FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION

DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

WAVE PROPAGATION IN LAYERED TRANSVERSELY ISOTROPIC FLUID-SATURATED POROUS MEDIA

Wang Yue-Sheng, Zhang Zi-Mao and Yu Gui-Lan

School of Civil Engineering, Northern Jiaotong University, Beijing 100044, China

Abstract Based on the Biot's theory, the propagation of elastic waves through transversely isotropic fluid-saturated porous media is examined. The dissipation due to fluid viscosity is considered. We first derived the characteristic equation of plane waves in an arbitrary layer, of which the closed-form solution is presented. This enable us to write the general form of waves in a layer. The results show that there are three kinds of waves propagating in a layer: the fast and slow quasi-longitudinal waves (QP1 and QP2) and the quasi-transverse wave (QSV). Then, by consideration of the continuity conditions at the interface, the transfer matrices between layers are derived. The expressions of amplitudes, phase velocities, attenuation coefficients and directions of the reflected and transmitted waves are presented.

1. INTRODUCTION

The propagation of elastic waves in layered media is of fundamental interest in many fields such as earthquake engineering, geophysics, civil engineering, etc. It is known that the geologic materials generally contains pores saturated with fluid. Biot[1,2] developed a wave theory on isotropic fluid-saturated porous media. Based on this theory, many researchers have studied the wave propagation through layered isotropic porous media[3–5]. In this paper, we consider the transversely isotropic case.

The basic equations for a general anisotropic porous medium were presented by Biot in 1962[6]. But the associated wave problems were not considered unitl late 1980's. See, for instance, Refs.[7–9] which discussed the waves in a transversely isotropic porous medium. The results show that four kinds of waves propagate in such a medium: the fast and slow quasi-longitudinal waves (QP1 and QP2); the quasi-transverse waves (QSV) and the anti-plane transverse wave (SH).

2. BASIC EQUATIONS

For a transversely isotropic medium with the principal axis as z-axis, the constitutive relation follows from Ref.[6] as

$$\begin{cases} \tau_{xx} = (2B_1 + B_2)e_{xx} + B_2 e_{yy} + B_3 e_{zz} + B_6 \varsigma \\ \tau_{yy} = B_2 e_{xx} + (2B_1 + B_2)e_{yy} + B_3 e_{zz} + B_6 \varsigma \\ \tau_{zz} = B_3 e_{xx} + B_3 e_{yy} + B_4 e_{zz} + B_7 \varsigma \\ \tau_{yz} = 2B_5 e_{yz}, \ \tau_{zx} = 2B_5 e_{zx}, \ \tau_{xy} = 2B_1 e_{xy} \\ p = B_6 e_{xx} + B_6 e_{yy} + B_7 e_{zz} + B_8 \varsigma \end{cases} \tag{1}$$

where p is the pore fluid pressure and ς is the increment of fluid content per unit volume. The eight material coefficients $B_1 \sim B_8$ can be calculated from the elastic coefficients of skeleton, c_{ij}, and the bulk moduli of constitutive grains and saturant fluid, K_s and K_f[10]. In terms of the displacement components of bulk material and saturant fluid, u_i and U_i (with $i = 1, 2, 3$ corresponding to x, y, z), e_{ij} and ς may be expressed as

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \quad \varsigma = -w_{i,i} \tag{2}$$

where $w_i = \phi(U_i - u_i)$ and ϕ is the porosity of the medium. For the motion in xz plane, the governing equations may be written as

