FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

WAVE LOCALIZATION IN HYDROELASTIC SYSTEMS

Abramian A.

Institute for Problems in Mechanical Engineering RAS,
St.Petersburg, Russia

The phenomenon of wave localization in hydroelastic systems leads to the strength concentration of radiation fields. The linear method considers the process of localization to be the formation of nonpropagation waves (trapped modes phenomenon). The presence of such waves in the total wave packet points to the existence of mixed natural spectrum of differential operators describing the behaviour of hydroelastic systems. The problem of liquid and oscillating structure interaction caused the trapped modes phenomenon has been solved (membranes, dies, plates and different liquid models). The areas of wave localization have been determined. This areas depend on both the parameters of oscillating structures and the liquid waveguide parameters (the linear dimensions of the channel filled with liquid; the height from the bottom to the liquid surface; the liquid density gradient according to the height, etc.).

1 INTRODUCTION AND STATEMENT OF THE PROBLEM

The linear problem on wave propagation in the finite depth water has been studied in [1]. The bottom topography influence on trapped modes characteristics thoroughly investigated in [2]. The problem on the standing wave formation localized in the area of dynamical inclusions on the bottom of a channel is seemed to be very interesting and of practical use. This paper is devoted to solving this problem.

There is a massive rigid die on the elastic foundation on the bottom of a three-dimensional channel filled with an ideal noncompressible liquid. Cartesian axes are chosen so that y is directed vertically upwards and x and z in the plane of the unperturbed bottom. The coordinate origin coincides with the middle of the die (its width $2a$). The motion of the liquid is described by velocity potential $\Phi(x, y, z, t)$ in the linear theory. Velocity potential is found from the following boundary problem:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \text{ in the domain occupied by liquid,} \tag{1}$$

$$\Phi_{tt} + g\Phi_y = 0, \text{ on the free surface } y = h, \tag{2}$$

$$\Phi_y = \begin{cases} w_t & \text{on the moving part of boundary } |x| < a \\ 0 & \text{on the rigid part } |x| > a, \ y = 0 \end{cases} \tag{3}$$

Where g is the gravity acceleration, h is the depth of the liquid in the channel, w is the die displacement determined by the equation

$$Mw_{tt} + Cw = \rho \int_{-a}^{a} (\Phi_t + gw)\, dx, \quad |x| < a, \quad y = 0. \tag{4}$$

Where M is die mass per unit length, C is the die rigidity, ρ is the liquid density. Let us consider the die as a wave surface having a constant cross-section in the z-direction and oscillating with frequency ω. The solution of problem (1)-(4) being found in the form:

$$\begin{aligned} w(x, t) &= \text{Re}\,\{w_0 e^{\mathrm{i}(mz-\omega t)}\}, \\ \Phi(x, y, z, t) &= \text{Re}\,\{\varphi(x, y) e^{\mathrm{i}(mz-\omega t)}\}. \end{aligned} \tag{5}$$

Substituting (5) into (1)-(4) we satisfy the following boundary value problem

$$\begin{cases} \varphi_{xx} + \varphi_{yy} = m^2\varphi, & -\infty < x < +\infty, \ 0 < y < h \\ \varphi_y = \frac{\omega^2}{g}\varphi, & y = h \\ \varphi_y = \omega w_0, \ |x| < a, & \varphi_y = 0, \ |x| > a \quad y = 0 \end{cases} \tag{6}$$

$$(\tilde{C} - M\omega^2) w_0 = \rho\omega \int_{-a}^{a} \varphi(x) dx, \quad |x| < a \quad y = 0. \tag{7}$$

Where m is wavenumber, $\tilde{C} = C - 2a\rho g$. For $\varphi(x, y)$ we require the "localization property"

$$\int_0^h \int_{-\infty}^{+\infty} |\nabla\varphi|^2 dx dy + \int_{-\infty}^{+\infty} \varphi^2(x, h) dx < +\infty. \tag{8}$$

In this case the energy of the wave mode is finite per unit length of the z-axis, and the mode (or wave) is said to be trapped. Without the die the problem (6)-(7) leads only to continuous spectrum starting with $\omega_b = \sqrt{gm \tanh mh}$ which is the cut-off frequency.

