FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

Invited Paper

TIME-DOMAIN COMPUTATION OF ACOUSTICS IN CONFINEMENTS

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ABSTRACT

Long-time stable time-domain numerical solution of wave propagation in a bounded domain even with one spatial dimension has been a formidable challenge despite the recent advances in algorithmic development and numerical analysis. The key issue is how the treatment of end conditions, which inevitably introduces inconsistencies, mismatches, and cumulative error to the inner scheme, would allow waves to rebound indefinitely from the ends and remain intact. The paper delineates the importance of formulation in characteristic variables, as opposed to the analytically equivalent primitive variables, demonstrates the cumulative error, and presents a long-time stable implicit scheme for computation of wave propagation in bounded domains with a distribution of wave sources and various types of forced and unforced boundary conditions, including the recently introduced time-domain impedance condition. Examples of duct and room acoustics, resonance, active noise cancellation and control will be given.

1. INTRODUCTION

The classical approach to solving wave propagation problems is by the reduction of the time-domain hyperbolic wave equation to the frequency-domain elliptic Helmholtz equation. This approach avoids the difficulty of enforcing physically consistent and mathematically admissible boundary conditions. Implicit in this approach is the assumption of space-time continuation and extension to infinity. This, however, excludes the finite-time solution of a wave system being forced at its resonance frequencies. A classical alternative to the frequency-domain approach is the method of characteristics, which handle the finite space-time problem correctly but is well known for its impracticality for problems in three dimensions. A majority of modern approaches employ computational methods such as FEM and BEM for solutions of the Helmholtz equation or its integral form via the Kirchhoff formulation. These basically frequency-domain methods have considerable difficulties in their extensions to finite-time hyperbolic problems.

Current advances in algorithm development for solution of wave systems focus on high-order methods [1,2,3]. These methods are capable of resolution of eight nodes per wavelength [4]. Which is a considerable saving in memory and corresponding processing time compared to the popular second order methods. If, however, the ultimate application of these advances involves distinct solution features and/or complex boundaries, accuracy, algorithm robustness, data structure simplicity, and solution efficiency must all be considered together for a fair assessment of their effectiveness. It is interesting to note that few of these methods have been applied and tested on problems whose solution domain is bounded by an exterior surface, such as room acoustics. As Carpenter et al. [5] pointed out, many high-order schemes closed with mathematically G-K-S [6] stable end schemes are not long-time stable, implying the inapplicability of these high-order algorithms to time-integration of boundary value problems.

Recently, Fung et al. [3] introduced a new class of efficient and accurate numerical schemes for computation of waves. They demonstrated successfully that convection and propagation of waves governed by the Euler equations in multi-dimensions involving solid boundaries can be efficiently and accurately solved using an **implicit** scheme (with unconditional stability) on a suitably finite but not asymptotically large domain without having to apply specific outgoing conditions at the domain boundaries. Their schemes have a simple data structure (two time levels and three spatial points including the implementation of characteristically exact boundary conditions). Their formulation allows the reduction of volumes of connected data to parallel arrays of unconnected data lines, which can be distributed dynamically for parallel processing. Most recently, Fung outlined the important extensions of this approach to include arbitrary geometry [7] and enforcement of impedance boundary condition [8].

This paper reports the implementation of this methodology for time-domain computation of acoustic waves in confinements, satisfying the general impedance boundary condition admissible to the wave equation.

2. FORMULATION

The groundwork for extensions to multi-dimensional wave problems has been laid in [3,7]. In [3] the two-dimensional Euler equations were split for each spatial dimension into systems of one-dimensional waves and solved using the system wave solver. In [7], the multi-dimensional wave equation is split into a system of uncoupled equations in each Cartesian coordinate, and a general boundary scheme was proposed to handle the enforcement of boundary condition on arbitrary surfaces embedded in a Cartesian grid. Solutions in two dimensions on domains bounded by interior surfaces were effected by directing one-dimensional arrays into the simple and system wave solvers developed in [3]. This approach is now extended to domains bounded by an external boundary.

