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SIMULATION OF FRICTION IN OSCILLATING MECHANISMS

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Friction is present in all mechanical systems that incorporate sliding contacts, but is often ignored when developing models to study system behaviour. This is largely because of the difficulties that arise in the numerical simulation of friction models, the equations of motion for which are discontinuous and implicit. The problem is exacerbated when considering oscillatory mechanisms, where multiple velocity zero crossings, and thus friction discontinuities, occur. However the inclusion of friction effects in the models of these systems can be very important when the models are used to quantitatively (rather than qualitatively) represent the system; for example, for the design and testing of system controllers, and for system parameter estimation. This paper begins with a brief summary of the types of friction models currently used. The difficulties of solving the models using numerical simulation are also discussed. A new method is then introduced, using the example of a simple oscillating mechanism, which has the potential to reduce computing time and to avoid certain other problems encountered with friction simulation.

1. INTRODUCTION

Friction is an important phenomenon in mechanical systems, but due to its complex behaviour, it is often ignored or grossly simplified in mathematical models. These simplified models are in many cases adequate, but in certain applications they are not and a better understanding of how friction affects the resulting motion is required. An example of the latter is the design of the controller for a robot. Friction in a robot's joints can consume a major portion of the applied force and can introduce significant transient behaviour near zero velocity, resulting in fidelity limits in position and force control [1]. With increased performance demands on such systems friction becomes increasingly difficult to ignore in analysis and design. As a consequence the problems associated with including friction in system analysis must be addressed. There are fundamentally two problems; firstly developing an accurate model for the friction, and then overcoming the difficulties in simulating a set of equations that include friction models. A brief discussion of the development of the classic friction model is given below, but the main purpose of this paper is to investigate a way of efficiently simulating a system model that includes friction. More importantly, the paper addresses the problem of simulating friction in a system that has an acceleration dependent normal force.

Friction opposes relative sliding motion, and the classic model assumes that the friction force is proportional to the normal force, and is independent of contact area [2]. This model was known to Leonardo da Vinci, but remained hidden in his notebooks for centuries. The model was rediscovered by Amontons in 1699 and was developed further by Coulomb among others, including Morin who introduced the idea of static friction in 1833. In 1866 Reynolds introduced the equation for viscous fluid flow, which lead to the model for viscous damping. A notable difference between friction and viscous damping is that friction, in terms of the classic model, is a discontinuous function and viscous damping is not. They are, however, both energy loss mechanisms. For this reason an equivalent viscous damping term is commonly used to represent friction in modern-day numerical simulations, and thus avoids the difficulties inherent with friction. However, equivalent viscous damping does not capture all the dynamic behaviour of friction.

Four phenomena that are consistently observed in lubricated mechanisms at low velocities, and which are not taken into account in the classic model, are Stribeck friction, rising static friction, frictional memory and presliding displacement [3]. Stribeck friction is characterised by a negatively sloped, non-linear friction-velocity feature occurring at low velocities. Rising static friction is a phenomenon associated with the breakaway force (the transition from non-sliding to sliding), and is a function of the time spent at zero velocity and the force application rate. Frictional memory is the lag observed between changes in velocity or normal force and the corresponding change in friction force, and presliding displacement, as the name suggests, is the displacement of the contacts before true sliding commences. Presliding displacement is associated with elastic and/or plastic deformation of the contacting asperities. The above phenomena mean the instantaneous friction is not simply a function of instantaneous velocity and normal force, but is also a function of dynamic friction effects. For a full description of these phenomena and their associated models see [2].