$$\begin{cases} (2B_1 + B_2)\dfrac{\partial^2 u_x}{\partial x^2} + B_5 \dfrac{\partial^2 u_x}{\partial z^2} + (B_3 + B_5)\dfrac{\partial^2 u_z}{\partial x \partial z} - B_6 \dfrac{\partial^2 w_x}{\partial x^2} - B_6 \dfrac{\partial^2 w_z}{\partial x \partial z} = \rho \ddot{u}_x + \rho_f \ddot{w}_x \\ (B_3 + B_5)\dfrac{\partial^2 u_x}{\partial x \partial z} + B_5 \dfrac{\partial^2 u_z}{\partial x^2} + B_4 \dfrac{\partial^2 u_z}{\partial z^2} - B_7 \dfrac{\partial^2 w_x}{\partial x \partial z} - B_7 \dfrac{\partial^2 w_z}{\partial z^2} = \rho \ddot{u}_z + \rho_f \ddot{w}_z \\ B_6 \dfrac{\partial^2 u_x}{\partial x^2} + B_7 \dfrac{\partial^2 u_z}{\partial x \partial z} - B_8 \dfrac{\partial^2 w_x}{\partial x^2} - B_8 \dfrac{\partial^2 w_z}{\partial x \partial z} = -\rho_f \ddot{u}_x - m_1 \ddot{w}_x - r_1 \dot{w}_x \\ B_6 \dfrac{\partial^2 u_x}{\partial x \partial z} + B_7 \dfrac{\partial^2 u_z}{\partial z^2} - B_8 \dfrac{\partial^2 w_x}{\partial x \partial z} - B_8 \dfrac{\partial^2 w_z}{\partial z^2} = -\rho_f \ddot{u}_z - m_3 \ddot{w}_z - r_3 \dot{w}_z \end{cases} \tag{3}$$

where $\rho = (1 - \phi)\rho_s + \phi\rho_f$ with ρ_s and ρ_f being, respectively, the mass densities of skeleton and fluid. m_j and $r_j (j = 1, 3)$ are the coefficients introduced by Biot and can be written as

$$m_j = \mathrm{Re}[\alpha_j(\omega)]\rho_f/\phi, \quad r_j = \eta/\mathrm{Re}[K_j(\omega)], \qquad j = 1, 3 \tag{4}$$

where ω is the angular frequency; η is the viscosity of the fluid; and $\alpha_j(\omega)$ and $K_j(\omega)$ are, respectively, the dynamic tortuosity and permeability with relation: $\alpha_j(\omega) = \mathrm{i}\eta\phi/[K_j(\omega)\omega\rho_f]$. Johnson *et al.*[11] presented an asymptotic expression for dynamic permeability in the isotropic case. Here we extend their results to the transversely isotropic case

$$K_j(\omega) = K_j(0)\left\{\left[1 - \frac{4\mathrm{i}\alpha_j^2(\infty)K_j^2(0)\omega\rho_f}{\eta a_j^2 \phi^2}\right]^{1/2} - \frac{\mathrm{i}\alpha_j(\infty)K_j(0)\omega\rho_f}{\eta\phi}\right\}^{-1} \tag{5}$$

where a_j is the characteristic length of the pores.

3. WAVE FIELDS IN A SINGLE LAYER

The problem considered in this paper is sketched in Fig.1. Two half-space are bonded through N layers. The materials are transversely isotropic fluid-saturated porous media with principal axes perpendicular to the interfaces. An elastic wave (QP1, QP2 or QSV) propagates from the lower half-space to the upper one through N layers. We first examine the wave fields in an arbitrary layer. Suppose that the waves in the nth layer are of the form

$$\{u_x, u_z, w_x, w_z\}_n = \{a_1, a_2, a_3, a_4\}_{(n)} \exp[\mathrm{i}(k_{(n)} + l_{(n)}z)], \quad n = 0 \sim N+1 \tag{6}$$

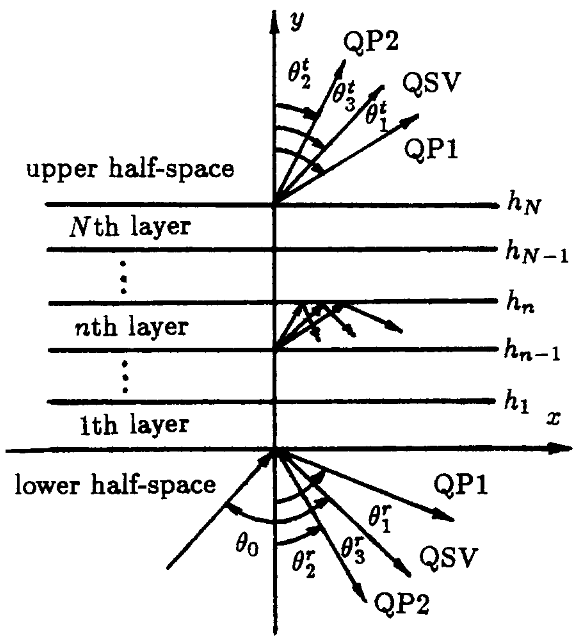

Figure 1: Wave propagation in layered porous media

where the subscript (n) will be suppressed in case of no confusion in the following analysis. it follows from the Snell's law that the complex apparent wavenumbers $k_{(n)}(n = 0 \sim N+1)$ are equal to that of the incident wave, denoted by k. Substituting (6) into (3), one may have

