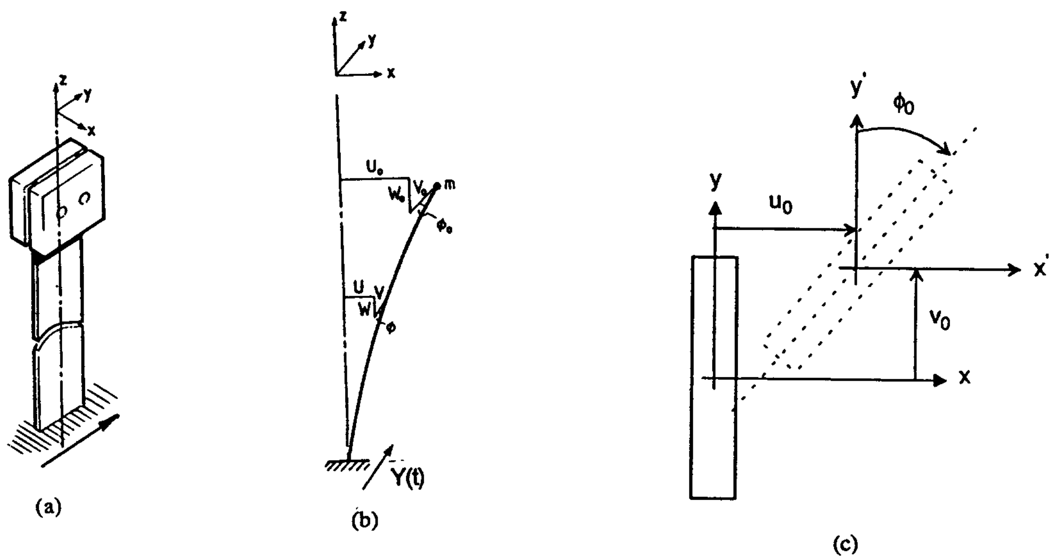

Figure 1 (a) Schematic diagram of the system and coordinate frame (b) kinematic coupling provided by v_0 displacement and (c) beam cross-section and its deflection.