The potential $\phi(x,y,t)$ of small acoustic disturbances satisfies the equation

$$\frac{\partial^2\phi}{\partial t^2}-c^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-f(x,y,t) \tag{1}$$

where x,y are the spatial Cartesian coordinates and f(x,y,t) a given forcing function. The domain and time can be so scaled to make the acoustic speed c=1. Let $u=\phi_x$ $v=\phi_y$ and $p=-\phi_t$ be the dimensionless velocity components and pressure, respectively. Equation (1) can be cast into a set of coupled first order equations in matrix form:

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\v\\p\end{pmatrix}+\begin{pmatrix}0,0,1\\0,0,0\\1,0,0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u\\v\\p\end{pmatrix}+\begin{pmatrix}0,0,0\\0,0,1\\0,1,0\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix}u\\v\\p\end{pmatrix}=\begin{pmatrix}0\\0\\f\end{pmatrix} \tag{2}$$

Here, u and v, by definition, are the spatial gradients of the wave and are known at any time, or given at initialization. Pressure p on the other hand is the temporal change that is about to occur at the rate determined by the sum of the spatial changes in u and v. Therefore, there is no loss of generality in splitting the last row of Eq. (2) into the time rate of change of p with respect to the variation in u and that with respect to v. But the time rate of change of u governed by the first row in Eq. (2) has no direct y-dependence, and that of v has no x-dependence. Hence, Eq. (2) can be split into the equivalent directional systems:

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\p\end{pmatrix}+\begin{pmatrix}0,1\\1,0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u\\p\end{pmatrix}=\begin{pmatrix}0\\f\end{pmatrix};\qquad \frac{\partial}{\partial t}\begin{pmatrix}v\\p\end{pmatrix}+\begin{pmatrix}0,1\\1,0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}v\\p\end{pmatrix}=\begin{pmatrix}0\\f\end{pmatrix} \tag{3}$$

Notice that the effect of the forcing function is wholly felt by the dependent variables in each direction. The systems in Eq. (3) are mathematically identical and can be further transformed into uncoupled systems of the form:

$$\frac{\partial}{\partial t}\begin{pmatrix}u^+\\u^-\end{pmatrix}+\begin{pmatrix}1,\ 0\\0,-1\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}u^+\\u^-\end{pmatrix}=\begin{pmatrix}f\\-f\end{pmatrix}\text{ through the linear relation : }\begin{bmatrix}u^+\\u^-\end{bmatrix}=\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}u\\p\end{bmatrix}. \tag{4}$$

The boundary conditions at the two ends of the computational domain in each coordinate direction depend on the physical model. In the x-direction for example, a given flux at the left end corresponds to $u(a,y,t)=u_0(y,t)$, the reflective condition relates p to the outward normal velocity component u_n through the general impedance condition $p(a,t)=Z(\omega)u_n(a,t)$, and the non-reflective condition corresponds to Z=1. It should be noted that in general impedance is a measured complex-valued function $Z=R+iX$ of the frequency ω. Initially, u, v and p are given as arbitrary functions, which can be set to zero with no loss of generality.

3. SOLUTION METHOD

The solution of these equation systems requires careful implementation of the boundary conditions. The conventional approach of solving Eq. (1) as a second order system on one dependent variable ϕ in general causes numerical instabilities and inaccuracy. Although Eq. (2) is a first-order system, the complexities in a numerical solution of Eq. (2) are comparable to those of Eq. (1). The major problem is the consistency of the end condition for closure of the inner scheme. As pointed out in [3,7] the uncoupled system Eq. (4) implies that the forward-propagating wave, $u^+(x,t)$, is independent of the backward-propagating wave, $u^-(x,t)$, everywhere except at the boundary points x=a and b. Thus, there is no coupling or exchange of error between u^+ and u^- except at the ends, which is not true for Eqs. (1) and (2). This decoupling also separates the stability and accuracy issues of a numerical scheme from those of implementing a boundary condition.

