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ENERGY FLOW MODELS FROM FINITE ELEMENTS: AN APPLICATION TO THREE COUPLED PLATES

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This paper is concerned with the prediction of the distribution of vibrational energy throughout a structure. The paper illustrates how the results from a finite element model can be rephrased in terms of subsystem powers and energies using component mode synthesis. Equations are derived for the response, and frequency averages are obtained from them analytically. The technique is then illustrated by considering a system comprising three coupled plates, each plate being coupled along two of its edges. Ensemble and frequency averaged numerical results are compared with SEA predictions. It is seen that SEA predictions can be in error, this being attributable to coherence effects which arise because the system is strongly coupled, and the effects of averaging over narrow frequency bands.

1. INTRODUCTION

The prediction of the vibrational response of a structure at higher frequencies presents particular challenges to the analyst. The effects of structural uncertainties, and the large number of modes contributing to the response at high frequencies, have led to the adoption of structural acoustic techniques such as statistical energy analysis (SEA). In SEA, a system is described in terms of the dynamic behaviour of a set of subsystems. The response of the system is determined by considering the interactions between these subsystems, averaged over an ensemble. SEA predictions (and consequently the assumption of coupling power proportionality) are found to be more accurate when the subsystems are weakly coupled and when the frequency bands of interest contain many interacting modes. Weak coupling requires that the local dynamic interaction of two subsystems is not significantly affected by the presence of additional subsystems [1], or that the effects of coherent power at the coupling between subsystems are negligible [2]. When the coupling becomes strong, coherence effects are important and the interaction of two subsystems can be dependent on the global properties of the system.

At low frequencies the response of a structure is normally predicted using numerical techniques such as the finite element (FE) method. A discrete description of the continuum leads to mass, stiffness and damping matrices which represent the spatial distribution of these quantities in terms of a large number of local shape functions. The vibrational response is then normally described in terms of frequency response functions, such as point or transfer receptances and mobilities. To simplify calculation of these quantities, a modal analysis is usually performed and mode shapes and natural frequencies found from an eigenvalue problem. Often, one of the criticisms directed at the use of finite elements at higher frequencies is the use of such response coordinates. Since frequency response functions are obtained from a summation of vectorial quantities (in a modal summation), they are sensitive to perturbations in the modal properties. The use of frequency response functions can also leave the analyst swamped with a vast amount of information. However, it is important to note that this criticism should be specifically directed at the use of an inappropriate response coordinate rather than at the finite element method in general. The results from a finite element model can equally be viewed in terms of spatial and temporal averaged energies and powers.

It is perhaps worth reiterating some of the specific problems associated with the use of FE to describe the vibrational response. The FE method can be computationally restrictive, the number of elements required to adequately describe the spatial fluctuations in the response increasing rapidly with frequency. The computational expense of the eigenproblem also increases rapidly with the size of the mass and stiffness matrices. Computational limitations aside, the resulting response is only as accurate as the information supplied to the model. Assumptions regarding boundary conditions and material properties can mean there is little point trying to obtain more accuracy out of something that is inherently uncertain. In the traditional FE approach the results are also deterministic, and therefore do not account for the effects of uncertainties.

There are, however, circumstances where a FE model can be useful in determining the vibrational response of a structure at higher frequencies. The next section discusses the form of various energy flow models. This is followed by a discussion of how FE and component mode synthesis (CMS) can be used to derive an energy flow model of a system. The vibrational behaviour of a system comprising three coupled plates is used as an example, and the effects of ensemble and frequency averaging the FE/CMS results is investigated. The results are also compared with an SEA prediction.

