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DIFFRACTION OF BENDING-GRAVITATION WAVES ON CRACK IN ICE FIELD

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ABSTRACT

We consider a reservoir with a finite depth, covered with continuous ice. There is a crack in the ice plate. We study and simulate diffraction of bending-gravitation waves on the crack, using the theory of shallow water. We study also energy, transferred with these waves. Correctness of this simulation in a sense of satisfying to the principle of energy conservation is shown. We obtained, how different parameters have an influence on diffraction, energy flows and so on.

1. BASIC ASSUMPTIONS

1.1 FOR LIQUID

We consider a reservoir with a constant depth H and ideal, noncompressible liquid. The Z-axis is directed upwards. Velocity vector: $\vec{V} = \operatorname{grad}\Phi = (\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z})$.

Potential $\Phi(x, y, z, t)$ satisfys to the Laplace equation: $\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$. The kinematic condition at the surface is as follows: $\frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x} - \frac{\partial\Phi}{\partial z} + \frac{\partial\Phi}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial\zeta}{\partial t} = 0$ (1).

The kinematic condition at the bottom is as follows: $\frac{\partial\Phi}{\partial x}\frac{\partial H}{\partial x} + \frac{\partial\Phi}{\partial z} + \frac{\partial\Phi}{\partial y}\frac{\partial H}{\partial y} = 0$ (2).

1.2 FOR AN ICE PLATE

The equation of free bending plate vibrations is as follows: $\rho_1\frac{\partial^2\zeta}{\partial t^2} + \frac{D}{h}\Delta^2\zeta = 0$ (3),

where $\zeta(x, y, t)$- plate deflection , ρ_1- ice density , h - ice thickness, D - plate regidity.

1.3 CONTINUOUS ICE ON THE WATER

Now let us consider an ice plate, lying on the water. In this case we have to add to the right part of (3) a force, acting from the water on per unit plate square - liquid pressure p. For the pressure in waves we have: $\frac{p}{\rho} = -\frac{\partial \Phi}{\partial t} - gz$, where ρ - water density, Φ - velocity potential. In our case $z = \zeta$.

So we get a dynamic boundary condition for the system «continuous ice on the water»:

$D\Delta^2\zeta + \rho_1 h\frac{\partial^2\zeta}{\partial t^2} + \rho(\frac{\partial \Phi}{\partial t} + g\zeta) = 0$ (4). Thus we get our model in the form of the Laplace equation and 3 boundary conditions:

$$\left|\begin{array}{ll}
\Delta\Phi = 0 & (|x| < \infty, -H < z < \zeta) \\
D\Delta^2\zeta + \rho_1 h\frac{\partial^2\zeta}{\partial t^2} + \rho(\frac{\partial \Phi}{\partial t} + g\zeta) = 0 & (z = \zeta) \\
\frac{\partial \Phi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 & (z = \zeta) \\
\frac{\partial \Phi}{\partial x}\frac{\partial H}{\partial x} + \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial y}\frac{\partial H}{\partial y} = 0 & (z = -H)
\end{array}\right.$$

If H=const then from (2) we have at the bottom: $\frac{\partial \Phi}{\partial z} = 0$. Neglecting the small terms of (1) we get for the liquid surface: $\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z}$.

2. SHALLOW WATER

2.1 PLANE WAVES ON SHALLOW WATER

Now let us use the theory and the equations of shallow water. For harmonic vibrations it was shown [1], that this approach requires $H << \lambda$, where λ- wave length. The potential $\Phi(x,t)$ doesn't depend on y (because we concider plane waves) and on z.

