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ENERGY FLOW AND S.E.A. AT LOW MODAL OVERLAP

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1. INTRODUCTION

This paper concerns energy flow and statistical energy analysis (SEA) models of structural vibrations when modal overlap is small. In this circumstance the accuracy of SEA predictions is often poor. Particular reference is made to equipartition of energy (i.e., uniform energy density or modal energy throughout a system) and it is seen, from both wave and modal analyses, that equipartition of energy does not occur in mechanical systems. Some wave analyses are reviewed and a modal analysis developed which is applicable in the low modal overlap limit. Numerical examples are presented.

In the situations discussed energy dissipation is low - this can be described in terms of small loss factor η, small bandwidth Δ, small modal overlap M or small wave attenuation rates. Whichever measure is taken, the point is that energy dissipation is small and energy flow relatively easy. In such circumstances the coupling is strong in the SEA sense (granted that no universally accepted definition of coupling strength exists).

It is as well to distinguish between the *ease of energy flow* between two subsystems (i.e., whether they are well or poorly coupled) and the *strength of coupling*. Two subsystems can be said to be well (or poorly) coupled if energy passes easily (with difficulty) through the coupling that connects them. In wave terms this means the coupling transmission coefficient is large (small). Strong/weak coupling, on the other hand, describes the dynamics of energy sharing between subsystems and typically involves both measures of the strength of transmission through the coupling (e.g., transmission coefficients) and rate of energy dissipation within the subsystems: strongly coupled subsystems share vibrational energy easily, while weak coupling involves the leakage of vibrational energy from one subsystem to another. The systems are strongly coupled in the low modal overlap situations considered here.

In the next section energy flow and SEA models in the low modal overlap/equipartition limit are discussed in general terms. Then results from existing wave analyses are reviewed and a global modal analysis performed for the case of low modal

overlap. Both wave and modal approaches have advantages: wave methods are well suited to the analysis of systems with relatively simple geometries and in these cases provide understanding of the underlying physical behaviour, while modal methods have the advantage of generality, this factor tending to obscure the underlying physics.

2. ENERGY FLOW AND SEA MODELS

In these models a system is divided into N subsystem and the response is described by the (time, frequency and ensemble) averaged subsystem energies E_i, which are given in terms of the (statistically independent) subsystem input powers $P_{in,i}$ by

$$\mathbf{E} = \mathbf{A}\mathbf{P}_{in} \tag{1}$$

where $\mathbf{A}$ is a matrix of energy influence coefficients. It is these coefficients that are predicted in this paper. SEA is an approximate method which approaches the problem by considering conservation of energy for each subsystem, such that

$$P_{in,i} = P_{diss,i} + P_{coup,i}; \quad P_{diss,i} = \omega\eta_i E_i; \quad P_{coup,i} = \sum P_{ij}; \quad P_{ij} = \omega\eta_{ij}E_i - \omega\eta_{ji}E_j \tag{2}$$

where P_{diss} and P_{coup} are the total dissipated and coupling powers, P_{ij} the coupling power from subsystem i to subsystem j, ω the centre frequency of excitation and η_i and η_{ij} the damping and coupling loss factors. The relation

$$n_i\eta_{ij} = n_j\eta_{ji} \tag{3}$$

is also assumed, n being the asymptotic modal density. If all the parameters and input powers are known, the responses can be predicted immediately. SEA relies on a number of implicit and explicit assumptions (some of which merely reflect desired properties of an SEA model), and the accuracy of its predictions can be poor, especially when the coupling is strong (or when the modal overlap is low enough).

Equipartition of energy occurs if the energy density $D=E/n$ (the frequency average energy per mode) is the same for all subsystems. If equipartition occurs, the matrix $\mathbf{A}$ is singular, the i'th row being proportional to n_i. In an SEA model, this requires that all the loss factors tend to zero but the coupling loss factors do not - they are assumed to be constants that depend on the coupling parameters and some of the subsystem properties (e.g., modal density) but (perversely?) not subsystem damping. This is perhaps valid for weak coupling but not for strong. As an example, if there are just two subsystems, with subsystem 1 being excited, then

$$\frac{D_2}{D_1} = \frac{n_1}{n_2}\frac{\eta_{12}}{\eta_{21}+\eta_2} \tag{4}$$

As $\eta_2 \to 0$, $D_2/D_1 \to 0$ but *only* if $\eta_{12},\eta_{21} > O(\eta)$. If not, then the energy density ratio depends on the ratio of coupling and damping loss factors and equipartition will not occur.