2. ENERGY FLOW MODELS

In an energy flow model of a structure, the vibrational energies within various component parts, or subsystems, of the structure are related to the input powers applied to each excited subsystem. If the excitations applied to each subsystem are statistically independent, then the time average input powers $\mathbf{P}_{in}$, and the time average response energies $\mathbf{E}$, in each subsystem are linearly related by

$$\mathbf{E} = \mathbf{A}\mathbf{P}_{in} \tag{1}$$

where $\mathbf{A}$ is a matrix of energy influence coefficients (EIC), A_{ij} indicating the amount of energy stored in the i'th subsystem due to excitation of subsystem j. Knowing the matrix of EIC's enables the response to a particular distribution of excitations to be found. Equation (1) can be written as

$$\mathbf{P}_{in} = \mathbf{HE} \tag{2}$$

where $\mathbf{H}=\mathbf{A}^{-1}$. In an SEA model and when the system is weakly coupled, the elements of $\mathbf{H}$ take on a particular physical significance, and are related to damping and coupling loss factors. If the system is strongly coupled, or if finite frequency averages are taken for a single system (as opposed to ensemble average powers and energies in SEA) then the coupling loss factors inferred from inverting $\mathbf{A}$ may be negative, zero or infinite [3]. Furthermore, in certain circumstances this inverse may not even exist. In this paper the emphasis is placed on estimation of the EIC matrix $\mathbf{A}$. The problem is then how to calculate this matrix for a particular structure, or a particular population of structures. The numerically efficient calculation of $\mathbf{A}$ from a finite element model is discussed in this paper and in [4], with an example of three coupled plates being considered here.

3. COMPONENT MODE SYNTHESIS

In a standard finite element model of a structure, the response is described in terms of a large number of local shape functions. The size of the finite element model can be reduced by choosing a more appropriate basis with which to span the space of possible responses. In the component mode synthesis (CMS) approach [5], this is achieved by first obtaining a set of local component modes. These can include the normal modes of a substructure (when the substructure interface is subject to certain boundary conditions), and constraint modes of the substructure. The constraint modes represent the shape of a substructure when a particular interface degree of freedom is given a unit displacement or rotation whilst all other interface degrees of freedom remain fixed. The component modes of the structure are then used as a basis with which to describe the global response. By choosing fewer component mode degrees of freedom than nodal degrees of freedom, the size of the model can be significantly reduced. The component modes can be thought of as representing the local dynamics of a substructure, whilst the constraint modes provide coupling between substructures. This philosophy fits well with SEA, since the local dynamic properties are used to describe the global dynamic properties. Choosing subsystems that correspond with substructures simplifies the calculation of the energy flow model.

4. KINETIC ENERGY AND INPUT POWER

In this section, expressions for the (time average) kinetic energy and input power to various subsystems are derived. The potential energy can be calculated in a similar manner but is not considered here. A fixed interface CMS model is used, with the response coordinates being the component mode amplitudes. The global mass and stiffness matrices are found using CMS, and a free vibration analysis performed to determine the global mode shapes and natural frequencies.

4.1 DISCRETE FREQUENCY RESPONSE

The time average kinetic energy in the a'th subsystem of a structure is given by

$$\langle E_k \rangle_{(a)} = \frac{\omega^2}{4} \sum_{j,k\in a} \mathbf{M}_{jk} \operatorname{Re}\{\mathbf{Q}_j^* \mathbf{Q}_k\} \tag{3}$$

where $\mathbf{M}$ is the component mass matrix associated with the a'th subsystem and $\mathbf{Q}_k$ is the response of the k'th component mode. The summation here runs over all component modes j,k in subsystem a. The component mode response is given by

$$\mathbf{Q}_j = \sum_m \mathbf{P}_{jm} \alpha_m \mathbf{F}_{Y,m} \tag{4}$$

where α_m is the receptance of the m'th global mode given by $1/\left(\omega_m^2(1+i\eta_m)-\omega^2\right)$, $\mathbf{P}$ is a matrix of global mode shapes, $\mathbf{F}_{Y,m}$ is the force applied to global mode m, ω_m is the m'th global natural frequency and η_m is the loss factor of the m'th global mode. Here the summation is over all global modes m that contribute to the response. The global modal forces can be related to the forces acting on each individual node in a subsystem b by