For this simulation for the surface of reservoir covered with ice we have:

$$\left|\begin{array}{ll}
D\frac{\partial^4\zeta}{\partial x^4} + \rho_1 h\frac{\partial^2\zeta}{\partial t^2} + \rho(\frac{\partial \Phi}{\partial t} + g\zeta) = 0 & (5) \\
\frac{\partial \zeta}{\partial t} = -H\frac{\partial^2\Phi}{\partial x^2} & (6)
\end{array}\right.$$

If an ice plate isn't infinite, then outside it at the free water surface we have:

$\frac{\partial^2\Phi}{\partial x^2} - \frac{1}{gH}\frac{\partial^2\Phi}{\partial t^2} = 0$ (7). We assume the ice plate to be enough light, so that the depth of dipping: $h^* = h\frac{\rho_1}{\rho} << H$

2.2 CONTINUOUS ICE ON THE WATER BY STEADY-STATE HARMONIC VIBRATIONS

From (5) we get: $D\frac{\partial^4\zeta}{\partial x^4} + \rho g\zeta(1 + \frac{\rho_1 h}{\rho g\zeta}\frac{\partial^2\zeta}{\partial t^2}) + \rho\frac{\partial \Phi}{\partial t} = 0$. We designate

$(\frac{\rho g}{\rho_1 h})^{\frac{1}{2}} \equiv \omega_b$. The processes are harmonic, thus: $\Phi = \Phi(x)e^{-i\omega t}$, $\zeta = \zeta(x)e^{-i\omega t}$.

We will leave out $e^{-i\omega t}$ hereafter. Boundary condition (6) gives us:

$\zeta = -\frac{H}{i\omega}\frac{\partial^2 \Phi}{\partial x^2}$ (7). So the potential Φ must satisfy to the following equation by appropriate radiation condition: $\frac{D}{\rho g}\frac{\partial^6 \Phi}{\partial x^6} + (1-(\frac{\omega}{\omega_b})^2)\frac{\partial^2 \Phi}{\partial x^2} + \frac{\omega^2}{gH}\Phi = 0$ (8).

Let us look for ζ as a travelling bending wave $\zeta = \zeta_j e^{\pm ikx}$. Accordingly, we get a solution of the Laplace equation in the form of a surface wave: $\Phi \equiv \Phi_j e^{\pm ikx}$.

Let us consider potential Φ as follows : $\Phi \equiv \Phi_j e^{\alpha_j x}$. From the equation (8) we get the algebraic equation, which has 6 roots: α_1 and α_2 are imaginary, the other are complex and $\mathrm{Re}(\alpha_{3,4}) > 0$, $\mathrm{Re}(\alpha_{5,6}) < 0$. The potential will be as follows: $\Phi = \sum_{j=1}^{6}\Phi_j e^{\alpha_j x}$. Solving equation (8) for potential $\Phi = \Phi_i e^{i(kx-\omega t)}$ by shallow water we could also get the relation $\omega = \omega(k)$: $\omega^2 = \frac{Dk^4 + \rho g}{\rho_1 h + \frac{\rho}{k^2 H}}$ (9)

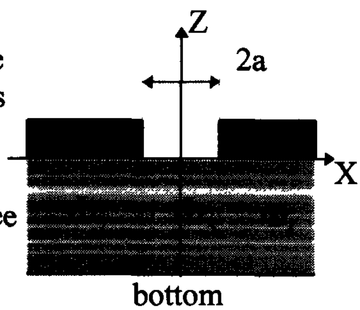
2.3 FREE WATER SURFACE BY STEADY-STATE HARMONIC VIBRATIONS

For a free water surface inside the crack we have (7). We consider steady-state harmonic vibrations, consequently we get: $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\omega^2}{gH}\Phi = 0$ (10). We could also get this equation from the equation (8), if $h \to 0$. Let us find a solution of (10) in the following form: $\Phi \equiv \Phi_j e^{\alpha_j x}$. Substituting Φ in (10) we get a characteristic equation, which has 2 roots. They are components of general solution of (10) :

$\alpha^{*2} + \frac{\omega^2}{gH} = 0$, $\alpha^*_{1,2} = \pm i\sqrt{\frac{\omega^2}{gH}}$ (11). The wave number for the free water surface will be as follows: $k^* = \frac{\omega}{\sqrt{gH}}$.