$$[d_{ij}]\{a_1, a_2, a_3, a_4\}^T = 0 \tag{7}$$

where $[d_{ij}]$ is a 4×4 square matrix with elements given by

$$\begin{cases} d_{11} = \omega^2\rho - [(2B_1 + B_2)k^2 + B_5 l^2], \ d_{12} = d_{21} = -(B_3 + B_5)kl \\ d_{13} = d_{31} = \omega^2\rho_f + B_6 k^2, \ d_{14} = d_{41} = B_6 kl \\ d_{22} = \omega^2\rho - (B_5 k^2 + B_4 l^2), \ d_{23} = d_{32} = B_7 kl \\ d_{24} = d_{42} = \omega^2\rho_f + B_7 l^2, \ d_{33} = \omega^2(m_1 + \mathrm{i}r_1/\omega) - B_8 k^2 \\ d_{34} = d_{43} = -B_8 kl, \ d_{44} = \omega^2(m_3 + \mathrm{i}r_3/\omega) - B_8 l^2 \end{cases} \tag{8}$$

The condition that (7) has non-trivial solution is

$$|d_{ij}| = 0 \tag{9}$$

from which we get an equation of 6th-order as follows

$$al^6 + b\omega^2 l^4 + c\omega^4 l^2 + d\omega^6 = 0 \tag{10}$$

where a, b, c and d are listed in Appendix. Set

$$\xi = \frac{l^2}{\omega^2} + \frac{b}{3a}, \; p = \frac{c}{a} - \frac{b^2}{3a^2}, \; q = \frac{d}{a} + \frac{2b^3}{27a^3} - \frac{cb}{3a^2} \tag{11}$$

Equation (10) then reduces to

$$\xi^2 + p\xi + q = 0 \tag{12}$$

of which the solution may be obtained by Cardon method[12] as

$$\xi_1 = R_1 + R_2, \; \xi_2 = \kappa R_1 + \kappa^2 R_2, \; \xi_3 = \kappa^2 R_1 + \kappa R_2 \tag{13}$$

with

$$R_j = \left[-\frac{q}{2} + (-1)^j\sqrt{\Delta}\right]^{1/3}, \; \Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, \; \kappa = \frac{-1 + \mathrm{i}\sqrt{3}}{2}$$

It follows from (11)

$$l_j = \omega[\xi_j - b/(3a)]^{1/2}, \quad j = 1, 2, 3 \tag{14}$$

which is generally complex. It is noted that $\mathrm{Re} l_j \leq 0$ for the reflected waves and $\mathrm{Re} l_j \geq 0$ for the refracted ones. The three roots of the equation correspond, respectively, to the three types of waves: QP1, QP2 and QSV. From (7) we arrive at

$$a_i^{(j)} = \Delta_i^{(j)} X_j, \quad j = 1, 2, 3; \quad i = 1, 2, 3, 4 \tag{15}$$

where $X_j = \left[\sum_{i=1}^{4} \left(\Delta_i^{(j)}\right)^2\right]^{1/2}$ with $\Delta_i^{(j)}$ given in Appendix. With the superscript r referring to the reflected waves and t to the refracted ones, the wave fields in the nth layer may be written as

$$\begin{aligned} \{u_x, u_z, w_x, w_z\}_n = &\sum_{j=1}^{3} \{a_{1r}^{(j)}, a_{2r}^{(j)}, a_{3r}^{(j)}, a_{4r}^{(j)}\} A_j^r \exp[\mathrm{i}(kx + l_j^r z)] \\ &+ \sum_{j=1}^{3} \{a_{1t}^{(j)}, a_{2t}^{(j)}, a_{3t}^{(j)}, a_{4t}^{(j)}\} A_j^t \exp[\mathrm{i}(kx + l_j^t z)] \end{aligned} \tag{16}$$

