FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

Invited Paper

DESCRIPTIONS OF TURBULENCE FOR HYDROACOUSTIC APPLICATIONS

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ABSTRACT

This paper describes the use of proper orthogonal decomposition for problems in hydroacoustics where structural response and sound radiation is caused by unsteady flow. For a complete solution the flow in the region of interest must be calculated using numerical solutions to the Navier Stokes equations and this paper will describe how proper orthogonal modes may be used to specify the unsteady flow at the upstream boundary of the computational domain. This will be shown to offer significant savings in computational effort for both linear and non-linear problems providing the correct modes can be determined.

INTRODUCTION

Recent advances in computational methods have resulted in improved techniques for solving problems in hydroacoustics which involve the interaction of incompressible turbulent flow with fluid loaded flexible structures. The primary interest is to calculate the response of the structure to the incoming flow and this requires a knowledge of the surface pressure fluctuations induced by the flow. A typical problem is illustrated in figure 1 which shows a plate like structure downstream of an appendage which protrudes into a non-uniform flow. Accurate CFD calculations can be carried out in the computational domain to obtain the blocked pressure on the plate surface and then structural response models are used to calculate the motion of the structure and, if necessary, the subsequent sound radiation. One of the difficulties with this approach is the correct definition of the inflow to the computational CFD domain. Clearly this must be specified accurately if the correct result is to be obtained but there are fundamental difficulties as to how this flow should be defined. In all problems of practical interest this flow will be turbulent and of finite extent. The fact that the flow is bounded implies that the turbulence must also be inhomogeneous and this complicates the problem because the statistics of the surface pressure fluctuations are not independent of location.

The objective of this paper is to discuss the optimal description of the flow at the upstream boundary of a finite computational domain. One approach is to assume linear equations of motion and specify a harmonic inflow disturbance, solving the problem in the wave number domain[1]. For the turbulent inflow, a wavenumber spectrum of the turbulence can be defined and, for homogeneous flows, this leads to a simple result for the surface pressure[1]. However for inhomogeneous flows, as will shown below, the wavenumber spectra do not simplify and leave complicated expressions for the surface pressure, which require enormous detail about the statistics of the inflow to be specified. An alternative approach is to use proper orthogonal decomposition [2,3] to describe the inflow. This specifies the inflow turbulence as a set of uncorrelated orthogonal modes and formal expressions exist for the optimum modal description which minimizes the number of modes required. Using this approach in a linear problem means that the computations need only be carried out once for each mode specified at the input. The time averaged output is then simply the independent sum of the mean square values computed for each mode. This can result in significant computational advantages for stochastic problems at the same time as providing a rigorous basis for the procedure being used.

This paper will describe the mathematical basis for proper orthogonal decomposition of linear and non-linear problems and then give an example of how a simple flow may be broken down into it's empirical orthogonal modes. The example chosen for this is an unsteady trailing tip vortex, which is typical of the type of flow which might be expected from a large coherent structure in a more complex turbulent flow. It will be shown that the modal summation converges very rapidly and that the modes for this flow can be estimated from a relatively small number of measurements.

PROPER ORTHOGONAL MODES FOR LINEAR PROBLEMS

We will start by considering the relatively simple problem of the pressure fluctuations induced by an unsteady turbulent flow next to a structure. We will assume that the linearized equations of motion may be used, which is a valid assumption when boundary layer pressure fluctuations are caused by the interaction of turbulent flow with mean flow shear[1]. We will also assume that the surface pressure at the location $\mathbf{x}$ may be calculated providing that the flow is specified on the inflow surface to the computational domain shown in figure 1.