In general, the more complex models for friction also suffer from the same simulation difficulties as the classic friction model. Thus, for simplicity, only the classic model will be used in this paper. It can be defined mathematically as,

$$F_f = \begin{cases} \eta\dot{x} + \mu_k \operatorname{sign}(\dot{x}) F_n & \dot{x} \neq 0 \\ \eta\dot{x} + \mu_k \operatorname{sign}(\mu_s) F_n & \dot{x} = 0 \ \& \ \mu_s > \mu_{s\max} \\ \eta\dot{x} + \mu_s F_n & \dot{x} = 0 \ \& \ \mu_s \leq \mu_{s\max} \end{cases} \tag{1}$$

where F_f is the friction force, η is the viscous damping coefficient, $\dot{x}$ is the relative joint velocity, F_n is the normal force, μ_k is the coulomb (or kinetic) friction coefficient, μ_s is the required static friction coefficient (determined from static equilibrium equations), and μ_{smax} is the actual breakaway static friction coefficient. The model is discontinuous, which is not necessarily the case in reality, but it is a reasonable approximation in many cases. Figure 1 shows a graphical representation of the classic friction model.

In the following sections the difficulties of numerically simulating a friction model are discussed, using the example of a pendulum. A pendulum was chosen because problems with friction simulation are more acute in oscillating systems, where the direction of motion is constantly changing. Also a pendulum has a normal force that is a function of the

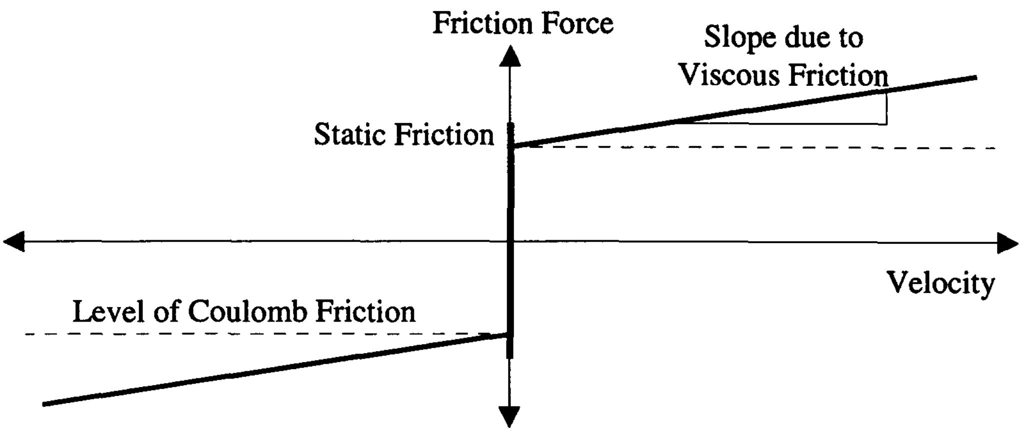

Figure 1. The classic friction model.

acceleration. A new method of structuring the equations to improve simulation of the pendulum is presented and the results are extended to a double pendulum.

2. NUMERICAL SIMULATION OF SYSTEMS WITH FRICTION

Simulation is a widely used tool for the investigation and prediction of behaviour of mechanical systems, but in general, simulation algorithms do not deal with discontinuities very well. This is due to the underlying assumption of most simulation algorithms that the models are continuous, and hence can be approximated by a Taylor series over a finite range. To help improve simulation efficiency most numerical integrators provide variable step size control through comparisons between different Taylor series approximations. Step size control does assist the numerical integrator when simulating a discontinuous system, by reducing the time step when a discontinuity is encountered until some error criterion is satisfied; however this process does not guarantee accuracy for the simulation of systems with friction [4]. A better approach is to determine the location of the discontinuity in time, integrate up to this time, and then stop and restart the integration on the other side of the discontinuity [5]. The method takes more time but is less prone to error.

The classic friction model is discontinuous at zero relative velocity, and at this crossover point three different events are possible: a joint may become stuck, a joint may become free or a joint may pass through zero velocity and keep moving. When a joint becomes stuck or free a degree of freedom is removed or added to the system respectively. Both events are potentially discontinuous because the friction force jumps between its static and kinetic values, and these may be considerably different in magnitude. When a joint moves through zero velocity the event is always discontinuous, because the friction force changes direction. Wiercigroch [6] addressed the difficulty associated with simulating across zero velocity by "softening" the discontinuity. This was done by using a function with a large but finite gradient through the zero velocity region, and effectively eliminated the static friction component of the friction model. As a result, models of this type exhibit creep; that is, small applied forces, which are less than the required breakaway force, result in low but steady velocities. Thus this approach is not useful when trying to study stick-slip behaviour.