3. CRACK OF FINITE WIDTH IN ICE PLATE

Let us consider diffraction of bending-gravitation waves on a crack, which is parallel to the wave front. We set the origin of co-ordinates in the center of the crack. There is a plane wave, travelling from the right to the left with unit amplitude.



General solutions for 2 half-infinite plates and for the free water surface are as follows:

$$\Phi_1 = \sum_{j=1}^{6}\Phi_{1j} e^{\alpha_j x},\ \Phi_2 = \sum_{j=1}^{6}\Phi_{2j} e^{\alpha_j x},\ \Phi_3 = \sum_{j=1}^{2}\Phi_{3j} e^{\alpha^*_j x}$$

By $x \to \pm\infty$ potentials Φ_1, Φ_2 are limited, therefore: $\Phi_{13} = \Phi_{14} = 0$, $\Phi_{25} = \Phi_{26} = 0$ because $\mathrm{Re}(\alpha_{3,4}) > 0$, $\mathrm{Re}(\alpha_{5,6}) < 0$. There is no incident wave from the left, therefore: $\Phi_{22} = 0$. Thus, taking into account (11):

$$\begin{cases} \Phi_1 = e^{-ikx} + Re^{ikx} + \Phi_{15} e^{\alpha_5 x} + \Phi_{16} e^{\alpha_6 x} \\ \Phi_2 = Te^{-ikx} + \Phi_{23} e^{\alpha_3 x} + \Phi_{24} e^{\alpha_4 x} \\ \Phi_3 = \Phi_{31} e^{ik^* x} + \Phi_{32} e^{-ik^* x} \end{cases}$$

So, we have 8 unknown quantities. Therefore we have to get 8 boundary conditions by 4 on every free boundary of the ice plate: equality to zero of bending moments and shearing forces on each boundary .Using (7) we get: $\Phi_1\big|_{x=a} = \Phi_3\big|_{x=a}$;

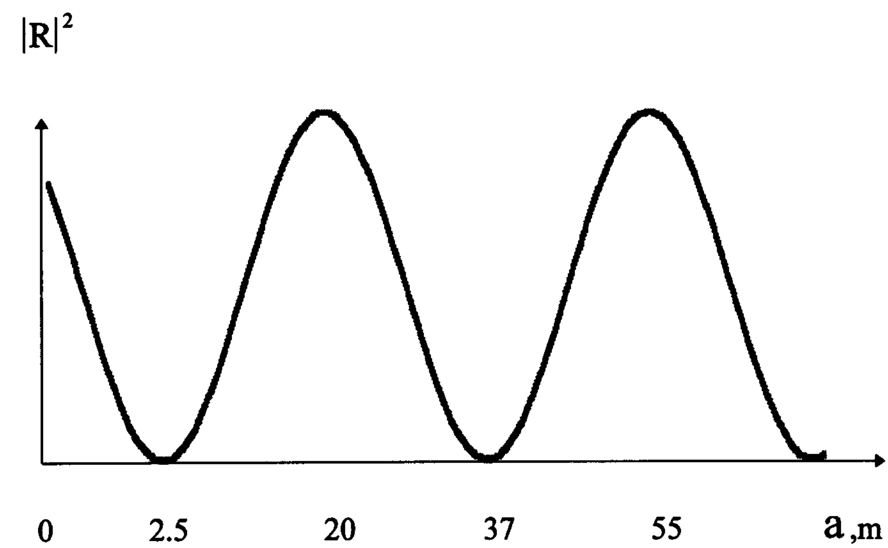
$\Phi_2\big|_{x=-a} = \Phi_3\big|_{x=-a}$; $\left.\dfrac{\partial \Phi_1}{\partial x}\right|_{x=a} = \left.\dfrac{\partial \Phi_3}{\partial x}\right|_{x=a}$; $\left.\dfrac{\partial \Phi_2}{\partial x}\right|_{x=-a} = \left.\dfrac{\partial \Phi_3}{\partial x}\right|_{x=-a}$; $\left.\dfrac{\partial^4 \Phi_1}{\partial x^4}\right|_{x=a} = 0$;

$\left.\dfrac{\partial^5 \Phi_1}{\partial x^5}\right|_{x=a} = 0$; $\left.\dfrac{\partial^4 \Phi_2}{\partial x^4}\right|_{x=-a} = 0$; $\left.\dfrac{\partial^5 \Phi_2}{\partial x^5}\right|_{x=-a} = 0$.