Each of the equations in Eq. (4) satisfies the simple wave equation:

$$\frac{\partial u^\pm}{\partial t}+c\frac{\partial u^\pm}{\partial x}=\pm f, \tag{5}$$

which has the solution in characteristic form $u^\pm(x-ct)$, where c equals +1 or -1 corresponds to the forward and backward propagating wave, respectively. Once initiated, given $u^\pm(x,0)$, the solution $u^\pm(x,t)$ of each wave component requires only the value at the wave entry points, i.e., $u^+(a,t)$ for c=1 and $u^-(b,t)$ for c=-1. The implementation of a boundary condition on a physical quantity, e.g., the velocity component $u_0(t)=(u^++u^-)/2$, involves the entry value of u^+ and the exit value u^-. Given u^-, causality dictates that locally $u^-(a,t)$ must not be affected by the implementation of u_0. Thus, $u^+(a,t)$ is computed as $2u_0(t)-u^-(a,t)$. Each wave component is represented by an array, e.g., $up_i^n=u^+(i\Delta x,n\Delta t)$, of size Ni and updated using the fourth order compact solver Wav1D of [3] for Eq. (5). Given the wave entry value up_1^{n+1}, the solver Wav1D returns the array up_i^{n+1} for all $1<i\le Ni$.

When the waves are coupled, the entry value of one wave is connected to the exit value of the other through the boundary conditions. The difficulty here is that the entry value $u^+(a,t)$ of the forward running wave is not only locally affected by the reflection condition, $u^+=2u_0-u^-$, but also globally by

$u^-(b,t)$ since it is the entry value for $u^-(a,t)$ and in turn by $u^+(a,t)$ itself through the boundary condition at x=b. As pointed out in [3], the correct implementation is to tie both arrays up_i^{n+1} and um_i^{n+1} through the end conditions and solve simultaneously as one cyclic array. Unlike the simple wave solver Wav1D which has been tested to allow escape of all waves and thus indefinite computation, the cyclic system tied at both ends must have perfect phase match, otherwise a slow buildup of phase mismatch will translate into high frequency noise. If an explicit boundary condition, e.g. $u(b,t)=u(b-c\Delta t,t-\Delta t)$, was used to predict the exit wave component, the resulting system is straightly tridiagonal requiring substantially less computing time than the corresponding coupled cyclic tridiagonal system for implicit boundary condition but at the expense of the inevitable appearance of spurious wiggles and instability when the CFL number $\nu=c\Delta t/\Delta x$ exceeds 1. The implicit boundary condition, however, is capable of communicating the growing (with t and ν) phase error throughout the domain without causing mismatches or spurious wiggles at the boundaries [9]. Thus, implicit methods can provide faster convergence towards a steady state provided the boundary condition is also implicit, which has been formidable for other methods known to this author.

For implementation of the impedance condition, Fung and Tallaprgada [8] recently pointed out that a direct implementation of the condition $p=Z(\omega)u$ by using a equivalent time-domain expansion $p=Z(d/idt)u$ is numerically unstable. They proposed instead to use the analytically equivalent and numerically unconditionally stable expression, $u^-(b,t)=Wu^+(b,t)$, where W is the reflection coefficient related to the impedance Z by the bilinear transformation $(1-Z)/(1+Z)$. Again, the uncoupled system offers a direct implementation of this general, time-domain, impedance boundary condition separate from stability consideration of the interior scheme. When Z is real, W ranges from –1 to 1, corresponding to enforcing u=0 and p=0, respectively. In general W can be expanded in ω as $W(\omega)=U+iV=U_0+U_2\omega^2+\dots+i(V_1\omega+V_3\omega^3+\dots+)$, or in time-domain derivatives as:

$$W(\frac{d}{idt}) = U_0 - U_2\frac{d^2}{dt^2} + U_4\frac{d^4}{dt^4} + \dots + V_1\frac{d}{dt} - V_3\frac{d^3}{dt^3} + \dots \tag{6}$$