$$\mathbf{F}_{Y,m} = \sum_{r\in b} \mathbf{P}_{rm} \mathbf{F}_{Q,r} = \sum_{r\in b} \mathbf{P}_{rm} \sum_{t\in b} \mathbf{T}_{tr} \mathbf{f}_t \tag{5}$$

where $\mathbf{F}_{Q,r}$ is the force applied to the r'th component mode in subsystem b, $\mathbf{f}_t$ is the force applied to the t'th node in subsystem b and $\mathbf{T}$ is the transformation matrix relating nodal and component modal degrees of freedom. The summation over r runs over all component modes in the excited subsystem b, whilst the summation over t runs over all nodal degrees of freedom in the excited subsystem. Substituting equations (4) and (5) into equation (3) and rearranging the order of summation gives

$$\langle E_k \rangle_{(a)} = \frac{\omega^2}{4} \sum_{m,p} \left(\sum_{j,k\in a} \mathbf{M}_{jk} \mathbf{P}_{jm} \mathbf{P}_{kp} \right) \sum_{r,s\in b} \mathbf{P}_{rm} \mathbf{P}_{sp} \sum_{t,u\in b} \mathbf{T}_{tr} \mathbf{T}_{us} \operatorname{Re}\{\mathbf{f}_t^* \mathbf{f}_u \alpha_m^* \alpha_p\} \tag{6}$$

4.1.1 ASSUMED EXCITATION - "RAIN-ON-THE-ROOF"

Let us now assume that the distribution of nodal forces is such that, when averaged over time, the product of nodal forces $\mathbf{f}_t^* \mathbf{f}_u$ is real and proportional to the (t,u)'th entry in the local nodal mass matrix $\mathbf{m}$, i.e.,

$$\mathbf{f}_t^* \mathbf{f}_u = R \mathbf{m}_{tu} \tag{7}$$

where R is a constant of proportionality. This assumption is similar to assuming that all nodal forces are uncorrelated and that the magnitude of the force squared applied to each node is proportional to the amount of mass lumped at that node. Such loading results in equal modal forces and is hence the equivalent of "rain-on-the-roof" when phrased in terms of component modes. Equation (6) then becomes

$$\langle E_k \rangle_{(a)} = R\sum_{m,p} \psi_{mp}^{(a)}\psi_{mp}^{(b)}\beta_{mp}; \quad \psi_{mp}^{(a)} = \sum_{j,k\in a} \mathbf{M}_{jk}\mathbf{P}_{jm}\mathbf{P}_{kp}; \quad \beta_{mp} = \frac{\omega^2}{4}\mathrm{Re}\{\alpha_m^{\ *}\alpha_p\} \tag{8}$$

The terms ψ are referred to as distribution factors [6], and quantify the degree to which each global mode is spatially distributed over the responding and excited subsystems, whilst the term β is frequency dependent. The (m,p)'th distribution factor can be found by pre- and post-multiplying the local component mass matrix **M** by the appropriate partitions of the m'th and p'th global modes respectively. The distribution factor $\psi_{mm}^{(a)}$ indicates the proportion of kinetic energy stored in the a'th subsystem, when the system vibrates in the m'th mode. The terms $\psi_{mp}^{(a)}(m \neq p)$ give an indication of the orthogonality of global modes m and p over subsystem a.

4.2 FREQUENCY AVERAGE RESPONSE

We are often interested in the response to broadband excitation rather than the discrete frequency response. This can be found by integrating equation (8) over frequency and noting that only the term β_{mp} is frequency dependent. If the power spectral density S_{ff} of the excitation is constant, then R is also constant and the frequency average value of β between frequencies ω_1 and ω_2 is given by