Another major problem with simulating friction occurs when the normal force acting at a joint is a function of the joint's position, velocity and acceleration. This is often referred to as load-dependent friction [2]. When the direction of the normal force is constant within a local co-ordinate frame, the normal force is a linear function of the acceleration terms. The equations of motion are then explicit and can be written in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{2}$$

which can be directly dealt with using the stop/start simulation technique recommended earlier. Equation (2) is valid when modelling friction in translational joints, such as X-Y positioning systems, and for axially loaded bearings when the axial load can be considered independent of radial load [5]. However, if the direction of the normal force is not constant in a local co-ordinate frame, then the normal force will be a function of acceleration. In this case the equations of motion are implicit in the accelerations and can only be written in the form

$$\mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, t) = 0 \tag{3}$$

To solve for the accelerations at each time step of the integration an iterative procedure, such as the Newton Raphson method, must be used. This drastically reduces the speed of simulation, and may introduce significant problems in the uniqueness of solution when the rigid-body assumption is made [7]. It may also generate numerical stability problem when stiff numerical integrators are used.

As an example, consider the pendulum shown in Figure 2.

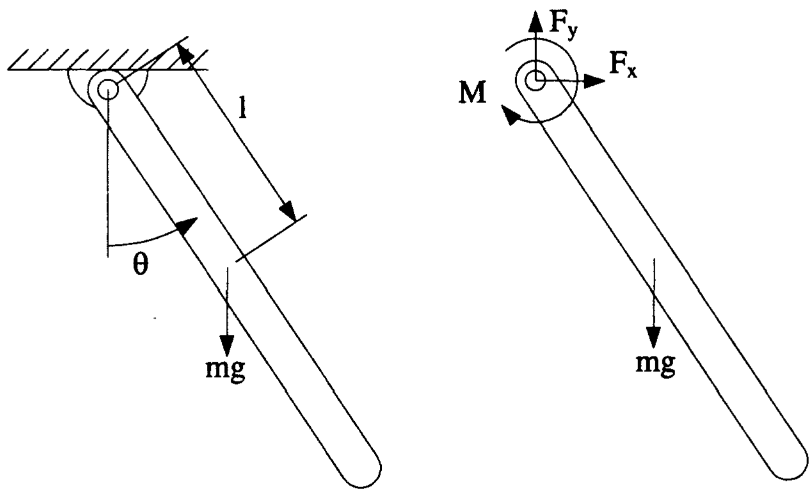

Figure 2. Physical Pendulum

If a friction moment is included, the equations of motion for the pendulum are given by

$$\left(ml^2 + I\right)\ddot{\theta} + mlg\sin(\theta) = -M \tag{4}$$

$$F_x = ml\left(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)\right) \tag{5}$$

$$F_y = ml\left(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\right) + mg \tag{6}$$

where

$$M = r_p \begin{cases} \eta\dot{\theta} + \mu_k \operatorname{sign}(\dot{\theta}) F_n & \dot{\theta} \neq 0 \\ \eta\dot{\theta} + \mu_k \operatorname{sign}(\mu_s) F_n & \dot{\theta} = 0 \ \& \ \mu_s > \mu_{s\max} \\ \eta\dot{\theta} + \mu_s F_n & \dot{\theta} = 0 \ \& \ \mu_s \leq \mu_{s\max} \end{cases} \tag{7}$$

$$F_n = \sqrt{F_x^2 + F_y^2} \tag{8}$$

$$\mu_s = \frac{-l\sin(\theta)}{r_p} \tag{9}$$

r_p is the radius of the support pin and μ_k is the kinetic friction coefficient.

It can be seen that F_n is a function of position, velocity and acceleration and consequently the pendulum's equations of motion can only be written in the form of equation (3). Note, however, if friction is ignored, M=0, and the equations can be expressed in the explicit form of equation (2).