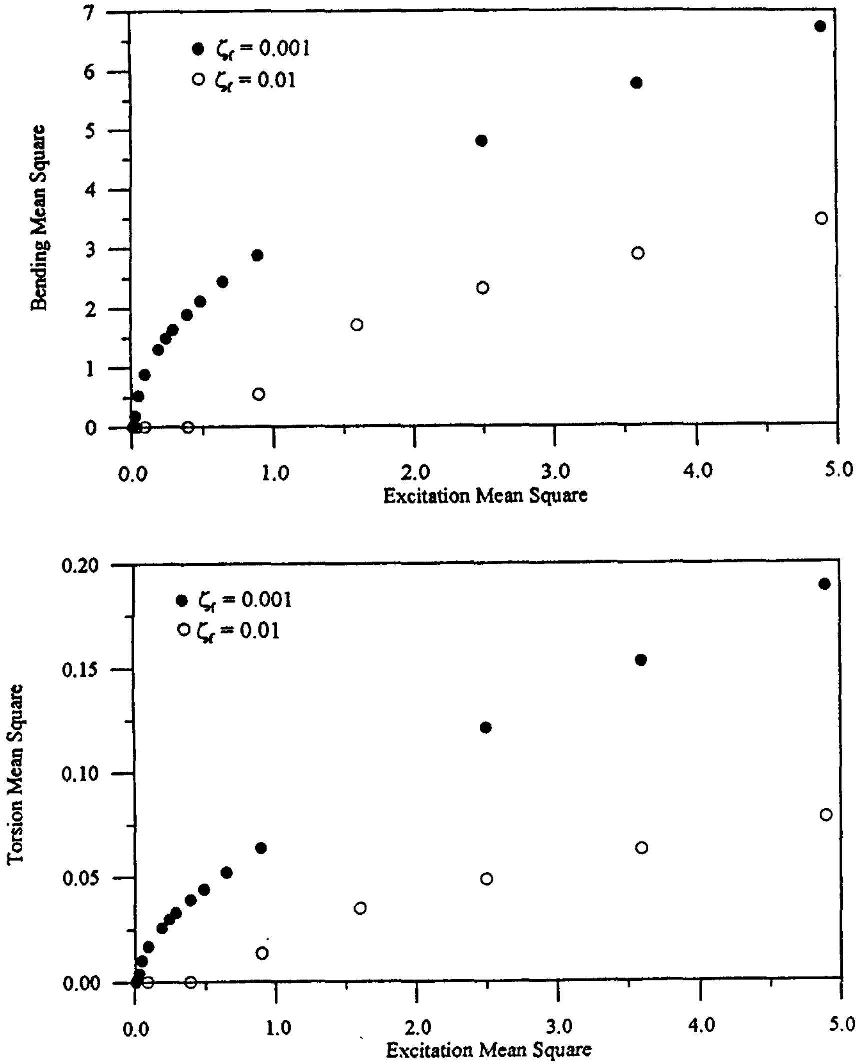

Figure 2 Dependence of response mean square on filtered excitation level

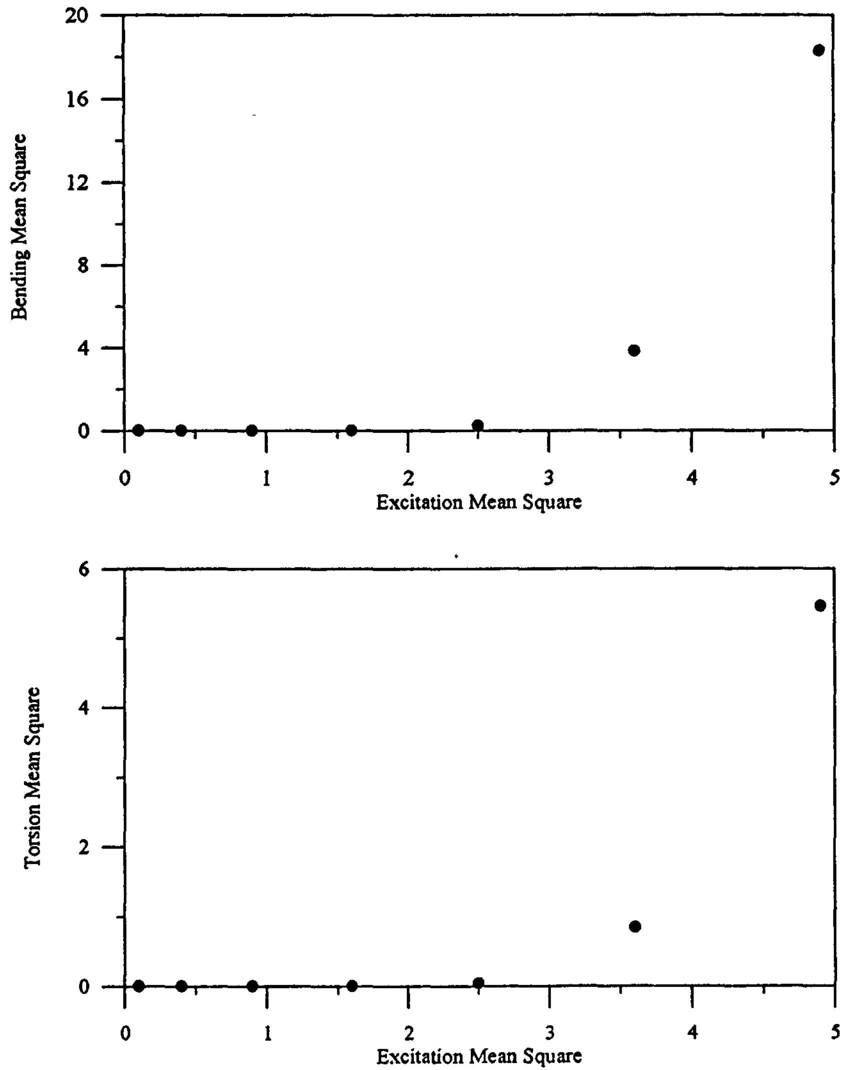

Figure 3. Dependence of response mean square on white noise excitation level