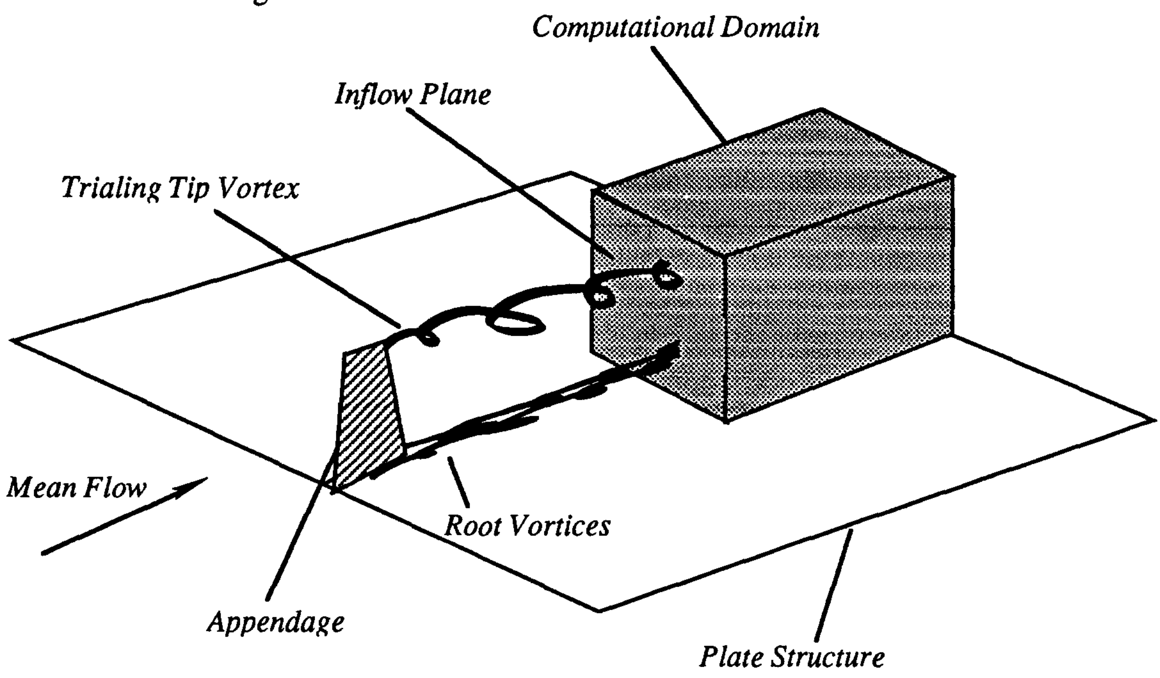

Figure 1: *A typical problem in hydroacoustics: an appendage disturbs the mean flow over a flexible structure and excites both root and trailing tip vortices. The unsteady flows excite the nearby structure.*

Since the problem is assumed to be linear a solution can be obtained by considering an inflow disturbance which is harmonic in space and time, and superimposing the results for each wavenumber and frequency to give the complete solution. Consequently if a vortical inflow disturbance $w_j exp(-i\mathbf{k}.\mathbf{y}-i\omega t)$ is defined at the inflow surface, the surface pressure can be specified in the form

(1)

$$p(\mathbf{x},t) = w_j F_j(\mathbf{x},\mathbf{k},\omega)\exp(-i\omega t)$$

where F_j is the response of the system to the j^{th} component of the incoming gust. For a general inflow disturbance $u_j(\mathbf{y},t)$ we define w_j using a space time Fourier transform so that

(2)

$$w_j(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\int_{-T}^{T}\int_{V} u_j(\mathbf{y},t)e^{i\omega t+i\mathbf{k}.\mathbf{y}}\,dtdV$$

and

$$p(\mathbf{x},t) = \int_{-\infty}^{\infty}\int_{\mathbf{k}} w_j(\mathbf{k},\omega)F_j(\mathbf{x},\mathbf{k},\omega)\exp(-i\omega t)d\mathbf{k}d\omega$$

If the inflow is turbulent we can only define it's mean or average statistics and so we are primarily interested in the cross power spectrum of the pressure fluctuations. This is obtained from the Fourier transform of the pressure fluctuations at **x** and **x'** in the form

(3)

$$C_{pp}(\mathbf{x},\mathbf{x}',\omega) = \frac{\pi}{T}Ex\left[p(\mathbf{x},\omega)p^*(\mathbf{x}',\omega)\right]$$

$$= \int_{\mathbf{k}'}\int_{\mathbf{k}} \frac{\pi}{T}Ex\left[w_i(\mathbf{k},\omega)w_j(\mathbf{k}',\omega)\right]F_i(\mathbf{x},\mathbf{k},\omega)F_j^*(\mathbf{x}',\mathbf{k}',\omega)d\mathbf{k}d\mathbf{k}'$$

This basic result shows that the surface pressure spectrum is defined in terms of the wavenumber transforms of the inflow disturbance. This can be related to the more easily interpreted cross correlation of the velocity fluctuations by