One can show, that the approach with a zero - crack is a special case of the approach with a crack of finite width, wenn a crack width $2a \to 0$.

4.DEPENDENCE OF THE REFLECTION COEFFICIENT ON THE CRACK WIDTH

The obtained results let us fulfil a computation, which let us find a dependence of reflection on the crack width:



where H=8 m, h=4 m, ω=0.4, $\max(|R|^2) = 0.045$

5. ENERGY FLOW

5.1 ENERGY FLOW THROUGH ANY RESERVOIR CROSS SECTION

Now let us consider an energy flow in liquid through any cross section. We assume, that water take up volume V, which is limited by surface S. Energy in volume V consist of kinematic and potential energy. For energy E in volume V we have:

$E=\rho\iiint_V\left\{\frac{1}{2}\left[\ (\frac{\partial\Phi}{\partial x})^2+(\frac{\partial\Phi}{\partial y})^2+(\frac{\partial\Phi}{\partial z})^2\ \right]+gz\right\}dxdydz$. Using the Bernoulli low:

$\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left(V_x^{\,2}+V_y^{\,2}+V_z^{\,2}\right)+\frac{p}{\rho}+gz=C(t)$ with $C(t)\equiv 0$, we get:

$E=-\iiint_V(p+\rho\frac{\partial\Phi}{\partial t})dxdydz$. Let us determine $\frac{dE}{dt}$. If any function

$F=\iiint_V f(x,y,z,t)dxdydz$, then: $\frac{dF}{dt}=\iiint_V\frac{\partial f}{\partial t}dxdydz+\iint_S f\,V_n dS$, where V_n is normal velocity of S. We assume, that S is fixed, therefore: $V_n=0$. Thus we get:

$$\frac{dE}{dt}=\rho\iiint_V(\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial y}+\frac{\partial\Phi}{\partial z}\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial z})dxdydz.$$

$\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial y}+\frac{\partial\Phi}{\partial z}\frac{\partial}{\partial t}\frac{\partial\Phi}{\partial z}=\text{grad}\,\Phi\,\text{grad}\frac{\partial\Phi}{\partial t}$. Using the Green formula:

$\iiint_V\{U\,\Delta W+(\text{grad}U,\text{grad}W)\}dV=\iint_{\partial V}U\frac{\partial W}{\partial n}dS$ and because $\Delta\,\Phi=0$ we get for $\frac{dE}{dt}$:

$\frac{dE}{dt}=\rho\iint_S\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial n}dS$. The principle of energy conservation in integral form is as follows: $\frac{dE}{dt}+\iiint_V\text{div}\,\vec{\pi}\,dV=0$, where $\vec{\pi}$ - vector of energy flow. Thus we have:

$\frac{dE}{dt}+\iint_S(\vec{\pi},\vec{n})dS=0$ and for an energy flow in direction 0X : $\pi_x=\rho\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial x}$.

π_x averages : $\left\langle\pi_x\right\rangle_p=\frac{1}{T}\rho\int_0^T\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial x}dt$

In order to determine an energy flow through any reservoir cross section under the ice plate we have to integrate energy flow per unit depth $\left\langle\pi_x\right\rangle_p$:

$$\left\langle\pi_x\right\rangle=\int_{-H}^{0}\left\{\frac{1}{T}\rho\int_0^T\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial x}dt\text{-}\right\}dz$$