2 FREQUENCY EQUATION

The Green function allows us to express the solution of problem (6) in the form

$$\varphi(x, y) = -\omega w_0 \int_{-a}^{a} G(x - \eta, y, \omega) d\eta. \tag{9}$$

Substituting (9) into (7) one can have the following equation to determine the natural frequency

$$\tilde{C} - M\omega^2 = -\rho\omega^2 \int\int_{-a}^{a} G(x - \eta, 0, \omega) d\eta dx. \tag{10}$$

The corresponding Green function $G(x,y,\omega)$ is

$$G(x,y,\omega)=\begin{cases} A_0(\omega)e^{-\zeta_0|x|}+\sum_{k=1}^{\infty}A_k(\omega)e^{-\zeta_k|x|}, & \omega<\omega_b \\ iA_0(\omega)e^{i\zeta_0|x|}+\sum_{k=1}^{\infty}A_k(\omega)e^{-\zeta_k|x|}, & \omega>\omega_b, \end{cases} \tag{11}$$

where

$$A_0=-\frac{2\xi_0\cosh y\xi_0}{\zeta_0(\sinh 2h\xi_0+2h\xi_0)},\quad A_k=-\frac{2\xi_k\cos y\xi_k}{\zeta_k(\sin 2h\xi_k+2h\xi_k)};$$

ξ_0 is the positive root of the below equation

$$g\xi\tanh h\xi=\omega^2,\quad \zeta_0=\begin{cases}\sqrt{m^2-\xi_0^2}, & \omega<\omega_b\\ \sqrt{\xi_0^2-m^2}, & \omega>\omega_b,\end{cases}$$

ξ_k is the positive root of the below equation

$$g\xi\tan h\xi=-\omega^2,\quad \zeta_k=\sqrt{\xi_k^2+m^2},\quad k=1,2,\dots .$$

Remark: It has been shown that having been integrated, the series obtained in the right parts of (9) and (10) converge.

Analyzing the obtained solution of the spectrum problem (9)-(11) one can come to the following conclusions.

3 MAIN RESULTS

1. In the region of low frequency ($\omega<\omega_b$) only localized waves can exist, and the condition of lack of radiation (8) is always met. The analysis of frequency equation (10) shows that its right part is positively determined increasing function. In this case, the unique natural frequency can exist, when the following inequality is fulfilled

$$C>2a\rho g.$$

Then $\omega_{\mathrm{I}n}$ satisfies the inequality

$$0<\omega_{\mathrm{I}n}<\min\left\{\omega_b,\sqrt{\frac{C-2a\rho g}{M}}\right\}.$$

2. Above the cut-off frequency ($\omega>\omega_b$) the Green function is the complex, with a consequent formation of surface travelling waves carrying the energy to infinity. The condition (8) is fulfilled when

$$\zeta_0=\frac{\pi n}{a},\quad n=0,1,\dots,$$

which is equal to the existence of the trapped wave discrete spectrum above the cut-off frequency

$$\omega_{\mathrm{II}n}^2=g\sqrt{m^2+\pi^2n^2/a^2}\tanh\left(h\sqrt{m^2+\pi^2n^2/a^2}\right),\quad n=0,1,\dots .$$

In order that some trapped mode $\omega_{\mathrm{II}n}$ is the eigenvalue, it is necessary that it satisfies the frequency equation (10).

When the frequency is equal to $\omega_{\mathrm{II}n}$ than the right part of equation (10) has negative values, and theoretically always possible to choose the parameters for the mechanical system C, M and a so that the equation of frequency can turn into the identity, and $\omega_{\mathrm{II}n}$ is the natural frequency

3. The existence of the discrete spectrum at certain parameters of the discrete spectrum of frequency leads to possible resonance oscillations in the given die-liquid system under the action of harmonic forces which frequency coincide with the system natural frequencies.