The associated stress components of the skeleton and pore fluid pressure are given by

$$\begin{aligned} \{\tau_{xz}, \tau_{zz}, p_f\}_n = &\sum_{j=1}^{3} \{b_{1r}^{(j)}, b_{2r}^{(j)}, b_{3r}^{(j)}\} \mathrm{i} A_j^r \exp[\mathrm{i}(kx + l_j^r z)] \\ &+ \sum_{j=1}^{3} \{b_{1t}^{(j)}, b_{2t}^{(j)}, b_{3t}^{(j)}\} \mathrm{i} A_j^t \exp[\mathrm{i}(kx + l_j^t z)] \end{aligned} \tag{17}$$

where

$$\begin{cases} b_{1r}^{(j)} = B_5(l_j^r a_{1r}^{(j)} + k a_{2r}^{(j)}) \\ b_{2r}^{(j)} = k B_3 a_{1r}^{(j)} + l_j^r B_4 a_{2r}^{(j)} - B_7(k a_{3r}^{(j)} + l_j^r a_{4r}^{(j)}) \\ b_{3r}^{(j)} = k B_6 a_{1r}^{(j)} + l_j^r B_7 a_{2r}^{(j)} - B_8(k a_{3r}^{(j)} + l_j^r a_{4r}^{(j)}) \end{cases} \tag{18}$$

$b_{1t}^{(j)}, b_{2t}^{(j)}$ and $b_{3t}^{(j)}$ are of the similar forms. It is noted that no reflected waves exit in the upper half-space and no refracted waves in the lower one.

4. TRANSFER MATRIX BETWEEN LAYERS

Deresiewicz *et al.*[13] discussed the boundary conditions on the interface between two fluid-saturated porous media. These conditions include the continuity of the following components: (*i*) the tangential displacement u_x of the skeleton; (*ii*) the normal displacement u_z of the skeleton; (*iii*) the average normal displacement $(u_z + w_z)$, or equivalently w_z; (*iv*) the tangential traction τ_{xz}; (*v*) the average normal traction $(\tau_{zz} - \phi p_f)$; and (*vi*) pore fluid pressure p_f (for the case of no resistance to interstitial flow across the interface).

Introduce a column matrix

$$\{S_n(z)\} = \{u_x, u_z, w_z, \tau_{xz}, \tau_{zz} - \phi p_f, p_f\}_n^T, \quad n = 0 \sim N+1 \tag{19}$$

The boundary conditions then lead to

$$\{S_n(h_n)\} = \{S_{n+1}(h_n)\} \tag{20}$$

Set

$$\{C_n\} = \{A_1^r, A_2^r, A_3^r, A_1^t, A_2^t, A_3^t\}_n^T \tag{21}$$

where $A_j^t = 0$ for $n = 0$ and $A_j^r = 0$ for $n = N+1$. Equations (16) and (17) may be written as

$$\{S_n(z)\} = [D_n(z)]\{C_n\}\mathrm{e}^{ikx} \tag{22}$$

which when substituted into (20) yields

$$[D_n(h_n)]\{C_n\} = [D_{n+1}(h_n)]\{C_{n+1}\} \tag{23}$$

where the 6×6 square matrix $[D_n(z)]$ is given in Appendix. Rewrite (23) as

$$\{C_{n+1}\} = [T_n]\{C_n\} \tag{24}$$

where $[T_n] = [D_{n+1}(h_n)]^{-1}[D_n(h_n)]$ is the transfer matrix between layers. Using the boundary conditions of the interface $z = h_1, h_2, \cdots h_{N-1}$ successively, we get the following relation between unknown coefficients of the 1th layer and the nth layer:

$$\{C_N\} = [T_{N-1}][T_{N-2}]\cdots[T_1]\{C_1\} = [E_N]\{C_1\} \tag{25}$$

Furthermore we have

$$\{S_N(h_N)\} = [D_N(h_N)][E_N][D_1(0)]^{-1}\{S_1(0)\} = [R_N]\{S_1(0)\} \tag{26}$$