$$\Gamma_{mp} = \frac{1}{\Omega}\int_{\omega_1}^{\omega_2} \beta_{mp} d\omega = -\frac{1}{4\Omega}\left(\mathrm{Re}\left\{a_1 b_1 \arctan\left(\frac{b_1\Omega}{b_1^2 + \omega_1\omega_2}\right)\right\} + \mathrm{Re}\left\{a_2 b_2 \arctan\left(\frac{b_2\Omega}{b_2^2 + \omega_1\omega_2}\right)\right\}\right) \tag{9}$$

where

$$a_1 = \frac{-i}{\left(\eta_m\omega_m^2 + \eta_p\omega_p^2\right) + i\left(\omega_p^2 - \omega_m^2\right)}, \quad a_2 = -a_1^*, \quad b_1 = i\omega_m\sqrt{1+i\eta_m}, \quad b_2 = i\omega_p\sqrt{1+i\eta_p} \tag{10}$$

and $\Omega = \omega_2 - \omega_1$. Whilst this expression is useful for calculating narrow band responses, or calculating responses where modes partially contribute (ie. where modes lie close to ω_1 or ω_2), it can be simplified for larger frequency bands. If it is assumed that a mode lies within the bandwidth of excitation, and the damping is light, then the limits of integration can be replaced by 0 and ∞ to give the approximate frequency averages

$$\Gamma_{mm,\infty} = \frac{1}{\Omega}\frac{\pi}{8\eta\omega_m}, \quad \Gamma_{mp,\infty} = \Gamma_{mm,\infty}\left(\Delta^2 \Big/ \left(\Delta^2 + \left(\omega_m - \omega_p\right)^2\right)\right) \tag{11}$$

As discussed in [6], the magnitudes of Γ_{mp} determine which cross modal pairs (m,p) give significant contributions to the response. The time average input power to subsystem b can be calculated in a similar manner to the kinetic energy and is given in terms of the distribution factors $\psi_{mm}^{(b)}$ associated with the excited subsystem by

$$\langle P_{in} \rangle_{(b)} = R \sum_m \psi_{mm}^{(b)} \beta'_{mm}; \quad \beta'_{mm} = -\frac{\omega}{2} \mathrm{Im}\{\alpha_m\}; \quad \Gamma'_{mm,\infty} = \frac{1}{\Omega} \int_0^\infty \beta'_{mm} d\omega = \frac{\pi}{4\Omega} \tag{12}$$

5. ENERGY DISTRIBUTION IN A THREE PLATE STRUCTURE

As an example of the application of the expressions derived in the previous section, consider a structure comprising three coupled plates. Each plate is rectangular and joined to the others along two of its edges, the uncoupled edges being unconstrained, as shown in Figure 1. The nominal plate dimensions and material properties are given in Table 1. A finite element model of the structure was implemented in Matlab using Heterosis plate elements [7]. The lines of coupling were assumed to remain fixed in space and be simply supported. Each plate was modelled with 100 elements, from which 35 interior modes were calculated (the highest uncoupled natural frequency being above 6 kHz). The first 100 global modes were calculated using subspace iteration, with the highest computed global natural frequency being approximately 6 kHz. The matrix of EIC's for the structure was then calculated, assuming that when frequency averaged, the total subsystem energy is twice the kinetic energy.

5.1 SINGLE REALISATION FREQUENCY AVERAGES

Figure 2 plots the 200 Hz and 1 kHz frequency average EIC's for a single realisation of the plate structure against frequency, and the results of an SEA prediction. There are approximately 3 global modes in a 200 Hz frequency band. When averaged over a narrow frequency band, the EIC's between subsystems show distinct resonant behaviour. When averaged over a large frequency band it is seen that SEA tends to over predict the response in the undriven subsystems. This is due in part to the effects of coherent reflections at the coupling [2] with the plates being strongly coupled (γ being approximately 2.5 between each pair of plates at 1 kHz).