3. EXPLICIT SYSTEM REPRESENTATION

The cost, in terms of simulation time, of solving the implicit equations of motion can be very high. The main reason is the iterative procedure used to determine the acceleration at each time step. To avoid this the equations can be differentiated with respect to time to obtain an explicit representation, the result being a set of equations that are linear in jerk (time rate of change of acceleration). An added advantage of representing the system in this form is that the acceleration is now available at each time step of the simulation, which may be important for decision processes in multiply degree-of-freedom systems. Note that, differentiation removes the implicit nature of the equations of motion, because the normal force components are linear functions of acceleration.

Differentiating the equations of motion of the pendulum produces

$$\left(ml^2 + I\right)\dddot{\theta} + m l g\cos(\theta)\dot{\theta} = -\dot{M} \tag{10}$$

$$\dot{F}_x = ml\left(\ddot{\theta}\cos(\theta) - 3\dot{\theta}\ddot{\theta}\sin(\theta) - \dot{\theta}^3\cos(\theta)\right) \tag{11}$$

$$\dot{F}_y = ml\left(\ddot{\theta}\sin(\theta) + 3\dot{\theta}\ddot{\theta}\cos(\theta) - \dot{\theta}^3\sin(\theta)\right) \tag{12}$$

where

$$\dot{M} = r_p \begin{cases} \eta\ddot{\theta} + \mu_k \operatorname{sign}\left(\dot{\theta}\right)\dot{F}_n & \dot{\theta} \neq 0 \\ \eta\ddot{\theta} + \mu_k \operatorname{sign}(\mu_s)\dot{F}_n & \dot{\theta} = 0 \text{ \& } \mu_s > \mu_{s\max} \\ \eta\ddot{\theta} + \left(\dot{\mu}_s F_n + \mu_s \dot{F}_n\right) & \dot{\theta} = 0 \text{ \& } \mu_s \leq \mu_{s\max} \end{cases} \tag{13}$$

$$\dot{F}_n = \frac{F_x\dot{F}_x + F_y\dot{F}_y}{\sqrt{F_x^2 + F_y^2}} \tag{14}$$

$$\dot{\mu}_s = \frac{-l\cos(\theta)\dot{\theta}}{r_p} \tag{15}$$

These equations are linear in terms of their highest derivatives and thus the jerk can be found explicitly by rearranging the equations.

There is a hidden cost to this procedure, and it is due to the resulting accelerations being discontinuous. This means, at the start of the simulation and whenever a discontinuous change in force occurs, the acceleration must be determined using of an iterative procedure. As a consequence, a simulation method that locates discontinuities must still be used. However, as this type of solution method has already been recommended for general friction simulation, it does not introduce any additional difficulties.

4. SIMULATION OF A PENDULUM

Simulations of the pendulum's motion were performed in Matlab for both implicit and explicit model representations. The solution method used was based on a Runge-Kutta 4-5 algorithm and was able to locate discontinuities associated with friction. The decision process used to determine the friction regime (kinetic or static) was based around a comparison between the maximum potential static friction force that can act at a joint and the applied forces acting at that joint. An additional criterion, that the velocity must be close to zero, was used to determine whether a transition between friction regimes was possible.

The results of the simulations for the implicit and explicit equations of motion were identical within the simulation tolerances, and an example set of results is shown in Figure 3 (acceleration is taken from the explicit simulation). Interesting features to note are the discontinuities encountered in the acceleration whenever the velocity passes through zero. These are associated with the change in the direction of the friction force as the velocity changes direction.