3. WAVE ANALYSES AT LOW MODAL OVERLAP

A system comprising two, coupled, one-dimensional subsystems, such as two coupled beams, was considered in [1]. In wave terms, two subsystems are well (poorly) coupled if the

transmission coefficient of the coupling which joins them is large (small). Strong (weak) coupling occurs when this transmission coefficient is large *compared to dissipation effects* [1].

In the low damping limit the (ensemble average) coupling power is given by

$$\frac{P_{12}}{P_{in,1}} = \left(\frac{P_{12}}{P_{in,1}}\right)_{equip} \left(1 + \frac{M_a M_b}{\left(M_a + M_b\right)^2} \frac{\left(1-T^2\right)}{T^2}\right)^{-1/2} \tag{5}$$

where $(.)_{equip}$ is the value the power ratio would take were equipartition of energy to hold, M is the subsystem modal overlap and T the coupling transmission coefficient. Note that equipartition of energy holds *only* if $T = 1$. If T is different from 1 the excited subsystem has a higher energy density - an energy density difference exists across the coupling and this is responsible for driving energy through the coupling.

An exact analysis of two, uniformly coupled, rectangular plates was considered in [2]. Non-uniform two-plate systems were discussed in [3,4]. However, when non-uniformities are introduced the analysis becomes complicated and can proceed only so far. In all these cases the same features are evident: well coupled systems have large transmission coefficients; strongly coupled systems are those where transmission effects are large compared to damping effects; equipartition of energy does not occur at low damping when T differs from unity: the coupling acts as a barrier to energy flow, the excited subsystem having higher energy density than the undriven subsystem; non-uniformity tends to spread the energy more uniformly through the system. Furthermore it is found that when the coupling is strong global properties determine energy flow through a coupling, while local properties (i.e., those properties used in traditional estimates of coupling loss factors) determine the flow when the coupling is weak.

4. LOW MODAL OVERLAP - MODES OF VIBRATION

In this section a modal description is developed. The analysis is general compared to the wave analyses described above, but the results perhaps less easy to understand.

The response of a structure at point x_2 due to a time harmonic excitation $F \exp(i\omega t)$ applied at point x_1 is given by a sum of modal contributions as

$$u(\omega, x_1, x_2) = \sum_j \alpha_j(\omega) \phi_j(x_1) \phi_j(x_2) F \tag{6}$$

where ω_j and $\phi_j(x)$ are the j'th natural frequency and mode shape and where

$$\alpha_j(\omega) = \frac{1}{\omega_j^2\left(1+i\eta_j\right) - \omega^2} \tag{7}$$

is the modal receptance. The mode shapes are assumed to be mass-normalised so that

$$\int \rho(x) \phi_j(x) \phi_k(x) dx = \delta_{jk} \tag{8}$$

where ρ is the mass density. The time average input power and kinetic energy density are then

$$P_{in}(\omega,x_1)=\tfrac{1}{2}\omega\sum_j \eta_j\alpha_j\alpha_j^*\phi_j^2(x_1)F^2$$
$$D_T(\omega,x_1,x_2)=\tfrac{1}{4}\omega^2\sum_{j,k}\mathrm{Re}\{\alpha_j\alpha_k^*\}\{\rho(x_2)\phi_j(x_2)\phi_k(x_2)\}\phi_j(x_1)\phi_k(x_1)F^2 \qquad (9)$$

Note that the input power is a sum of modal contributions while the kinetic energy density depends also on cross-modal terms. If the damping is light then these terms are very small unless either, and particularly if both, of modes j and k are resonant at ω.

4.1. MODES AND ENERGY FLOW

The previous expressions can be used to form an energy flow model of a system. Suppose there are N subsystems, the e'th being excited by "rain-on-the-roof" of unit amplitude. This is defined here to be broad-band, uncorrelated, distributed random excitation whose power spectral density is proportional to the local mass density, i.e., equals $S_{ff}\,\rho(x)$, where S_{ff} is a constant, here taken to be 1. The time and frequency average input power can be found by integrating equation (9) over all points x_1 which lie in the subsystem e and over all frequencies ω which lying in the band of excitation Ω. If the damping is light then

$$\langle P_{in}^{(e)}\rangle = 2\sum_j \eta_j\omega\Gamma_{jj}\psi_{jj}^{(e)} \qquad (10)$$

where $\langle.\rangle$ denotes time, space and frequency averaging and ω is now the centre frequency of Ω. The total kinetic energy in a responding subsystem i can be found similarly, together with a further integration over all response points x_2 which lie in i. It is given by

$$\langle T^{(i)}\rangle = \sum_{j,k}\Gamma_{jk}\psi_{jk}^{(e)}\psi_{jk}^{(i)} \qquad (11)$$