5.2 ENSEMBLE AND FREQUENCY AVERAGES

In order to investigate the effects of ensemble averaging on the EIC's, a monte-carlo simulation was undertaken, with the material properties of each subsystem, and the leading dimensions taken as uncorrelated random variables. Whilst this approach is computationally impractical for larger problems, it provides useful insights into the effects of ensemble averaging for this example. The ensemble was defined by perturbing the material properties of each subsystem, and the leading dimensions of the structure, uniformly between ± 2.5% of their nominal values. The frequency average EIC's, averaged over a 200 Hz frequency band, were obtained for 100 separate finite element calculations. The statistics of the EIC's across the ensemble, such as the median, lower and upper quartiles were then obtained and are shown in Figure 3, along with the SEA prediction. Distinct resonant behaviour is still observed even when ensemble averaged. This can be attributed to the effects of averaging over a narrow frequency band and mid-frequency effects. Mid-frequency effects occur when the population of plate structures under investigation does not contain enough uncertainty to be adequately modelled by the SEA ensemble.

6. CONCLUDING REMARKS

In this paper a computationally efficient method for determining an energy flow model from a finite element model was described. Expressions for discrete frequency and frequency average responses were given in terms of component mode synthesis. The method was used to calculate the energy distribution in a structure comprising three coupled plates, and ensemble and frequency average results were compared with SEA predictions. It was observed that SEA can be in error, due to the effects of strong coupling and finite frequency band averaging.

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7. REFERENCES

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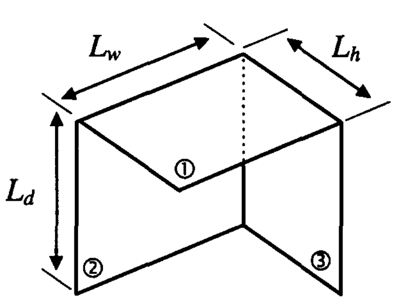

Figure 1. System comprising 3 coupled plates

L_w	255 mm
L_h	225 mm
L_d	275 mm
η	0.05
Plate thickness	10 mm
Young's Modulus	4.6 GN/m^2
Density	1130 kg/m^3
Poisson's ratio	0.25