(4)

$$\frac{\pi}{T}Ex\left[w_i(\mathbf{k},\omega)w_j^*(\mathbf{k}',\omega)\right] = \frac{1}{(2\pi)^7}\int_{-T}^{T}\int_{V}\int_{V} R_{ij}(\mathbf{y},\mathbf{y}',\tau)e^{i\omega\tau+i\mathbf{k}.\mathbf{y}-i\mathbf{k}'.\mathbf{y}'}\,d\tau dV dV'$$

where

$$R_{ij}(\mathbf{y},\mathbf{y}',\tau) = \frac{1}{2T}\int_{-T}^{T} u_i(\mathbf{y},t)u_j(\mathbf{y}',t-\tau)dt = Ex\left[u_i(\mathbf{y},t)u_j(\mathbf{y}',t-\tau)\right]$$

This shows that a double volume integral and a Fourier transform need to be evaluated to correctly specify the random inflow to the computational domain. Furthermore the interpretation of equation (4) is far from obvious and it is not clear how the features of the flow are coupled to the acoustic pressure fluctuations. Inevitably, simplifying assumptions have been used to help with the interpretation of this problem and the most useful of these is to assume a homogeneous turbulent flow so that the cross correlation is a function of **y** - **y'** only. Then equation (4) yields

$$\frac{\pi}{T} Ex\left[w_i(\mathbf{k},\omega)w_j^*(\mathbf{k}',\omega)\right] = \frac{1}{(2\pi)^4} \int_{-T}^{T} \int_V R_{ij}(\mathbf{y}-\mathbf{y}',\tau)e^{i\omega\tau + i\mathbf{k}.(\mathbf{y}-\mathbf{y}')} d\tau dV \delta(\mathbf{k}-\mathbf{k}')$$

$$= \Phi_{ij}(\mathbf{k},\omega)\delta(\mathbf{k}-\mathbf{k}')$$

so that

$$C_{pp}(\mathbf{x},\mathbf{x}',\omega) = \int_{\mathbf{k}} \Phi_{ij}(\mathbf{k},\omega)F_i(\mathbf{x},\mathbf{k},\omega)F_j^*(\mathbf{x}',\mathbf{k},\omega)d\mathbf{k} \tag{5}$$

The surface pressure is then defined in terms of the wavenumber energy spectrum $\Phi_{ij}(\mathbf{k},\omega)$ of the turbulent flow. Various models are available for this function which allow it to be specified in terms of a turbulence intensity and integral length scale (or for anisotropic turbulence, lengthscales in each orthogonal direction). Flow measurements therefore need only be directed towards evaluating these parameters which simplifies the measurement problem. The wavenumber spectrum model also allows the integral in equation (5) to be carried out analytically in some cases and so a closed form solution can be obtained. While this leads to a relatively attractive result, it imposes a very severe condition on the description of the inflow, namely that the flow is homogeneous. Unfortunately this assumption is unrealistic for almost all flows of interest, and, unless conditions such as "local homogeneity" can be applied, the full wavenumber integral defined in equation (4) must be used to evaluate equation (3).

An alternative approach is to use a modal expansion of the inflow which was originally proposed by Lumley[2]. The concept is to expand the unsteady velocity at the inflow plane as a set of uncorrelated orthogonal modes in the form

$$u_i(\mathbf{y},t) = \sum_n a_n(t)\phi_i^{(n)}(\mathbf{y}) \tag{6}$$

The requirement that the modes are uncorrelated is imposed by specifying

$$\mathrm{Ex}\left[a_n(t)a_m(t-\tau)\right] = \begin{cases} \lambda_n(\tau) & m = n \\ 0 & m \neq n \end{cases} \tag{7}$$

and so

$$R_{ij}(\mathbf{y},\mathbf{y}',\tau) = \sum_n \lambda_n(\tau)\phi_i^{(n)}(\mathbf{y})\phi_j^{(n)}(\mathbf{y}') \tag{8}$$

By taking Fourier transforms over time and space we find

$$\frac{\pi}{T} Ex\left[w_i(\mathbf{k},\omega)w_j^*(\mathbf{k}',\omega)\right] = \sum_n \lambda_n(\omega)\phi_i^{(n)}(\mathbf{k})\phi_j^{(n)}(\mathbf{k}') \tag{9}$$

and so

$$C_{pp}(\mathbf{x},\mathbf{x}',\omega) = \sum_n \lambda_n(\omega)\left[\int_{\mathbf{k}} \phi_i^{(n)}(\mathbf{k})F_i(\mathbf{x},\mathbf{k},\omega)d\mathbf{k}\right]\left[\int_{\mathbf{k}} \phi_j^{(n)}(\mathbf{k})F_j^*(\mathbf{x},\mathbf{k},\omega)d\mathbf{k}\right]^* \tag{10}$$

The advantage of this approach is that the computations of the surface pressure can be carried out for each mode individually and the acoustic power spectrum will be the independent sum of the mean square output from each modal calculation. Using the modal expansion (5) the flow has not been restricted in any sense and we can allow for inhomogeneous turbulence without difficulty. Furthermore we can identify dominant modes and their coupling efficiency, and this may lead to a better understanding of the features of the inflow which affect the structural response.