In the above expressions

$$\psi_{jk}^{(i)}=\int_i \rho(x)\phi_j(x)\phi_k(x)dx;\quad \Gamma_{jk}=\frac{1}{\Omega}\int_\Omega \tfrac{1}{4}\omega^2\,\mathrm{Re}\{\alpha_j\alpha_k^2\}d\omega \qquad (12)$$

The integral in $\psi_{jk}^{(i)}$ is taken only over those points which lie in subsystem i. Note that the terms Γ_{jk} tend to be very small except for those mode pairs which are resonant in Ω and which overlap, in the sense that their natural frequencies lie in each others bandwidths. The terms Γ_{jj} are necessarily large if ω_j lies in Ω. A similar expression can be developed for the subsystem potential energy - this can be assumed equal to the kinetic energy when frequency averaged.

The input power and subsystem energies depend on the mode shapes through the terms $\psi_{jk}^{(i)}$, these being termed the kinetic energy distribution factors for the mode pair (j,k). The modal orthonormality relations become

$$\sum_i \psi_{jk}^{(i)}=\delta_{jk} \qquad (13)$$

For $j=k$, $\psi_{jj}^{(i)}$ is positive and indicates the proportion of the total kinetic energy stored in subsystem i when the system vibrates in mode j - a large value indicates that the j'th mode is substantially localised within that subsystem. For $j\neq k$, $\psi_{jk}^{(i)}$ indicates the orthogonality of the (j,k)'th mode pair over subsystem i and may be positive or negative - small values indicate that the two mode shapes are, by-and-large, orthogonal within that subsystem.

4.2. ENERGY FLOW WITH LOW MODAL OVERLAP

When the modal overlap is small (i.e., $M \ll 1$), the subsystem energies are dominated by the terms involving Γ_{jj}. If it is assumed that Γ_{jj} is the same for all modes in Ω and that all modes and all subsystems have the same loss factor η, then the dissipated power per unit input power in subsystem i is given by

$$\frac{\langle P_{diss,i} \rangle}{\langle P_{in,e} \rangle} = \frac{\sum_j \psi_{jj}^{(e)} \psi_{jj}^{(i)}}{\sum_j \psi_{jj}^{(e)}} \tag{14}$$

Note that the total dissipated power, found by summing equation (14) over all subsystems i, equals the input power. Furthermore, large contributions to the response in subsystem i, and hence large contributions to the power dissipated in this subsystem, arise from those modes for which the product $\psi_{jj}^{(e)}\psi_{jj}^{(i)}$ is relatively large. Such modes are not only well excited ($\psi_{jj}^{(e)}$ large) but also respond well in subsystem i ($\psi_{jj}^{(i)}$ large). Modes which are localised within one subsystem tend to be weakly excited or respond weakly and contribute relatively little to the response - the exception, of course, is that a mode that is localised within the excited subsystem gives rise to a large response in the excited subsystem.

The dissipated powers can be related to the equipartition levels by noting that the asymptotic modal density of subsystem i, n_i, as a proportion of the total modal density of the system, is equal to the expected value of $\psi_{jj}^{(i)}$, and hence approximately equals $\langle \psi_{jj}^{(i)} \rangle_{j\to\infty}$, namely the mean value of $\psi_{jj}^{(i)}$ over many (all) modes. The dissipated power ratio can therefore be written as

$$\frac{\langle P_{diss,i} \rangle}{\langle P_{in,e} \rangle} = \lambda^{(i)} \left(\frac{P_{diss,i}}{P_{in,e}} \right)_{equip} \quad ; \quad \lambda^{(i)} = \frac{\langle \psi_{jj}^{(e)} \psi_{jj}^{(i)} \rangle_j}{\langle \psi_{jj}^{(e)} \rangle_j (n_i / n_{tot})} = \frac{\langle \psi_{jj}^{(e)} \psi_{jj}^{(i)} \rangle_j}{\langle \psi_{jj}^{(e)} \rangle_j \langle \psi_{jj}^{(i)} \rangle_{j\to\infty}} \tag{15}$$