Now let us consider an incident wave in the lower half-space with the form

$$\{u_x^{(i)}, u_z^{(i)}, w_x^{(i)}, w_y^{(i)}\} = \{a_{x0}, a_{z0}, b_{x0}, b_{z0}\}A_0 \exp[ik_0(x\sin\theta_0 + z\cos\theta_0)] \tag{27}$$

where k_0 is a complex wavenumber which, together with a_{x0}, a_{z0}, b_{x0} and b_{z0}, can be obtained as follows: Insert $k = k_0 \sin\theta_0$ and $l = k_0 \cos\theta_0$ into (10). We then have a cubic equation for k_0^2, of which the three roots can be derived as before. With k_0 in hand, we can compute a_{x0}, a_{z0}, b_{x0} and b_{z0} with equation (15). The phase velocity of the incident wave is given by $c_0 = \omega/\mathrm{Re}(k_0)$, and the attenuation coefficients by $\alpha_0 = \mathrm{Im}(k_0)$. For three different values of k_0, we have three types of waves: QP1, QP2 and QSV. The needed stress components associated with the incident wave are

$$\{\tau_{xz}^{(i)}, \tau_{zz}^{(i)}, p_f^{(i)}\} = \{t_{x0}, t_{z0}, s_0\}\mathrm{i}A_0 \exp[\mathrm{i}k_0(x\sin\theta_0 + z\cos\theta_0)] \tag{28}$$

where

$$\begin{cases} t_{x0} = k_0 B_5(a_{x0}\cos\theta_0 + a_{z0}\sin\theta_0) \\ t_{z0} = k_0(B_3 a_{x0}\sin\theta_0 + B_4 a_{z0}\cos\theta_0) - k_0 B_7(b_{x0}\sin\theta_0 + b_{z0}\cos\theta_0) \\ s_0 = k_0(B_6 a_{x0}\sin\theta_0 + B_7 a_{z0}\cos\theta_0) - k_0 B_8(b_{x0}\sin\theta_0 + b_{z0}\cos\theta_0) \end{cases} \tag{29}$$

Introduce a column matrix

$$\{D^{(i)}(0)\} = \{a_{x0}, a_{z0}, b_{z0}, \mathrm{i}t_{x0}, \mathrm{i}(t_{z0} - \phi s_0), \mathrm{i}s_0)\}^T \tag{30}$$

The boundary conditions on the interfaces: $z = 0$ and $z = h_N$ can be written as

$$A_0\{D^{(i)}(0)\} + \{S_0(0)\} = \{S_1(0)\}, \quad \{S_{N+1}(h_N)\} = \{S_N(h_N)\} \tag{31}$$

Substitution of (22) and (26) into (31) yields a linear equation group

$$([R_N][D_0(0)] - [D_{N+1}(h_N)])(\{C_0\} + \{C_{N+1}\}) = -[R_N]\{D^{(i)}(0)\}A_0 \tag{32}$$

from which we can obtain the coefficients $\{C_0\}$ and $\{C_{N+1}\}$. The angle and phase velocity for the reflected waves in the lower half-space are given by

$$\theta_j^r = \tan^{-1}[\mathrm{Re}(k)/\mathrm{Re}(l_j^r)_0], \; c_j^r = \omega/\sqrt{[\mathrm{Re}(k)]^2 + [\mathrm{Re}(l_j^r)_0]^2} \tag{33}$$

and those for the transmitted waves in the upper half-space are

$$\theta_j^t = \tan^{-1}[\mathrm{Re}(k)/\mathrm{Re}(l_j^t)_{N+1}], \; c_j^t = \omega/\sqrt{[\mathrm{Re}(k)]^2 + [\mathrm{Re}(l_j^t)_{N+1}]^2} \tag{34}$$

The attenuation coefficients in $z-$direction associated with the reflected and transmitted waves are

$$\alpha_j^r = -\mathrm{Im}(l_j^r)_0, \quad \alpha_j^t = \mathrm{Im}(l_j^r)_{N+1} \tag{35}$$

The results in other layers can be obtained analogously.

Acknowledgments—Support by the China National Natural Science Foundation under Grant No.19372001 is acknowledged.

References

[1] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. *J. Acoust. Soc. Am.* **28**, 168–178 (1956).

[2] M. A. Biot. Theory of propagation of elastic waves in a fluid-saturated porous solid. II. High-frequency range. *J. Acoust. Soc. Am.* **28**, 179–191 (1956).