4 LIQUID OSCILLATION IN A PLANE CHANNEL WITH RIGID SIDES AND AN INCLUSION IN THE FORM OF A RIGID DIE.

A die of mass m and width $2l$ executes vertical oscillation on a plane channel filled with inviscid compressible liquid of density ρ. The speed of sound in the liquid c_0. The general equation of the die motion and the state equation has the form:

$$-(P_{xx}+P_{yy})=\left(\frac{\omega}{c_0}\right)^2 P, \qquad -\infty<x<\infty, \qquad 0\le y<H$$

$$-M\,\omega^2\,w_0=\int_{-l}^{+l} P(H,\xi)\,d\xi \tag{12}$$

$$Q=\int_{-l}^{+l} P(H,\xi)\,d\xi, \quad P_{y|y=0}=0, \quad P_{y|y=H}=\begin{cases}\rho_0\,\omega^2\,w_0, & |x|<l\\ 0, & |x|>l\end{cases}$$

where $w_0\,e^{i\omega t}$ — die displacement; H — channel height; $P\,e^{i\omega t}$ — liquid radiation pressure.

To get the uniqueness of a solution one should use the concepts of limiting absorption. Using Fourier transform in the equation along coordinate x one can get for $P(H,\xi)$:

$$P(H,\xi)=\rho_0\,\omega^2\,w_0\int_{-l}^{+l} G(H,|x-\xi|)\,d\xi \tag{13}$$

where

$$G(H,|x-\xi|)=-0.5\,i\frac{c_0}{H\,\omega}e^{-i\omega|x-\xi|/c_0}+\frac{1}{H}\sum_{k=1}^{N}\frac{1}{\gamma_k}e^{-|x-\xi|\gamma_k}$$

$$\gamma_k=\left(\left(\frac{\pi\,k}{H}\right)^2-\left(\frac{\omega}{c_0}\right)^2\right)^{1/2}$$

The Green function is complex because the system has travelling waves that transfer the energy of the oscillating die to infinity. With every other travelling wave harmonic the frequency ω is translated above the corresponding boundary frequency $\omega_k = \pi\, c_0/H$.

$$P(H,x) = \begin{cases} \rho_0 \omega^2 w_0 \left[-i \frac{c_0^2}{H\omega^2} \sin \frac{\omega l}{c_0} e^{-i\omega x/c_0} + \frac{2}{H} \sum\limits_{k=1}^{\infty} \frac{1}{\gamma_k^2} e^{-\gamma_k x} \mathrm{sh}\gamma_k l \right], & x \geq l \\ \rho_0 c_0^2 \frac{w_0}{H} \left[2\frac{\omega^2}{c_0^2} \sum\limits_{k=1}^{\infty} \frac{1}{\gamma_k^2} (1 - e^{-\gamma_k l} \mathrm{sh}\gamma_k x) - \left(1 - e^{-i\omega a/c_0} \cos \frac{\omega x}{c_0}\right) \right], & 0 \leq x \leq l \end{cases} \tag{14}$$

Substituting it in the die motion equation one can have a frequency equation. Integrating the pressure along the die length, one obtains:

$$Q = 2\rho_0 c_0^2 \frac{w_0}{H} \left[l \frac{\omega^2}{c_0^2} \left(2 \sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} - \frac{c_0^2}{\omega^2} \right) - 2 \frac{\omega^2}{c_0^2} \sum_{k=1}^{\infty} \frac{1}{\gamma_k^3} e^{-l\gamma_k} \mathrm{sh}\gamma_k l + \frac{c_0}{w} \sin \frac{l\omega}{c_0}\, e^{-i\omega l/c_0} \right] \tag{15}$$

After some manipulations the expression has the form.

$$Q = -W_0 \left[2\rho_0 c_0 \mathrm{ctg} \frac{wH}{c_0} + 4\rho_0 \frac{\omega^2}{H} \sum_{k=1}^{\infty} \frac{1}{\gamma_k^3} e^{-\gamma_k l} \mathrm{sh}\gamma_k l - \frac{\rho_0 c_0^3}{Hw} \sin \frac{l\omega}{c_0} e^{-i\omega l/c_0} \right]. \tag{16}$$

One has another form of the frequency equation:

$$-M\omega^2 + 2\rho_0 c_0 l\omega \mathrm{ctg} \frac{\omega H}{c_0} + 4\rho_0 \frac{\omega^2}{H} \sum_{k=1}^{\infty} \frac{1}{\gamma_k^3} e^{-\gamma_k l} \mathrm{sh}\gamma_k l + \frac{\rho_0 c_0^3}{H\omega} \sin \frac{l\omega}{c_0} e^{-i\omega l/c_0} = 0 \tag{17}$$