where $\langle . \rangle_j$ is the average over modes j. The localisation parameter $\lambda^{(i)}$ indicates by how much the response in the low modal overlap limit deviates from equipartition of energy. It depends on the global behaviour of the system through the global mode shape statistics. It is related to subsystem irregularity and how well the subsystems are coupled. For narrow band excitation that contains few modes the dissipated power ratio varies substantially, since it depends on the detailed properties of relatively few, specific modes and it can be greater or less than unity. For broad band excitation that contains many modes, on the other hand, $\lambda^{(i)}$ becomes sensibly constant. It is generally then less than unity for unexcited subsystems and consequently

greater than unity for the driven subsystem. In general terms $\lambda^{(i)}$ becomes closer to unity for subsystems which are more irregular and is larger the better the coupling.

4.3. NUMERICAL EXAMPLES

Two examples are discussed below. The case of two coupled plates with irregular shapes is considered in [5], the response being seen to depend on the shape of the plates at low M.

4.3.1. Simply supported, rectangular plate.

As a simple (although possibly perverse) example, consider a uniform, rectangular, simply-supported plate. Define x and y axes along two edges, whose lengths are a and b. Let a line along $x=c$ (where c is arbitrary) divide the plate into two subsystems. These subsystems are well coupled, in that the transmission coefficient of the coupling is 1, whatever the angle of incidence. The region $0 \le x \le c$ forms subsystem 1. The mass-normalised mode shapes are

$$\phi_j(x,y) = \frac{2}{\sqrt{\rho ab}} \sin\left(\frac{m_j \pi x}{a}\right) \sin\left(\frac{n_j \pi y}{b}\right) \tag{16}$$

where m_j and n_j are integers and where ρ is the plate mass per unit area. The kinetic energy distribution factors are

$$\psi_{jj}^{(1)} = \frac{c}{a} - \frac{1}{2m_j\pi} \sin\frac{2m_j\pi c}{a}; \quad \psi_{jj}^{(2)} = 1 - \frac{c}{a} + \frac{1}{2m_j\pi} \sin\frac{2m_j\pi c}{a} \tag{17}$$

The first terms are constant, independent of mode number j, while the sin terms are small, mode number dependent and reflect the fact that the central region of the plate typically has a slightly greater vibration amplitude than average (due to the fact that the ends are supported and have zero amplitude). To a very good approximation then

$$\psi_{jj}^{(1)} = \frac{c}{a}; \quad \psi_{jj}^{(2)} = 1 - \frac{c}{a} \tag{18}$$

The localisation factors λ of equation (15) equal 1, and the global modal description therefore predicts that equipartition of energy applies, as one would expect. Note also that the asymptotic modal densities of the subsystems are indeed in the ratio predicted by $\psi_{jj}^{(1,2)}$, since the asymptotic modal density of a plate is proportional to its area.

4.3.2. Three coupled plates

Consider the three, edge-coupled, rectangular plates shown in Figure 1. The material is steel (E = 185 GN/m^2, ρ = 7630, ν = 0.3). All plates have the same thickness (1.6 mm) and the sides are of lengths L_w = 280 mm, L_h = 355 mm and L_d = 310 mm. The modes are found using the methods described in [6]. The upper, middle-sized plate is excited, subsystems 2 and 3 being the smaller and larger plates.

The distribution factors $\psi_{jj}^{(i)}$ for the first 100 modes are shown in Figure 2. Clearly most modes tend to be localised within one subsystem or another, or maybe within two, and there seems to be a somewhat greater degree of localisation at higher frequency.

Figure 3 shows the powers dissipated in each subsystem per unit input power. These powers are frequency averaged over a 200 Hz bandwidth with centre frequencies at multiples of 50 Hz. Also shown are the dissipated powers that correspond to equipartition of energy as predicted by SEA. The bandwidth contains from 11 to 14 modes (12.75 on average). SEA tends to underpredict the response of the driven subsystem and to overpredict the responses of the undriven subsystems - the localisation of the modes acts to trap energy within the excited subsystem. This is especially true at higher frequencies, where the modes tend to be more localised. There are significant fluctuations with frequency, these being proportionately larger for the undriven subsystems, whose responses are dominated by fewer, non-localised modes. When averaged over the whole frequency band the localisation parameters $\lambda^{(i)}$ of equation (15) are respectively 1.780, 0.658 and 0.565. When averaged only over those higher frequency modes above 500 Hz the corresponding values are 1.9716, 0.5993 and 0.4388.