This formula is applied directly to an array of backward values of $u^+(b,t-k\Delta t)$, $0\le k\le K$, and multiplying them with the difference operator W constructed by, e.g., differentiation of Kth order Lagrange interpolants L_k, i.e., $u^-(a,t)=\Sigma(WL_k)u^+(a,t-k\Delta t)$. The order of the interpolants depends on the number of terms needed to represent the variation of W over the range of frequencies of interest in a particular application. For the simple case of a monochromatic wave of $\omega=\Omega$, only the real and imaginary parts, U and V respectively, evaluated at Ω is needed and Eq. (5) can be simplified to;

$$W(\frac{d}{idt}) = U(\Omega) + V(\Omega)\frac{d}{\Omega dt} \tag{7}$$

More terms would be needed if the variations of U and V were large over the range of frequencies of interest.

4. Numerical Examples

The first case tested was the long time retention of a Gaussian acoustic pulse of $u(x,0)=0$ and $p(x,0) = e^{-\ell n 2(\frac{x-0.5}{0.15})^2}$ in a duct of unit length with the perfectly reflective condition, i.e., W=-1, at both ends. After each time unit the split pulse bounces from the ends and recomposes itself. The numerical process can be cast in the matrix form as $[P]^n=A^n[P]^0$. Each advancement of the solution is the operation of the matrix A on the current solution array, $[P]^{n+1}=A[P]^n$. The perfect scheme will have exactly unity as the largest eigenvalue and preserve the solution for indefinite n, which is not possible for as long as there is numerical dispersion and round off. The dotted line in Fig. 1 is the pulse computed on a 21-point grid at t=10 using a CFL value of $\nu=0.125$, or the equivalence of n=1600 steps. Since the scheme is fourth order accurate and the truncation error is proportional to $\varepsilon=(1/20)^4\sim 6.25\times10^{-6}$, the total error, i.e. $(1+\varepsilon)^n$, is 0.01 after 1600 matrix multiplications. The computed solution (dotted line) shows a maximum error of less than 5% with no appreciable phase error. If the grid is refined to 41 points, the computation (dashed line) can be extended to t=100, or 32,000 steps without much appreciable error compared to the exact solution (solid line). These results demonstrate the expected accuracy and stability needed for acoustic computation. A slight damping, from either the boundary scheme or inner scheme, added to the system will smooth the solution to a constant and wipe out all information in the original signal.

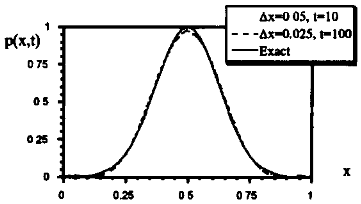

Figure 1, Long-Time Computation of Gaussian Pulse in a Closed Duct

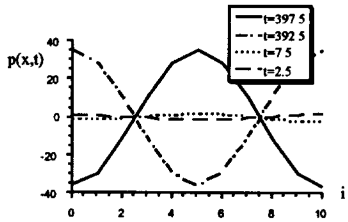

Figure 2, Computed Pressure Distributions at Resonance

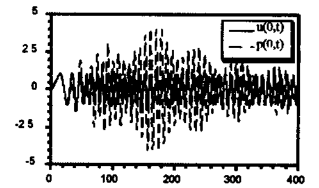

Figure 3, Computed Piston Pressure at Resonance Frequency with Impedance Condition

The next demonstration is the classical acoustics problem of a duct closed at one end and driven harmonically at the other by a piston. The domain chosen is a duct of length 0.12m, corresponding to the second resonance frequency of 2.8333 kHz. A grid of Δx=0.012m was chosen to resolve one wave length, which is slightly more than the eight points needed for a fourth order method. The time step used was ν=0.125. The piston starts moving harmonically at t=0 when both p(x,0) and u(x,0) are zero. Figure 2 shows perfectly phased pressure distributions at the first and last extrema at t=2.5, 7.5, 392.5, and 397.5 computed on a rather coarse grid. It should be noted that to compute beyond t>100, a small damping ε=0.001 [3] is added to offset the growth of error due to truncation. This causes the peak pressure to be slightly lower than if the resonance is perfectly undamped. Figure 3 shows the corresponding pressure for driving at resonance frequency but with the impedance model in [8] replacing the solid end. Compared to the linear growth for the solid end case, the magnitude is substantially modulated and reduced due to the phase shift after each bounce from the open end. Due to the sensitivity of the impedance boundary condition, the initial wave front has been slightly stretched to avoid spurious reflections, which are proportional to the derivatives, and the damping increased to ε=0.002 for smoothness.