The condition for existence of real roots in the equation is the lack of radiation behind the die, when $|x| > l$. The last expression is complex because of the two last components only. It is easy to verify that the imaginary part of them is determined by integrals that have the form:

$$\int_{-l}^{+l} e^{i\omega\xi/c_0}\, d\xi = 0,$$

$$\int_{-l}^{+l} e^{i\xi\left((\omega/c_0)^2 - (\pi/H)^2\right)^{1/2}}\, d\xi = 0, \tag{18}$$

$$\int_{-l}^{+l} e^{i\xi\left((\omega/c_0)^2 - (\pi n/H)^2\right)^{1/2}}\, d\xi = 0,$$

or

$$\sin \frac{l\omega}{c_0} = 0, \ldots, \sin \gamma_n l = 0$$

The last mentioned expression determines traveling waves of the pressure. It means that if they vanish, it will lead to a real frequency equation. Show the existence of parameters

l and H which allow us to find the real frequency ω from the equation. Solving the last equation step by step one obtains:

$$\omega_m = \frac{\pi m c_0}{l}; \quad (l/H)^2 = m^2 - k_1^2; \quad 4(l/H) = m^2 - k_2^2, \ldots; \quad N^2(l/H)^2 = m^2 - k_N \quad (19)$$

$$m = 0, 1, 2, \ldots; \quad k_j = 1, 2, \ldots; \quad j = 1, 2, \ldots N$$

The domain of the parameter m variation is found from the inequality determining the frequency ω_m location:

$$l\frac{N}{H} < m < l\frac{N+1}{H} \quad (20)$$

m — is integers in the last inequality. It needs to solve the system of equation for unknown l/H, k_1, k_2, k_{N-1} which must be expressed in terms of parameters m and k_N. In the general case it is rather awkward procedure. We can demonstrate it with some examples.

When $N = 0$ the location of the frequency is the following:

$$0 < \omega_m < \frac{\pi c_0}{H}$$

Considering $\omega_m = \frac{\pi m c_0}{l}$ one can obtain for the parameter m:

$$0 < m < \frac{l}{H}$$

Obviously the relation $l/H > 1$ must be correct since m is an integer. The last mentioned relation is a necessary condition for the real discrete spectrum existence between the first and the second boundary frequencies. A sufficient condition is the solution to the equation for mass M. When $\omega_m = \frac{\pi m c_0}{l}$ the expression for the radiation pressure has the form of a standing wave, because the travelling modes coefficient vanishes. When $y = H$, $x > l$, the pressure has the form:

$$P_m(H,x) = \frac{\rho_0 \omega_m^2 w_0^2}{H} \sum_{k=1}^{\infty} \frac{1}{\left(\frac{\pi k}{H}\right)^2 - \left(\frac{\omega_m}{c_0}\right)^2} e^{-\left(\left(\frac{\pi k}{H}\right)^2 - \left(\frac{\omega_m}{c_0}\right)^2\right) x^{1/2}} \mathrm{sh} \left(\left(\frac{\pi k}{H}\right)^2 - \left(\frac{\omega_m}{c_0}\right)^2\right)^{1/2} \quad (21)$$

Having determined the frequencies (name them trapping ones), one can find the value of m, when these frequencies become natural. It is from the equation one finds:

$$M_m \omega_m^2 = 2\rho_0 c_0 l \omega_m \mathrm{ctg} \frac{\omega_m H}{c_0} + \frac{4\rho_0}{H} \omega_m^2 \sum_{k=1}^{\infty} \frac{1}{\gamma_{km}^3} e^{-\gamma_{km} l} \mathrm{sh} \gamma_{km} l \quad (22)$$

where

$$\gamma_{km} = \frac{\pi}{l} \left(\frac{k^2 l^2}{H} - m^2\right)^{1/2}, \quad m = 1, 2, \ldots$$

The domain of integer value of m must ensure the inequality:

$$\frac{lm}{H\pi^2}\sum_{k=1}^{\infty}\frac{1-e^{-\pi\sqrt{k^2(l/H)^2-m^2}}}{[k^2(l/H)^2-m^2]^{3/2}}+\text{ctg}z_1\geq 0,\quad 0<z_1<\pi \tag{23}$$