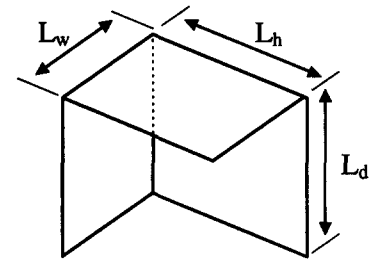

Figure 1. System comprising 3 coupled plates

5. CONCLUDING REMARKS

Energy density differences in a system arise, in wave terms, when transmission coefficients differ from unity and, in modal terms, from localisation of global modes. Both these tend to contain energy in the driven subsystem. Effects tend to decrease for subsystems that are well coupled and irregular and if the system has many subsystems, many of which are excited. Equipartition of energy has its roots in statistical mechanics. The lack of equipartition in the steady-state of SEA arises from the input power being non-zero, there being small but non-zero damping. There are, however, some ameliorating features regarding SEA predictions in this strong coupling limit. Equipartition provides an upper bound on subsystem response, no dissipated power can be less than zero and the input power (and hence the total dissipated power) is known relatively accurately. Other, possibly larger, errors arise from other sources, e.g., inaccuracy in the system parameters and finite frequency band averaging.

6. REFERENCES

1 Mace, B. R. (1993) *J. Sound Vib.* **166**, 429-461. The statistical energy analysis of two continuous one-dimensional subsystems.

2 Wester, E C N and Mace, B. R. (1996) *J. Sound Vib.* **193**, 793-822. Statistical energy analysis of two edge-coupled rectangular plates: ensemble averages.

3 Wester, E C N. (1996) *Proc. Internoise 96, Liverpool.* 2909-2914. A wave approach to the SEA of a coupled rectangular plate system involving a non-uniform boundary condition.

4 Wester, E C N and Mace, B. R. (1997) *Proc. Internoise 97, Budapest.* 643-646. A wave analysis of energy flow between two, non-uniformly coupled rectangular plates.

5 Mace, B R and Shorter, P J. (1997) *Proc. IUTAM Symp. on SEA, Southampton.* Irregularity, damping and coupling strength in S.E.A.

6 Shorter, P J and Mace, B R. (1997) *Proc 5'th Int Con Sound Vib., Adelaide.* Energy flow models from finite elements: an application to three coupled plates.