The validity of the method for multidimensional problems needs to be firmly established. The concept of splitting a multidimensional problem into directional steps is not new. Techniques such as ADI, fractional steps, operator factorization, etc., are well known for solving elliptic and parabolic problems. Their applications are restricted to steady and quasi-steady problems mainly for their ambiguity or inability in handling time-accurate boundary conditions. The next demonstration is the reflection of an acoustic pulse in a rectangular domain bounded by four solid walls. Unlike in one dimension, the solution once initiated never attains a periodic state, even though all components are harmonic. This requires especially space-time consistency and accuracy. Figure 4 shows pressure contours at t=20, 40, 80, and 480 after the initiation of a Gaussian pressure pulse $p(x,y,0) = e^{-\ell n\, 2 \frac{x^2+y^2}{25}}$ at the center of the 65x65 grid. The solution is well preserved after multiple bounces, confirming the validity of the approach, long-time stability, phase and magnitude accuracy of the solver Wav1D. This result implies that modal information is accurately preserved and can be extracted for accurate definition of mode shapes. Since the boundary condition is of Neumann type, any damping added for whatever reason will eventually lead to the steady state of a constant. Figure 5 shows pressure contours at t=1, 3, 5, and 10 after the initiation of a Gaussian pressure pulse $p(x,y,0) = e^{-\ell n\, 2 \frac{(r-r_0)^2}{0.04}}$ in the lower right quadrant. The complex diffraction by the cylinder and reflections from the walls are well captured on the 101x101 rectangular uniform grid. Figure 6 shows pressure contours at t=0.5 and t=74.42 of a square acoustic cavity forced by the source $f(x,y,t) = e^{-\ell n 2 \frac{x^2+y^2}{0.04}} \sin(\sqrt{2}\,t)$ at the resonance condition of the (1,1) mode computed on a 41x41 grid. The pressure continues to build up linearly while the modal pattern is formed soon after t>1.0. This again implies that accurate modal information can be extracted through a direct time-domain method, instead of through iterative methods in the frequency domain.

5. CONCLUDING REMARKS

Analysis of acoustics has so far relied on the frequency-domain approach. The viability of time-domain approach for boundary value problems has not been established due to the stringent requirements for such computations. The necessary ingredients for long-time computation with realistic boundary conditions are presented and demonstrated. The simplicity, accuracy, and computational efficiency of the schemes, including the implementation of boundary condition, are the enabling features for a viable time-domain approach. The examples here demonstrate one of the problems for which the frequency-domain approach is known to be inapplicable, i.e., forcing a system at a resonance condition.