The computation time for the evaluation of either equation (3) or equation (10) will typically be dominated by the calculation of the response functions $F_i(\mathbf{x},\mathbf{k},\omega)$. Consequently the formulation given using orthogonal modes does not necessarily offer major computational advantages if the calculation of the response function is carried out in the wavenumber domain. However if the computations are carried out for each modal velocity vector $\phi_i^{(n)}(\mathbf{y})$ defined on the inflow surface and the downstream response function to this mode is defined as $F^{(n)}(\mathbf{x},\omega)$ then, since the contribution from each mode is uncorelated, the surface pressure spectrum is simply the linear sum of the modal response functions in the form

$$C_{pp}(\mathbf{x},\mathbf{x}',\omega) = \sum_n \lambda_n(\omega) F^{(n)}(\mathbf{x},\omega)\left(F^{(n)}(\mathbf{x}',\omega)\right)^* \tag{11}$$

This clearly provides a major reduction in the computational effort required to solve this problem and since the modal description is optimal the number of terms required to be evaluated in (11) will be the minimum possible, providing the most efficient computational approach.

However we still need to define the optimal set of modes and these can be obtained from the theory of proper orthogonal decomposition[2,3]. This theory shows that a set of orthogonal modes which maximize the averaged projection of u_i onto the modes $\phi_i^{(n)}$ is obtained from the solutions to

$$\int_V R_{ij}(\mathbf{x},\mathbf{x}',\tau)\phi_i^{(n)}(\mathbf{x}')dV(\mathbf{x}') = \lambda_n(\tau)\phi_j^{(n)}(\mathbf{x}) \tag{12}$$

Therefore to evaluate the modes required in (11) we must, in principle, specify the cross correlation function everywhere in space as required by equation (4). It would appear therefore that the modal decomposition approach has not provided any reduction in detail required for the description (or measurement) of the flow. However the modes provide a more rigorous basis on which to interpret the flow. Furthermore the number of modes required to describe the flow [4] can be significantly less than the number of wavenumbers required for the same flow and so computational savings in evaluating (11) rather than (3) may be significant.

NON- LINEAR PROBLEMS

In the previous section we considered linear problems where the flow could be described using the linearized equations of motion. However the Navier Stokes equations are fully non-linear and the approach described above is not adequate for problems where non-linear effects can be important. In these cases Fourier analysis or the linear sum of uncorrelated modes can not be used. Furthermore the inflow boundary conditions must be specified in the time domain. The optimum descriptor of the inflow for non-linear problems is therefore quite different from those required for linear problems. One approach is to extend the upstream boundary sufficiently far upstream that the flow may be considered as either quiescent or uniform and completely steady. Direct numerical simulation can be used to describe the flow and the turbulence is allowed to evolve naturally until a statistically stationary mean and unsteady flow is established. However this approach is computationally intensive and here we propose an alternative which uses proper orthogonal modes to describe the inflow on a surface which is in the unsteady flow region. Clearly the choice of this surface will depend on the particular problem and the information available but the concept is to use a description such as equation (6) to define the inflow on that surface. This would appear to be the only or at least optimum flow description because the modes are chosen to be optimal by definition. To define the time history of the inflow, the coefficients $a_n(t)$ are specified using pseudo random sequences which have the same higher order statistics (i.e. mean, standard deviation, skewness and kurtosis etc.) as the measured or estimated inflow. When a steady state situation has been established, the statistics of the parameters of interest will then depend on $\text{Ex}[u_i(t)u_j(t+\tau)]$, $\text{Ex}[u_i(t)u_j(t+\tau)u_k(t+\tau')]$ and $\text{Ex}[u_i(t)u_j(t+\tau)u_k(t+\tau')u_m(t+\tau'')]$ at the inflow boundary. Since each mode in the sequence given in (6) is uncorrelated the results will depend on $\text{Ex}[a_n(t)a_n(t+\tau)]$, $\text{Ex}[a_n(t)a_n(t+\tau)a_n(t+\tau')]$ and $\text{Ex}[a_n(t)a_n(t+\tau)a_n(t+\tau')a_n(t+\tau'')]$. These parameters may be hard to measure or model but in principle they can be established, especially if the inflow can be defined with a limited number of dominant modes. This approach is very attractive because it provides a non-arbitrary method to define the inflow and may lead to significant reductions in the computational volume required to define the problem without any loss of accuracy and hence a major reduction in computational effort.