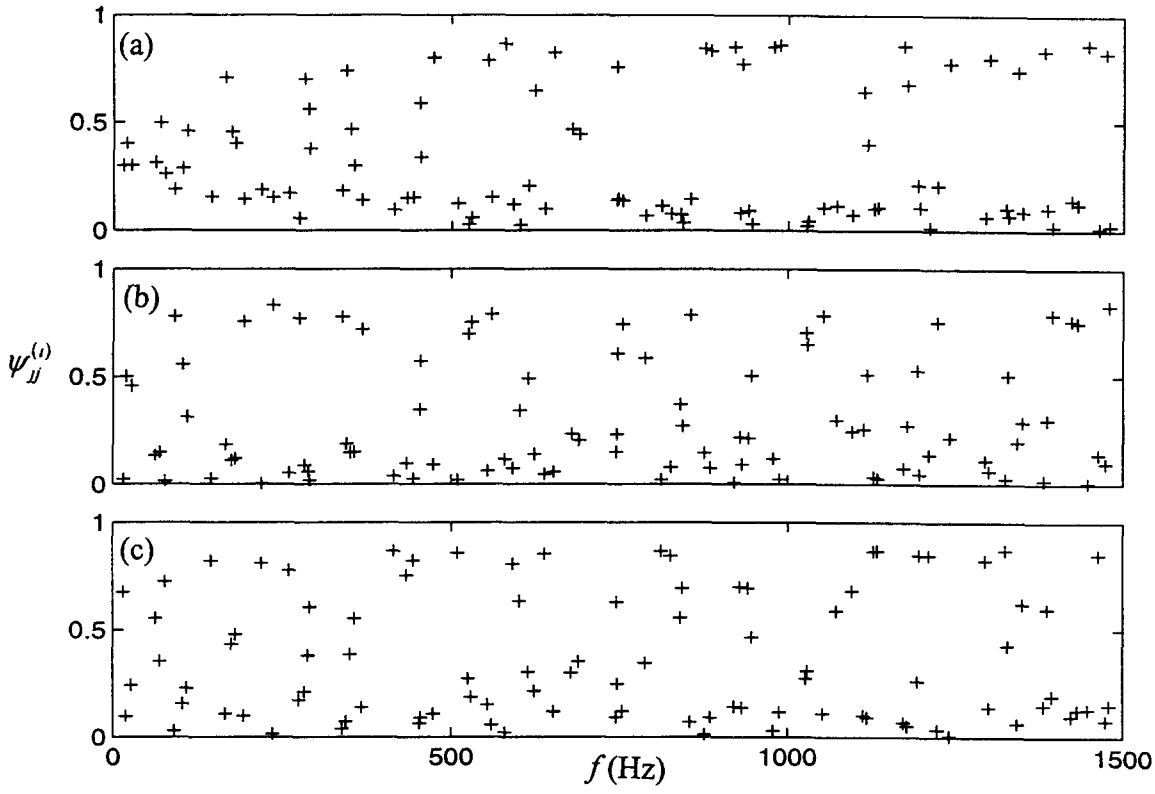

Figure 2. Distribution factors $\psi_{jj}^{(i)}$ for (a) excited, (b) smaller and (c) larger plate.

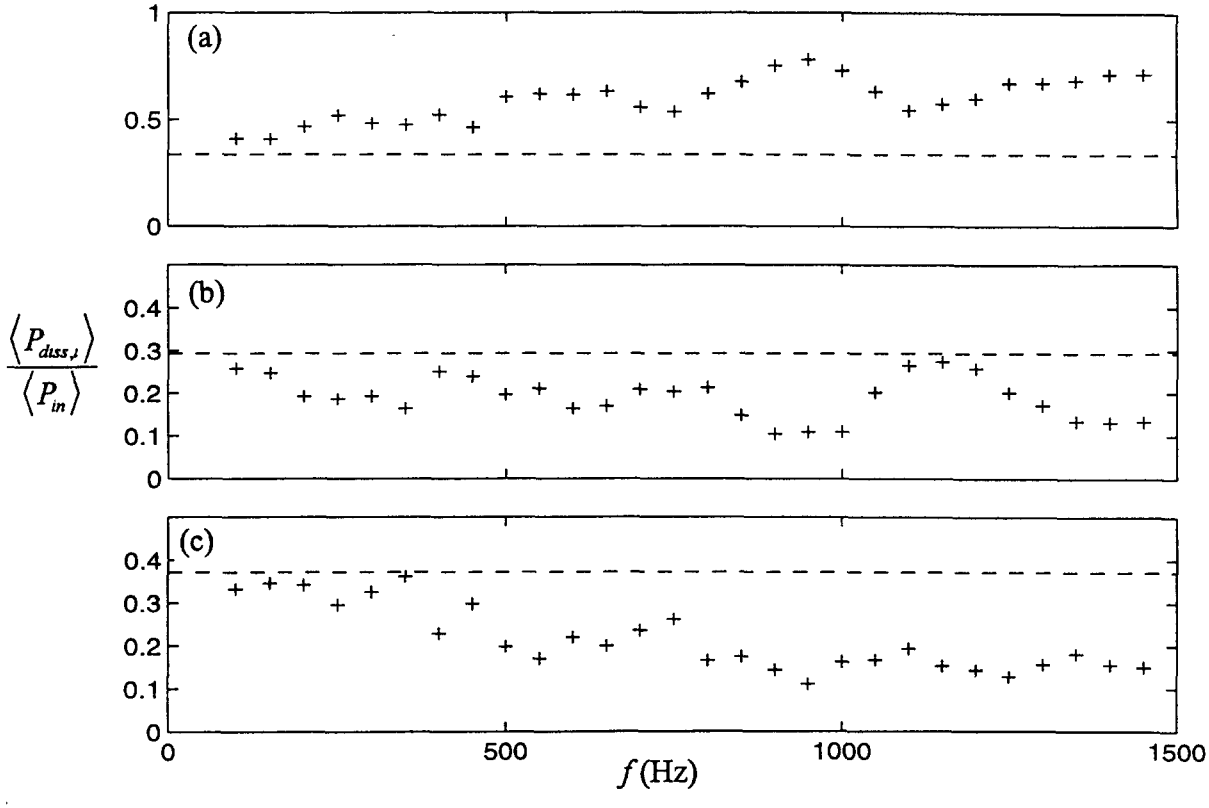

Figure 3. Dissipated power per unit input power for (a) excited, (b) smaller and (c) larger plate: + 200 Hz averages, - - - SEA, equipartition estimates.