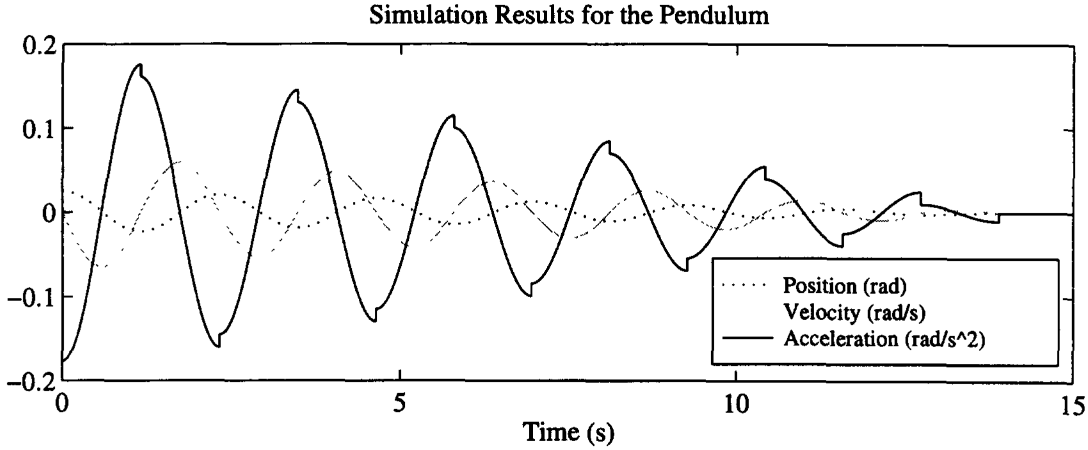

Figure 3. Simulation results for the Single Pendulum.

5. SIMULATION OF A DOUBLE PENDULUM

A schematic representation of a double pendulum is shown in Figure 4, where θ is the angular displacement of the top link, and β is the relative angular displacement between the top and bottom links. The implicit equations of motion for this configuration were also derived and then differentiated with respect to time to obtain an explicit representation of the system. The resulting equations are similar in form to those of the single pendulum and are not included here.

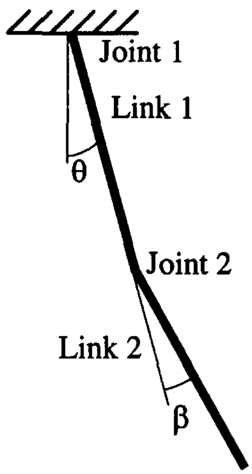

Figure 4. Double Pendulum

Simulations were performed for both the implicit and explicit representations, and once again the results were identical within simulation tolerance. An example set of results is shown in Figure 5 (the accelerations are taken from the explicit simulation). It can be seen that the accelerations of links 1 and 2 are highly discontinuous. Also evident is the strong coupling between the two links; in particular, a discontinuity encountered by one of the links affects the acceleration of both links. This is evident in the acceleration of link 1 when joint 2 becomes stuck 0.7 seconds into the simulation, and again 0.1 seconds later when joint 2 becomes free again.