[3] H. Deresiewicz and A. Levy. The effect of boundaries on wave propagation in a liquid-filled porous solid. X. Transmission through a stratified medium. *Bull. Seism. Soc. Am.* **57**, 381–391 (1967).
[4] J. F. Allard, R. Bourdier and C. Depollier. Biot waves in layered media. *J. Appl. Phys.* **60**, 1926–1929 (1986). [5] G. E. Baird, P. D. Thomas and G. Sang. The propagation of elastic waves through a layered poroelastic medium. *J. Acoust. Soc. Am.* **99**, 3385–3392 (1996).
[6] M. A. Biot. Mechanics of deformations and acoustic propagation in porous media. *J. Appl. Phys.* **33**, 1482–1489 (1962).
[7] M. D. Sharma and M. L. Gogna. Wave propagation in anisotropic liquid-saturated porous solids. *J. Acoust. Soc. Am.* **90**, 1068–1073 (1991).
[8] J. M. Carcione. Wave propagation in anisotropic, saturated porous media: plane-wave theory and numerical simulation. *J. Acoust. Soc. Am.* **99**, 2655-2666 (1996).
[9] Y. S. Wang and Z. M. Zhang. Propagation of plane waves in transversely isotropic fluid-saturated porous media (in Chinese). *Acta Mech. Sinica.* **29**, 257–268 (1996).
[10] M. N. Kazi-Aoual, G. Bonnet and P. Jouanna. Green's functions in an infinite transversely isotropic saturated poroelastic medium. *J. Acoust. Soc. Am.* **84**, 1883–1889.
[11] D. L. Johnson, J. Koplik and R. Dashen. Theory of dynamic permeability and tortuosity in fluid-saturated porous media. *J. Fluid Mech.* **176**, 379–402 (1987).
[12] G. A. Kern and T. M. Kern. *Handbook of Mathematics.* MCGraw-Hill, New York (1974).
[13] H. Deresiewicz and R. Skalak. On the uniqueness in dynamic poroelasticity. *Bull. Seism. Soc. Am.* **53**, 783–788 (1963).

Appendix

The coefficients of equation (10) are as follows

$$a = -\tilde{m}_1 D_2 B_5 \tag{A1}$$

$$b = -(\tilde{m}_3 B_5 D_2 + \tilde{m}_1 E)k^2 + [-\rho_f^2 D_2 + 2\rho_f \tilde{m}_1 B_5 B_7 + \rho\tilde{m}_1(D_2 + B_5 B_8) + \tilde{m}_1\tilde{m}_3 B_4 B_5] \tag{A2}$$

$$\begin{aligned} c = &-(\tilde{m}_1 B_5 D_1 + \tilde{m}_3 E)k^4 + \{-2\rho_f^2[(B_3 + 2B_5)B_8 - B_6 B_7] \\ &+ 2\rho_f\tilde{m}_1[(2B_1 + B_2)B_7 - (B_3 + B_5)B_6] + 2\rho_f\tilde{m}_3[B_4 B_6 - (B_3 + B_5)B_7] \\ &+ \rho\tilde{m}_1(D_1 + B_5 B_8) + \rho\tilde{m}_3(D_2 + B_5 B_8) + \tilde{m}_1\tilde{m}_3[(2B_1 + B_2)B_4 \\ &- (B_3 + 2B_5)B_3]\}k^2 + [2\rho_f^3 B_7 + \rho\rho_f^2 B_8 + \rho_f^2\tilde{m}_1 B_5 + \rho_f^2\tilde{m}_3 B_4 \\ &- \rho^2\tilde{m}_1 B_8 - 2\rho\rho_f\tilde{m}_1 B_7 - \rho\tilde{m}_1\tilde{m}_3(B_4 + B_5)] \end{aligned} \tag{A3}$$

$$\begin{aligned} d = &(\rho\tilde{m}_1 - \rho_f^2)(\rho\tilde{m}_3 - \rho_f^2) + [2\rho_f^3 B_6 + \rho\rho_f^2 B_8 + \rho_f^2\tilde{m}_1(2B_1 + B_2) \\ &+ \rho_f^2\tilde{m}_3 B_5 - \rho^2\tilde{m}_3 B_8 - 2\rho\rho_f\tilde{m}_3 B_6 - \rho\tilde{m}_1\tilde{m}_3(2B_1 + B_2 + B_5)]k^2 \\ &+ [-\rho_f^2 D_1 + 2\rho_f\tilde{m}_3 B_5 B_6 + \rho\tilde{m}_3(D_1 + B_5 B_8) + \tilde{m}_1\tilde{m}_3(2B_1 + B_2)B_5]k^4 \\ &- \tilde{m}_3 D_1 B_5 k^6 \end{aligned} \tag{A4}$$