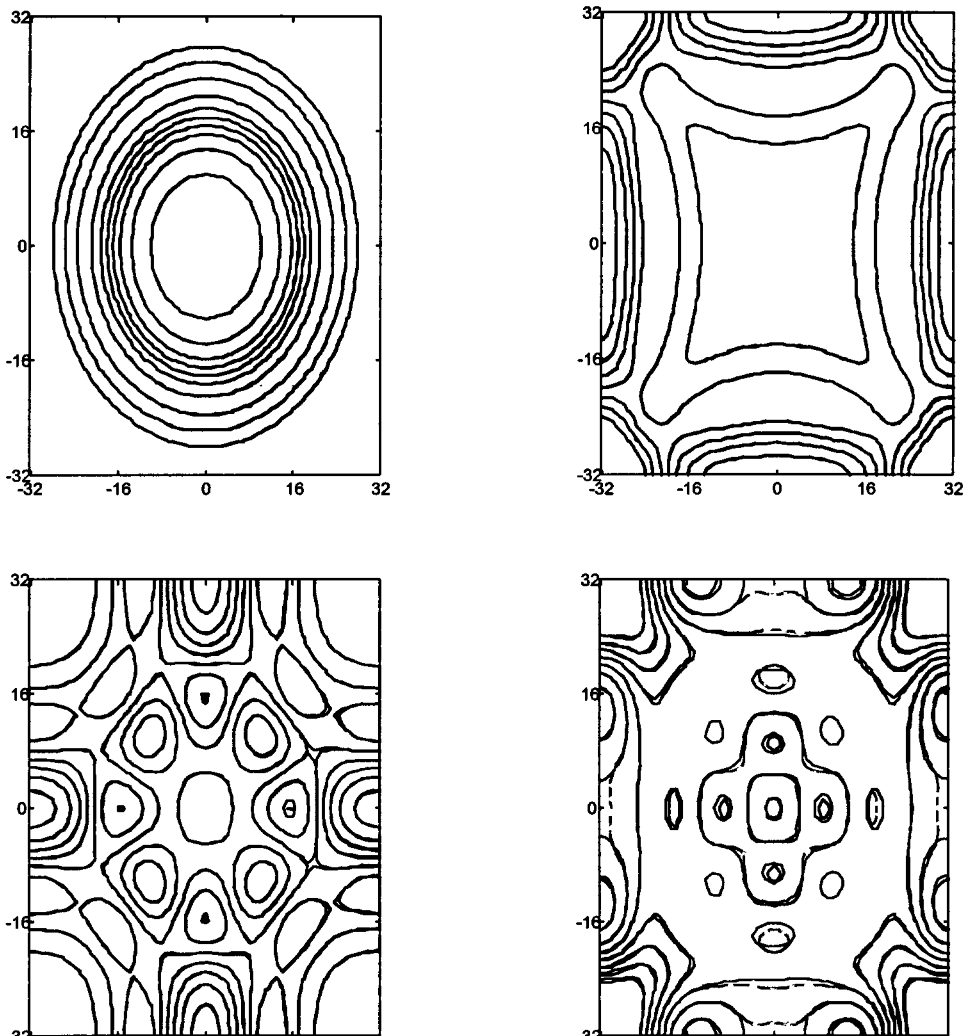

Figure 4. Computed (solid) and Exact (dashed) Pressure Contours (-0.2 + eight increments of 0.05) at t=20 (top left), 40 (top right), 80 (bottom left), and t=480(bottom right).