PROPER ORTHOGONAL DECOMPOSITION OF A TRAILING TIP VORTEX

There are very few examples in the literature where the proper orthogonal decomposition of a turbulent flow has been achieved[3]. The difficulty is that equation (12), which must be inverted to obtain the proper orthogonal modes, requires the cross correlation function to be defined for all reference points **x** and all displaced points **x'**. This type of detail is hard to obtain from measurements and the only successful evaluations of proper orthogonal modes have used direct numerical simulations to define the flow[4]. However Devenport et al[5] measured the details of the unsteady flow around a blade tip vortex in a free stream. It was found that the apparent low frequency turbulence in this flow was caused by the unsteady motion of the vortex core relative to a fixed probe. It was also shown that the statistics of the low frequency turbulence could be defined using the known mean flow and the probability density function of the vortex location relative to the probe. Assuming Gaussian statistics the mean and standard deviation of the vortex core location were then evaluated from measurements. The probability model described by Devenport et al [5] enables all the second order statistics of the unsteady flow to be completely defined. Consequently the cross correlation function required for the evaluation of (12) can be specified exactly and this enables the proper orthogonal decomposition of this flow to be carried out.

As an illustrative example we will show the proper orthogonal modes of the axial velocity components of a *q* vortex whose core location is a random function. The standard deviation of the core displacement is assumed to be small compared with the core size and the mean axial velocity is defined as

$$U_{Dm} = \frac{U_D \exp(-\alpha R^2 / Cd^2)}{1 + 2\alpha\sigma^2 / Cd^2}$$

where R is the distance from the core of the vortex, d is the radial scale of the axial profile and $\alpha=1.25643$. Figure 2 shows the mean square turbulence intensity in the axial direction obtained from this model and figure 3 the first four proper orthogonal modes. Finally in figure 5 the eigenvalues λ_n are shown for each mode, and it is seen that rapid convergence occurs and only a few modes are required to describe the flow.

CONCLUSIONS

In this paper we have outlined how proper orthogonal decomposition can be used to formally define the inflow boundary conditions to a computational domain when the inflow includes a turbulent or unsteady component. The proper orthogonal modes are the optimum description of the unsteady flow in as much that they minimize the number of modes required to define the problem. Using the approach discussed above they are also the optimum computational approach for linear problems and provide a rational basis for the time histories at the inflow for non-linear problems. The difficulty with defining the modes is obtaining sufficient information about the flow to carry out the proper orthogonal decomposition. A model problem has been considered in which the unsteady flow was caused by the wandering of a trailing tip vortex. The results show that there are a very limited number of proper orthogonal modes required to define the flow, and this implies that the measurement detail required to define the flow using modes, as in (8), is much less than the measurement detail required to define the cross wavenumber spectra defined in equation (4).

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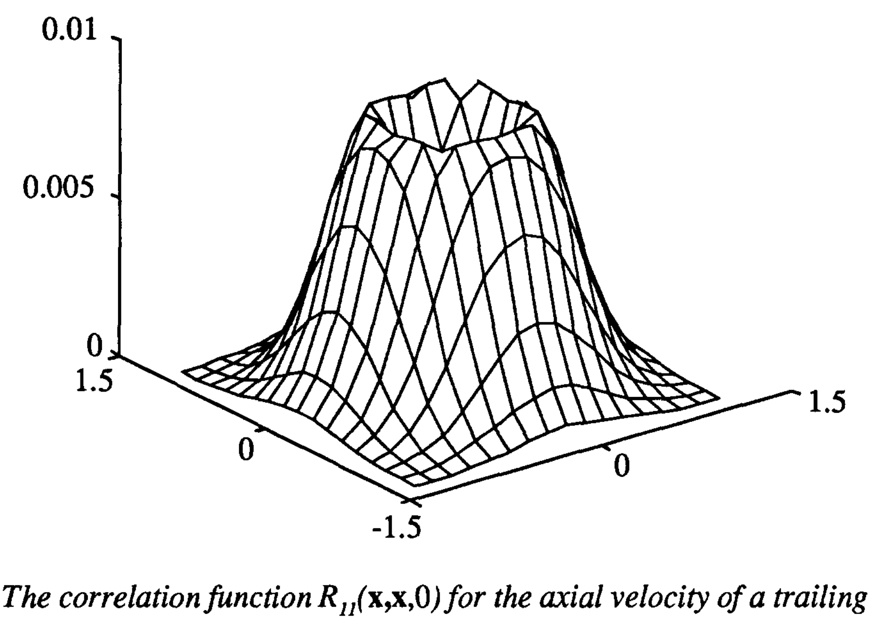