¿From one can find z_1^* such that the relation becomes zero:

$$\frac{\pi}{2}<z_1^*<\pi$$

Here, the natural frequencies are located in the domain:

$$0<\omega_m<\frac{\pi m c_0}{H}$$

It is one value of m only exits ($m=1$) and one value of the discrete frequency only when $l/H=2$.

$$\omega_1=\frac{\pi c_0}{l}$$

If $l/H>2-m$ real discrete frequencies exist. The value of m is located in the domain:

$$0<m<m_*$$

where m_* is defined by the relation a/H previously. When $N=1$ the real discrete frequencies are:

$$\omega_2=\frac{2\pi c_0}{l},\qquad l/H=\sqrt{3}$$

$$\omega_3=\frac{3\pi c_0}{l},\qquad (l/H)_1=\sqrt{5};\qquad (l/H)_2=\sqrt{8}$$

The existence of the real discrete natural frequencies in the region of the continuous spectrum is proven. It should be noted that, if a spring of rigidity c is connected to the die it will change the domain of values m that satisfy the equation. In that case one can always find some values of parameters m and c that satisfy, if $n=0$. Besides, there will be some other values of m which satisfy the inequality:

$$\omega_m<\pi c_0/H\qquad (l/H>1)$$

5 ELASTIC NONLINEAR SYSTEMS

Consider the following simplified model (the beam with nonlinear and linear inclusions)

$$Du_{xxxx}+mu_{tt}+Ku+f_\epsilon(x)u+\gamma g_\epsilon u^3=P(t,x) \tag{24}$$

where $x\in(-\infty,\infty)$. To simplify, firstly let us assume the external force $P=0$. It is well known that, in the linear case ($g=0$), for appropriate coefficients f, there exist "localized" and almost localized modes [3] describing special beam oscillating. For example, if we take two inclusions $f=\delta(x)+\delta(x-l)$, then one obtains the solution of the form

$$u=[A\cos(\omega t)+B\sin(\omega t)]\Psi(x,\omega) \tag{25}$$

where $a(\omega) = (m\omega^2 - K)^{\frac{1}{4}}$, $\Psi = \exp(-a|x|) + \exp(-a|x-l|) + \lambda(\omega)(\cos(ax) + \cos(a(x-l)))$ and λ is small, if the "resonance condition" is satisfied: $la \approx \pi(2n+1), n \in \mathbf{Z}$. This mode is the sum of exponentially damping contribution and the small harmonic tail $\cos(ax) + \cos(a(x-l))$. The pure localized mode corresponds to the case $\lambda = 0$.

The aim of this subsection is to investigate nonlinear evolution of such modes and show the possibility of the coherent structure formation.

6 CASE OF SINGLE MODE

We begin, to simplify, with the case of a single almost localized mode (2). The natural and classical approach for small nonlinearities, (assume, γ is a small parameter), is to suppose that these amplitudes depend on slow time $\tau = \gamma t$. Then one can apply the well known Whitham principle [4]. For the localized modes, there exists a simple approach leading to *ordinary* differential equations for slowly oscillating amplitudes $A(\tau)$ and $B(\tau)$ [5].

Let define the Lagrangian

$$L = \frac{1}{2}\int_0^1 (mu_t^2 - Du_{xx}^2 - Ku^2 - f_\epsilon(x)u^2 - \frac{1}{2}\gamma g_\epsilon u^4)dx.$$

Substituting (2) into this expression and integrating over "fast" variables x and t, one obtain the averaging Whitham lagrangian $\bar{L}$. Up to small corrections, one finds

$$\bar{L} = c\frac{1}{2}[\omega(dA/d\tau B - dB/d\tau A) + c_1(A^2+B^2)^2 + c_2(A^2+B^2)].$$

The corresponding motion equations can be easily resolved and describe slow oscillations of A and B:

$$A = \alpha\cos(\bar{\omega}(\tau - \tau_0)), \quad B = \alpha\sin(\bar{\omega}(\tau - \tau_0)).$$

The frequency $\bar{\omega}$ is proportional to the quantity $A^2 + B^2$ (that does not depend on time τ).