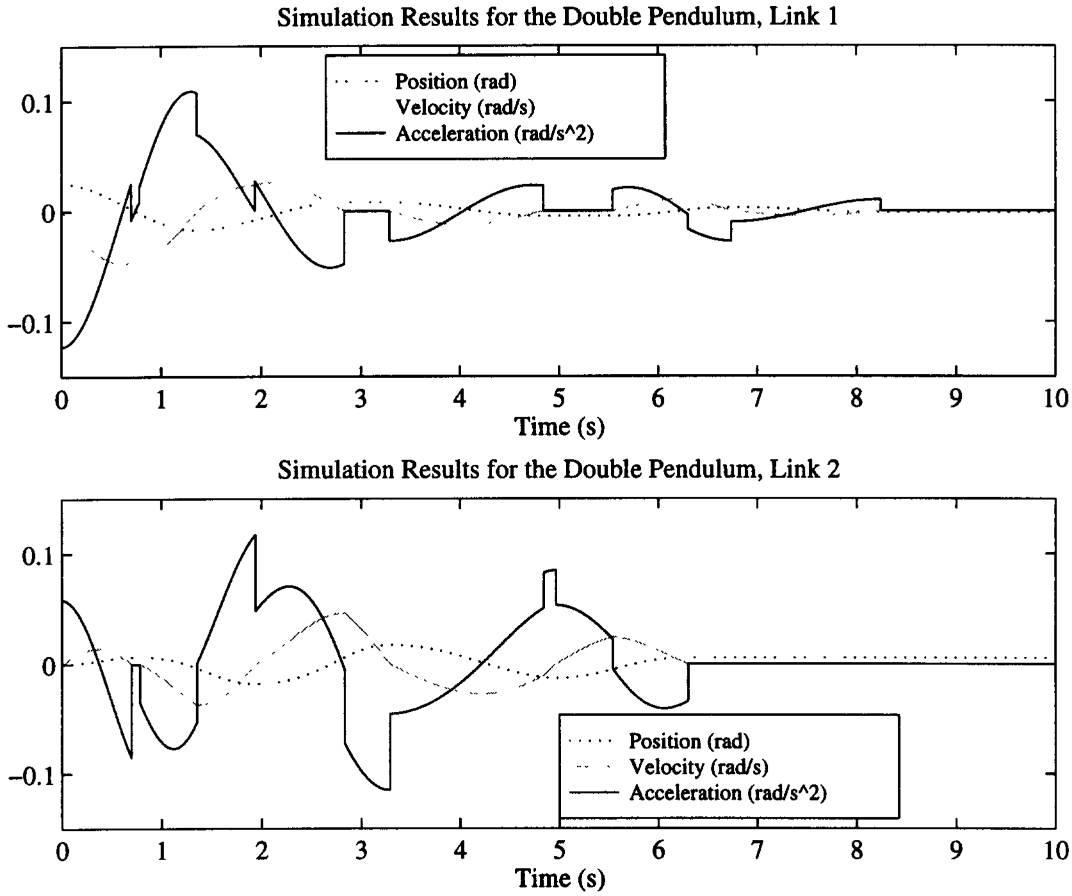

Figure 5. Simulation results for the Double Pendulum.

6. DISCUSSION

The simulation results confirm that the explicit and implicit system representations produce results that are identical within simulation tolerances. However, the times required to simulate each representation differ significantly. To illustrate this, the times required for the simulation runs shown in Figure 3 and 5, for both the explicit and implicit cases, are presented in Table 1. For a comparison the frictionless is also included. The last entry in the table is given in order to indicate the cost of locating the discontinuities (i.e. the zero velocity crossings). The times stated are for Matlab m-files running on a 200 MHz Pentium Pro PC.

	Simulation Times (seconds)	
Simulation Method	Pendulum	Double Pendulum
Explicit	14.66	38.84
Implicit	58.00	146.76
No Friction (ignoring velocity zero crossings)	1.76	2.58
No Friction (finding velocity zero crossings)	4.40	13.74

Table 1. Simulation times for the Single and Double Pendulums.

The first obvious point of note is that inclusion of friction into any system model increases the simulation time. For the double pendulum example, the explicit model is 15 times slower than the frictionless case, while the implicit model is 57 times slower. This increase in simulation time is due to locating the discontinuities and the iterative method required to find the accelerations. Both the implicit and explicit models must locate the discontinuities; thus the difference in the simulation times for these two cases is the result of how often each must solve the implicit acceleration equations. The explicit case need only do this at each discontinuity, but the implicit case must implicitly solve for the acceleration during each time step. As a result the explicit representation reduces the simulation time substantially. By comparing the last two entries in Table 1 the cost of locating the velocity zero crossings may be evaluated, and it can be seen that this may also be significant. The method used to locate the discontinuities for the above simulation results was fairly crude and this is an area for potential improvement. In general, the number of discontinuities encountered will dictate the simulation time for the explicit model.

Both the explicit and implicit cases depend on being able to determine the acceleration with sufficient accuracy at the start of the simulation and whenever discontinuities are encountered. Thus they are both prone to problems, such as invalid rigid-body assumptions, that can lead to non-unique solutions in the implicit representations. If the accelerations can not be found with sufficient accuracy or at all, then the simulation will fail. As a guide it is recommended that the acceleration should be found with tolerances of at least two orders of magnitude smaller than those used in the integration.

7. CONCLUSIONS

The inclusion of friction in system models is becoming more important in the analysis and design of mechanical systems. However, models including friction are not easily simulated due to friction's apparent discontinuous nature and the implicit models that may result. To address these problems, a method has been presented that allows explicit equations of motion for a system to be obtained. It has been shown that this representation may significantly reduce simulation times, which may have serious implications if the model is to be used for real time simulation applications. An extra advantage of this explicit representation is that the acceleration of each rigid-body is available during simulation.

8. REFERENCES

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