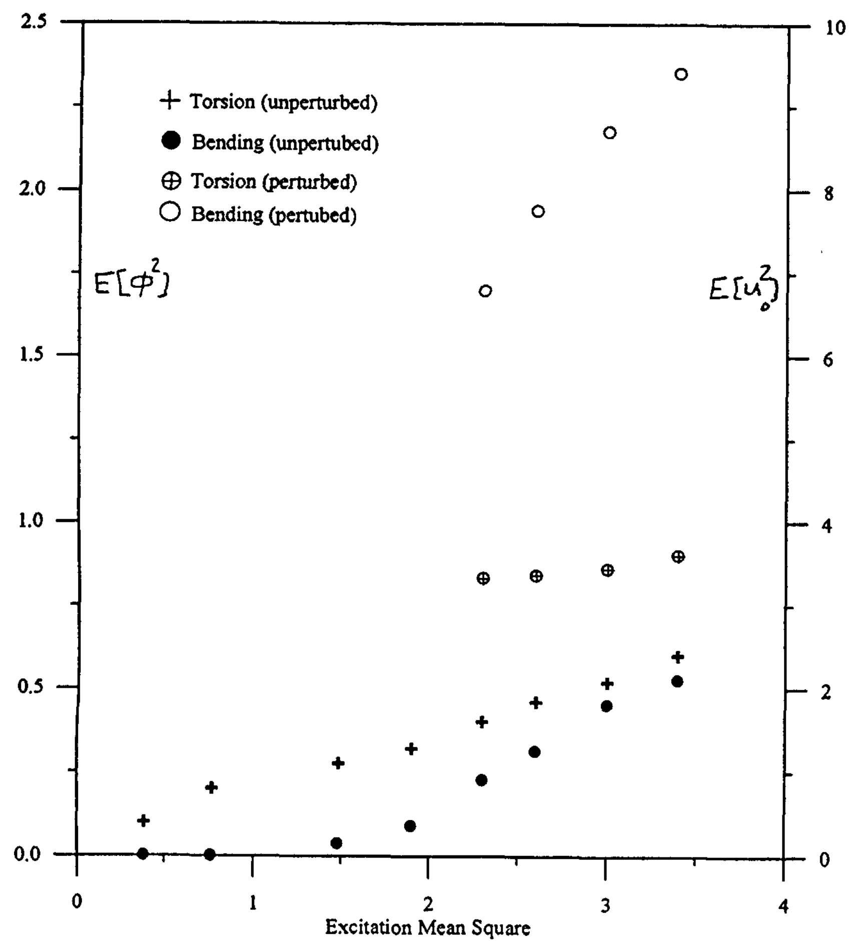

Figure 4 Dependence of response mean square on excitation level showing perturbation effect (excitation bandwidth 2Hz and center frequency 45 Hz)