Figure 2: The correlation function $R_{11}(\mathbf{x},\mathbf{x},0)$ for the axial velocity of a trailing tip vortex.

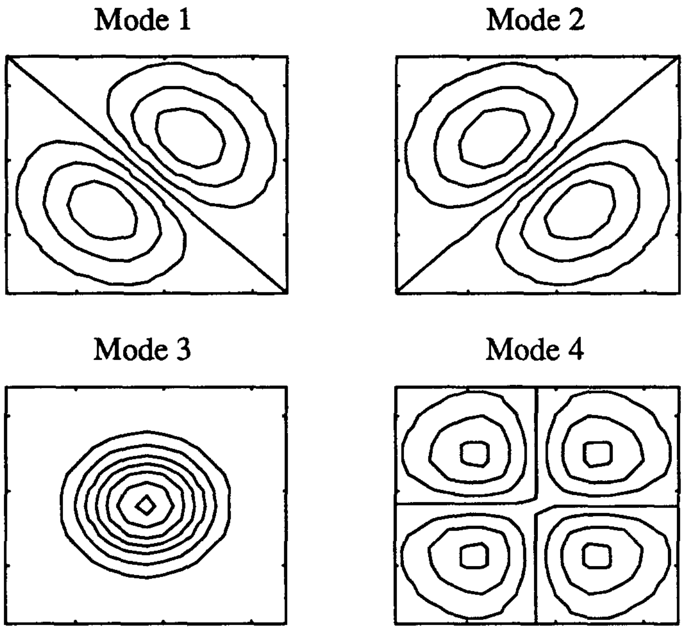

Figure 3: The first four proper orthogonal modes of a trailing tip vortex, $-1.5d<y<1.5d$, $-1.5d<z<1.5d$

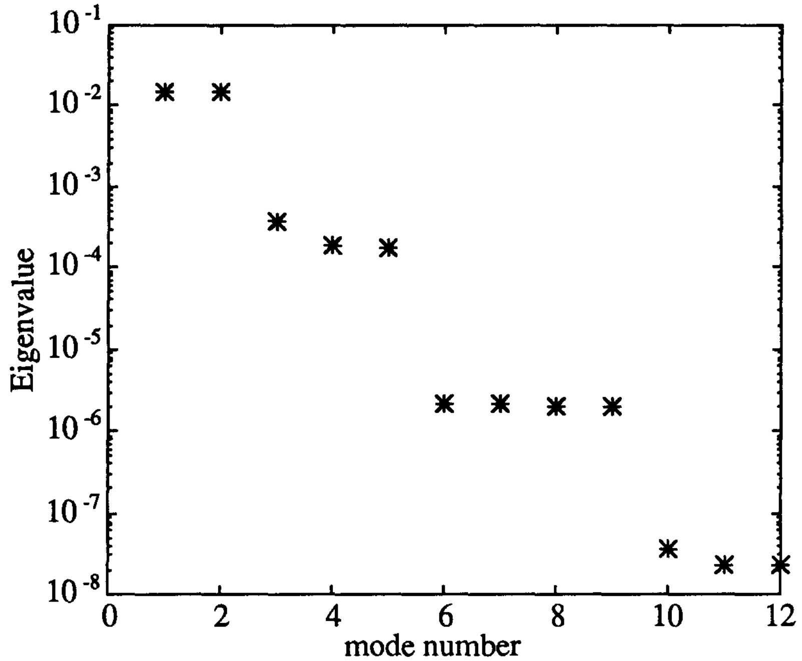

Figure 4: The eigenvalues of the proper orthogonal modes of a trailing tip vortex.