Table 1 : Properties of system

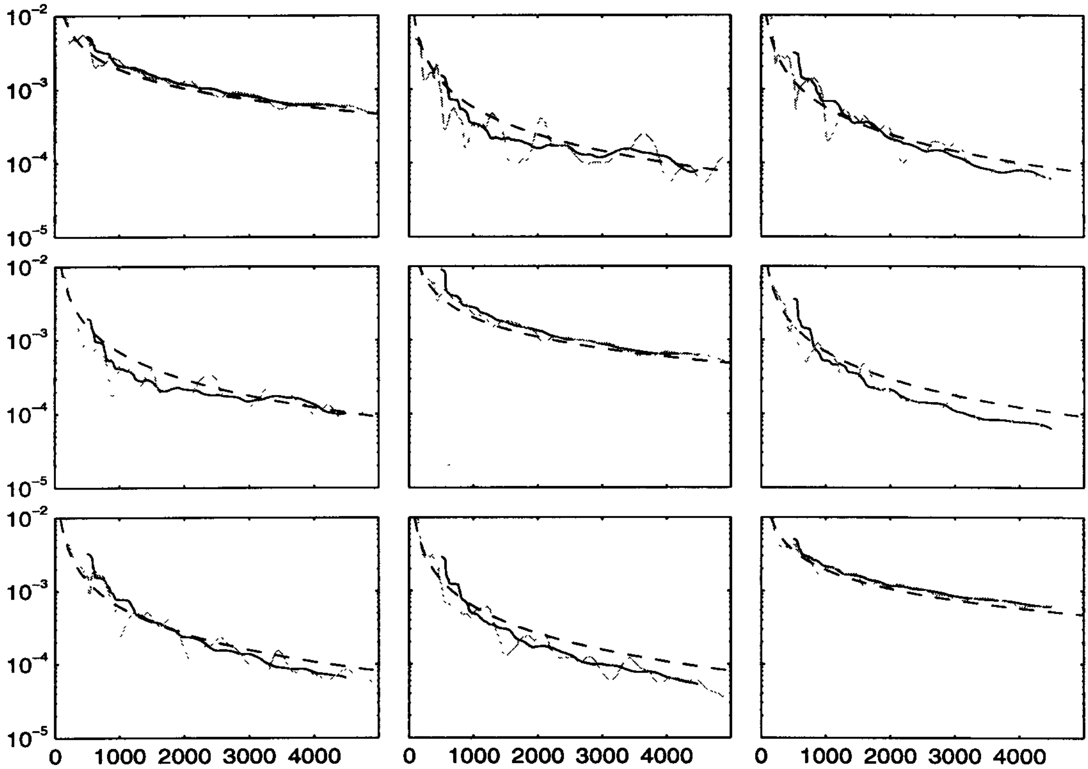

Figure 2. Plot of EIC's against frequency (in Hz). Position of plot relates to position of EIC in matrix **A**.
--- SEA prediction; FE/CMS single realisation, 200 Hz frequency average, — 1 kHz frequency average.

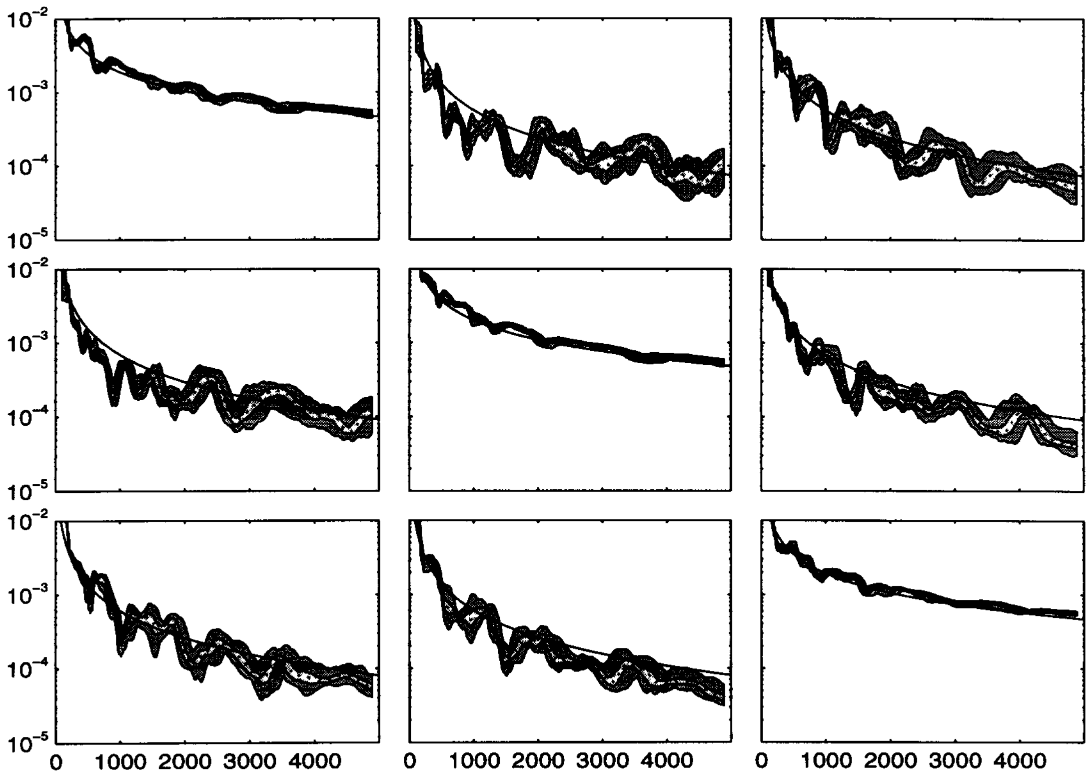

Figure 3. Ensemble and frequency averaged EIC's; dark grey, maximum and minimum values; light grey, upper and lower quartiles; ···· median; — SEA prediction