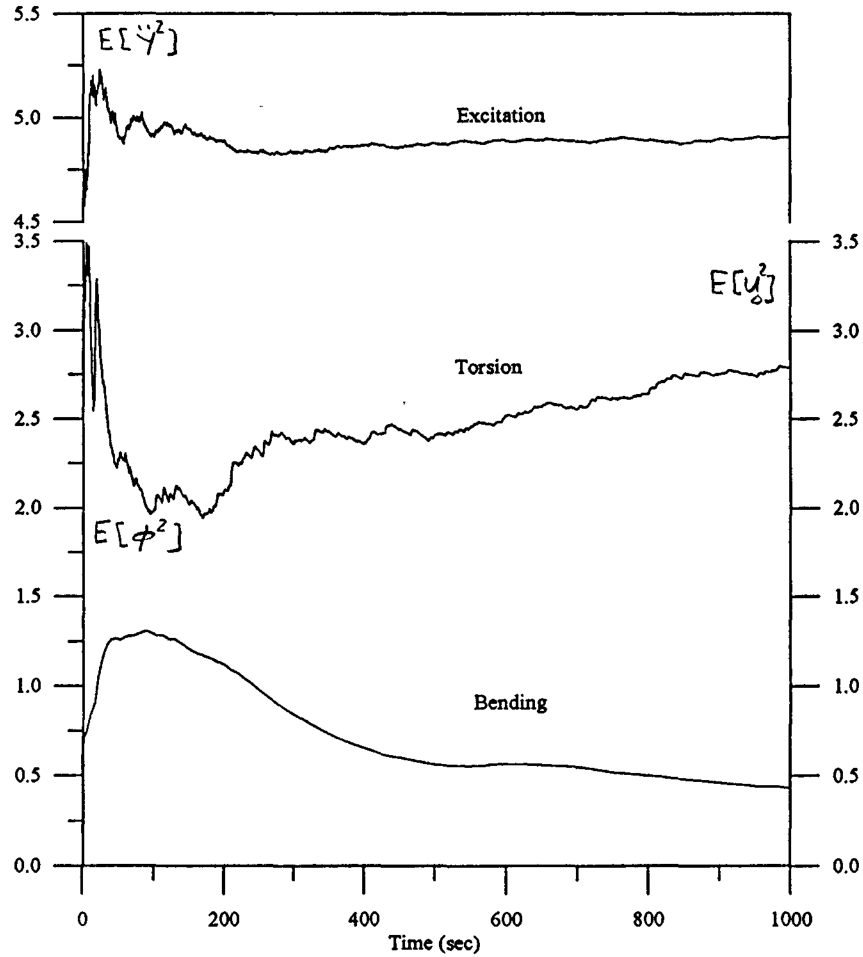

Figure 5a Possible time history records of excxitation and response mean square under high excitation level