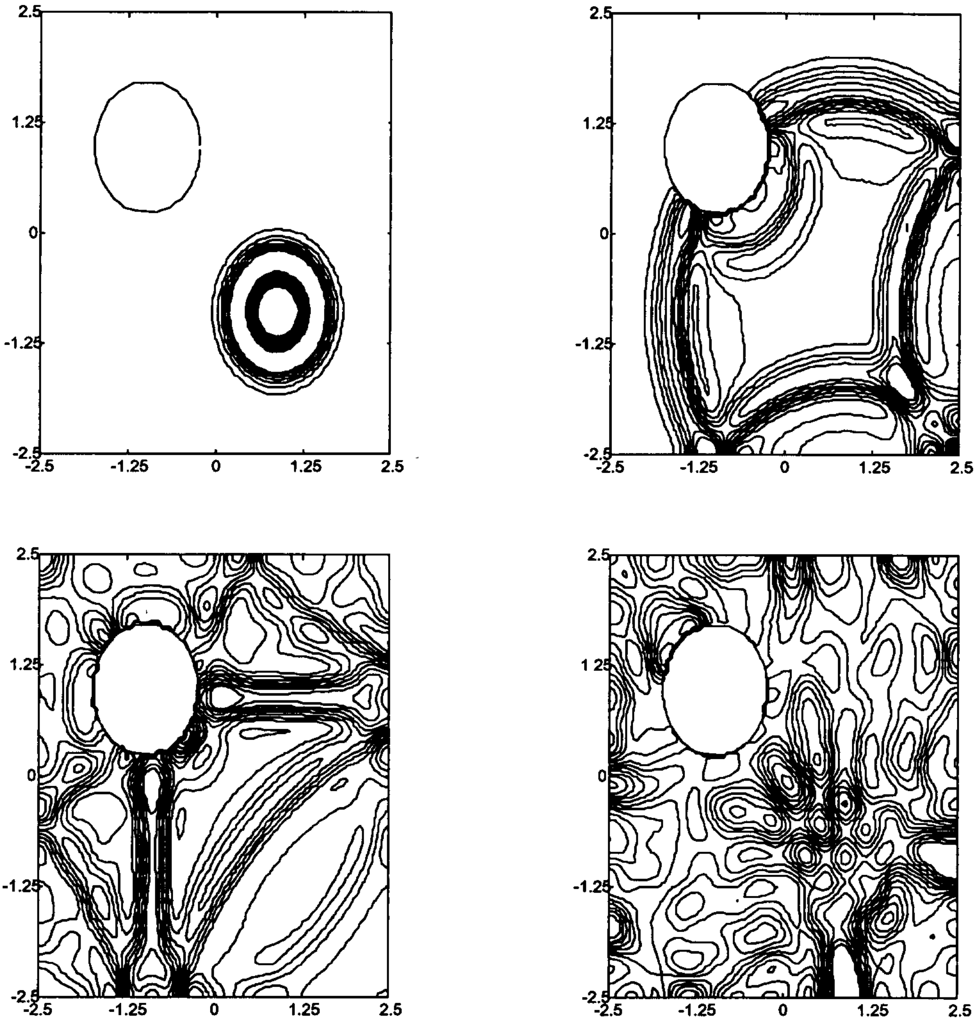

Figure 5. Computed Pressure Contours (-0.2 + eight increments of 0.05) at t=1 (top left), 3 (top right), 5 (bottom left), and t=10 (bottom right).

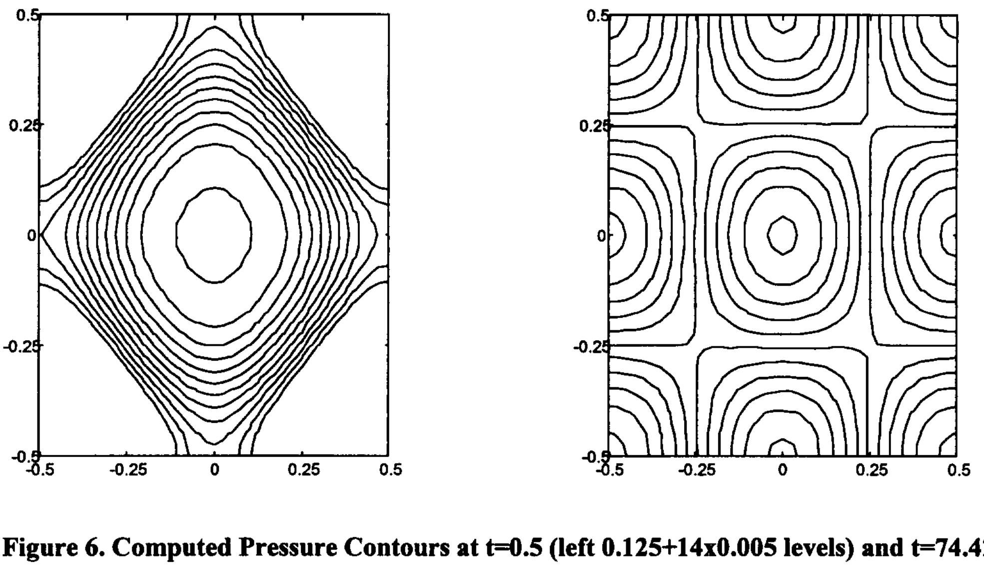

Figure 6. Computed Pressure Contours at t=0.5 (left 0.125+14x0.005 levels) and t=74.42 (right 14.74+10x0.47 levels) with the Solid Wall Condition (W=-1).

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