Finally, we see that, in this simplest case, the slow time evolution is defined by an integrable Hamiltonian system with the single freedom degree.

7 CASE OF SOME LOCALIZED MODES

If, in system (1), the coexistence of some modes is possible, then the mathematical approach 2 holds but the results are more interesting. Suppose modes $\Psi_1, \Psi_2, \ldots \Psi_n$ coexist, with corresponding frequencies $\omega_1, \omega_2, \ldots$. Let $A_i(\tau)$ and $B_i(\tau)$ be the corresponding slowly oscillating modes. A nontrivial Whitham lagrangian (with nonlinear contributions that define the mode interactions) can occur if the time resonance condition $\sigma = \omega_{i_1} + \omega_{i_2} + \omega_{i_3} + \omega_{i_4} = 0$ or $\sigma = \omega_{i_1} + \omega_{i_2} - \omega_{i_3} - \omega_{i_4} = 0$ holds, for some ω_{i_k} from the frequency set.

Repeating the calculations from the previous point, we find that the Whitham lagrangian has the following general form

$$\bar{L} = \frac{1}{2}\sum_{i=1}^{n} c[\omega(dA_i/d\tau B_i - dB_i/d\tau A_i) + c_1(A_i^2+B_i^2)^2 + +c_2(A_i^2+B_i^2)] +$$

$$+\lambda^4 \sum_{i_1,i_2,i_3,i_4:\sigma=0} c_{i_1,i_2,i_3,i_4} A_{i_1} A_{i_2} A_{i_3} A_{i_4} + \ldots + d_{i_1,i_2,i_3,i_4} B_{i_1} B_{i_2} B_{i_3} B_{i_4}$$

where the second sum describe different contributions connected with the 4-order resonances $\sigma = 0$. The corresponding equations for A_i and B_i, in general, are not integrable. They can be analyzed by the KAM theory which can be used due to the condition $\lambda << 1$. Due to classical results, we know that they can describe quasiperiodic motions and moreover, different chaotic occurrences (for example, Arnold's web, homoclinic structures and etc.)

To conclude this subsection, let us note the important point. We can change the form of $\bar{L}$ by a frequency and coefficient f, g choice. Thus, in a sense, these coherent structures are controllable.

8 NONLINEAR ELASTIC BODIES INTERACTING WITH FLUID

As a simplest model, let us consider the beam (1) in a fluid flow. This flow can be considered (to simplify) in acoustic approximations

$$\Delta\phi(x,y) = c_0^{-2}\phi_{tt}, \quad \phi'_y(x,y)|_{y=0} = u_t,$$

and the force P in (1) has the form $P = \rho_0\phi_t(x,0)$. The influence of the fluid can be easily taken into account in the limit of large frequencies $\omega_j >> 1$. Then for ϕ one has the following asymptotic expansion $\phi = \sum_j \phi_j(x,\omega,y,\tau)\cos(\omega_j t)$, where $\phi_j \approx c\Psi_j(x)\exp(i\omega_j y) + \ldots$. Thus, one can show, that for high frequencies, the term P in (1) can replaced to the usual simple dissipative term cu_t.

Repeating all usual procedure of two scale expansion, we can obtain some complicated equations for amplitudes A_i, B_i. They contains, in addition to case 3, some dissipative contributions.

The investigation shows that the coherent structures lead to slowly damping quasi-periodic waves (with slowly oscillating amplitudes) which exist in the beam and the fluid.

The nontrivial structures occur as a result of the interaction of localized modes through their tails.

REFERENCES

[1] Ursell, F.: Mathematical aspects of trapping modes in the theory of surface waves; J. Fluid Mech. 183 (1987), 421–437.

[2] Bonnet-Ben Dhia, A-S., Joly, P.: Mathematical analysis of guided water waves; SIAM J. Appl. Math. 53 No. 6 (1993), 1507–1550.

[3] Abramyan, A.K.: Trapping modes in elastic structures, J. of Technical Acoustic. V.2, N3, p.5–18, 1995.

[4] Whitham G.B.: Linear and Nonlinear Waves, J. Wiley, N. Y. and etc. 1974.

[5] Molotkov I.A. & Vakulenko S.A.: Localized nonlinear waves, St.Petersburg University, 1988.