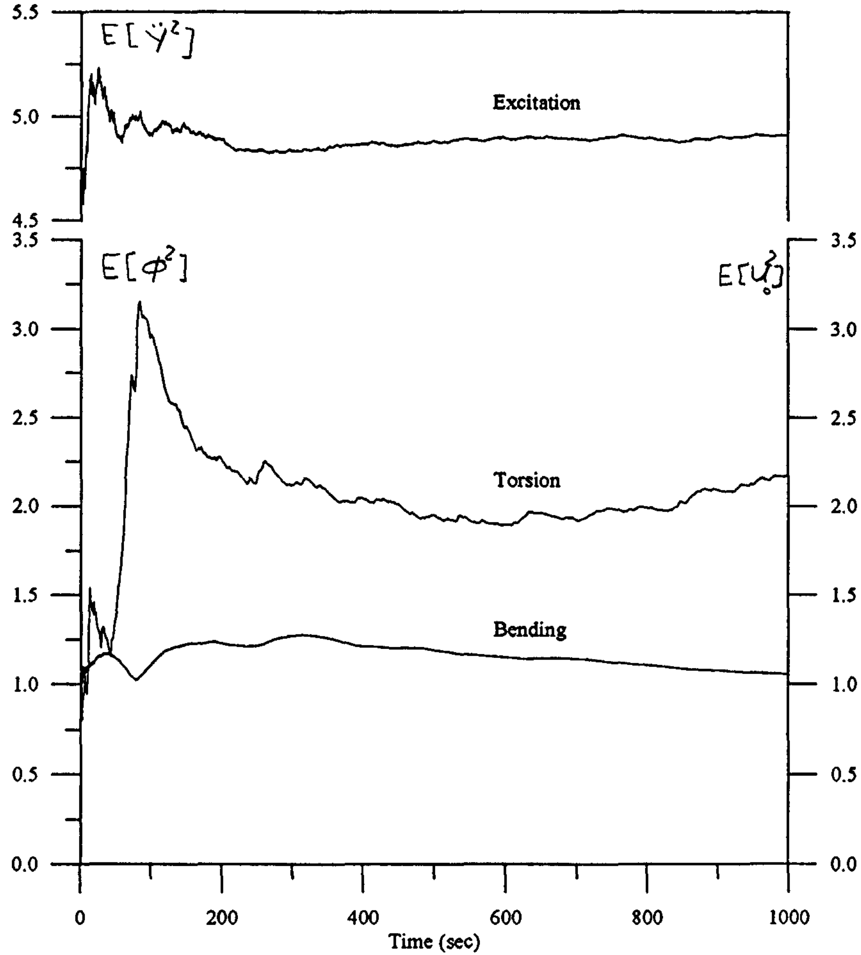

Figure 5b Possible time history records of excitation and response mean squares under same excitation level of Fig. 5a but with perterbation.

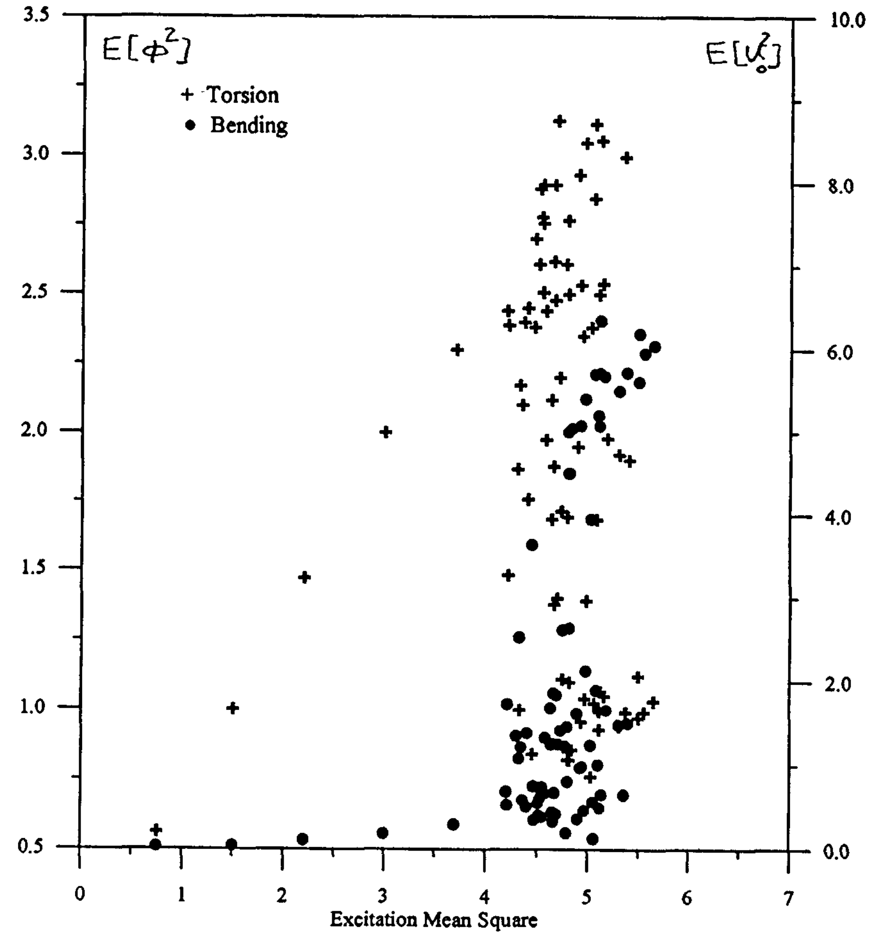

Figure 6 Dependence of response mean square on excitation level (excitation bandwidth 10 Hz and center frequency 45 Hz)