where

$$E = (2B_1 + B_2)D_2 - (B_3 + 2B_5)B_3B_8 + 2(B_3 + B_5)B_6B_7 - B_4B_6^2 \tag{A5}$$

$$D_1 = (2B_1 + B_2)B_8 - B_6^2, \qquad D_2 = B_4B_8 - B_7^2 \tag{A6}$$

$$\tilde{m}_1 = m_1 + \mathrm{i}r_1/\omega, \qquad \tilde{m}_3 = m_3 + \mathrm{i}r_3/\omega \tag{A7}$$

$\Delta_k^{(j)}$ in equation (15) is given by

$$\Delta_1^{(j)} = -\begin{vmatrix} d_{14}^{(j)} & d_{12}^{(j)} & d_{13}^{(j)} \\ d_{24}^{(j)} & d_{22}^{(j)} & d_{23}^{(j)} \\ d_{34}^{(j)} & d_{32}^{(j)} & d_{33}^{(j)} \end{vmatrix}, \Delta_2^{(j)} = -\begin{vmatrix} d_{11}^{(j)} & d_{14}^{(j)} & d_{13}^{(j)} \\ d_{21}^{(j)} & d_{24}^{(j)} & d_{23}^{(j)} \\ d_{31}^{(j)} & d_{34}^{(j)} & d_{33}^{(j)} \end{vmatrix} \tag{A8}$$

$$\Delta_3^{(j)} = -\begin{vmatrix} d_{11}^{(j)} & d_{12}^{(j)} & d_{14}^{(j)} \\ d_{21}^{(j)} & d_{22}^{(j)} & d_{24}^{(j)} \\ d_{31}^{(j)} & d_{32}^{(j)} & d_{34}^{(j)} \end{vmatrix}, \Delta_4^{(j)} = \begin{vmatrix} d_{11}^{(j)} & d_{12}^{(j)} & d_{13}^{(j)} \\ d_{21}^{(j)} & d_{22}^{(j)} & d_{23}^{(j)} \\ d_{31}^{(j)} & d_{32}^{(j)} & d_{33}^{(j)} \end{vmatrix} \tag{A9}$$

where $d_{kl}^{(j)} = d_{kl}\omega^2$ and d_{kl} is given by (8) with l replaced by l_j.

The elements of matrix $[D_n(z)]$ are

$$\begin{cases} (D_n)_{1,j} &= a_{1r}^{(j)} \exp(\mathrm{i}l_j^r z), \quad (D_n)_{1,j+3} = a_{1t}^{(j)} \exp(\mathrm{i}l_j^t z) \\ (D_n)_{2,j} &= a_{2r}^{(j)} \exp(\mathrm{i}l_j^r z), \quad (D_n)_{2,j+3} = a_{2t}^{(j)} \exp(\mathrm{i}l_j^t z) \\ (D_n)_{3,j} &= a_{3r}^{(j)} \exp(\mathrm{i}l_j^r z), \quad (D_n)_{3,j+3} = a_{3t}^{(j)} \exp(\mathrm{i}l_j^t z) \\ (D_n)_{4,j} &= \mathrm{i}b_{1r}^{(j)} \exp(\mathrm{i}l_j^r z), \quad (D_n)_{4,j+3} = \mathrm{i}b_{1t}^{(j)} \exp(\mathrm{i}l_j^t z) \\ (D_n)_{5,j} &= \mathrm{i}(b_{2r}^{(j)} - \phi b_{3r}^{(j)}) \exp(\mathrm{i}l_j^r z), \; (D_n)_{5,j+3} = \mathrm{i}(b_{2t}^{(j)} - \phi b_{3r}^{(j)} \exp(\mathrm{i}l_j^t z) \\ (D_n)_{6,j} &= \mathrm{i}b_{3r}^{(j)} \exp(\mathrm{i}l_j^r z), \quad (D_n)_{6,j+3} = \mathrm{i}b_{3t}^{(j)} \exp(\mathrm{i}l_j^t z) \end{cases} \tag{A10}$$

For $n = 0$ the last three columns are zero, while for $n = N + 1$ the first three ones are zero.