For stead-state harmonic vibrations an energy flow under ice plate will be as follows:

$$\left\langle\pi_x\right\rangle=\int_{-H}^{0}\{\ \rho\langle\ \frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial t}\ \rangle\ \}dz=\int_{-H}^{0}\rho\frac{\omega}{2}\text{Im}\frac{\partial\Phi}{\partial x}\overline{\Phi}dz=H\rho\frac{\omega}{2}\text{Im}\frac{\partial\Phi}{\partial x}\overline{\Phi}dz\ (12).$$

5.2 ENERGY FLOW IN ICE PLATE

From [2] we get the formula for an energy flow of plate bending vibrations:

$$\vec{\pi}=D\{\ \text{grad}\Delta\zeta\frac{\partial\zeta}{\partial t}-(1-\sigma)(\text{grad}\frac{\partial\zeta}{\partial t},\nabla)\text{grad}\zeta-\sigma\,\Delta\zeta\,\text{grad}\frac{\partial\zeta}{\partial t}\ \}\ (13)$$

We consider stead-state harmonic vibrations. (13) averages :

$\langle\vec{\pi}\rangle=\frac{D\omega}{2}\text{Im}\{\sigma\,\Delta\,\zeta\,\text{grad}\overline{\zeta}+(1-\sigma)(\text{grad}\overline{\zeta},\nabla)\text{grad}\zeta-\overline{\zeta}\,\text{grad}\Delta\zeta\ \}$ (14). Let us determine components of (14): $\text{grad}\zeta=\frac{\partial\zeta(x)}{\partial x}\vec{i}$; $\overline{\zeta}=\overline{\zeta}(x)$; $\text{grad}\overline{\zeta}=\frac{\partial\overline{\zeta}(x)}{\partial x}\vec{i}$;

$$(\text{grad}\overline{\zeta},\nabla)\text{grad}\zeta=(\frac{\partial\overline{\zeta}(x)}{\partial x}\vec{i},\vec{i}\frac{\partial}{\partial x})\frac{\partial\zeta(x)}{\partial x}\vec{i}=\frac{\partial\overline{\zeta}(x)}{\partial x}\frac{\partial^2\zeta(x)}{\partial x^2}\vec{i}$$

Consequently for $\left\langle\vec{\pi}\right\rangle$ we get:

$$\langle \vec{\pi} \rangle = \frac{D\omega}{2} \operatorname{Im}\left\{ \frac{\partial \overline{\zeta}(x)}{\partial x} \frac{\partial^2 \zeta(x)}{\partial x^2} \vec{i} - \overline{\zeta}(x) \frac{\partial^3 \zeta(x)}{\partial x^3} \vec{i} \right\} \quad (15).$$

For shallow water, using (7) , we have:

$$\langle \vec{\pi} \rangle = \frac{D\omega}{2} \operatorname{Im}\left\{ \frac{\partial^4 \Phi(x)}{\partial x^4} \frac{\partial^3 \overline{\Phi}(x)}{\partial x^3} \frac{H^2}{\omega^2} \vec{i} - \frac{\partial^5 \Phi(x)}{\partial x^5} \frac{\partial^2 \overline{\Phi}(x)}{\partial x^2} \frac{H^2}{\omega^2} \vec{i} \right\} =$$

$$= \frac{DH^2}{2\omega} \operatorname{Im}\left\{ \frac{\partial^4 \Phi(x)}{\partial x^4} \frac{\partial^3 \overline{\Phi}(x)}{\partial x^3} - \frac{\partial^5 \Phi(x)}{\partial x^5} \frac{\partial^2 \overline{\Phi}(x)}{\partial x^2} \right\} \vec{i} \quad (16).$$

In order to get a full energy flow through any cross section of reservoir covered with ice we have to add an energy flow through any cross section of ice plate and energy flow through any cross section of reservoir with water under ice plate. Thus, from (16) and (12) we can determine a full energy flow through any cross section of reservoir covered with ice in direction X-axis:

$$\langle \pi \rangle = \frac{DH^2}{2\omega} \operatorname{Im}\left\{ \frac{\partial^4 \Phi(x)}{\partial x^4} \frac{\partial^3 \overline{\Phi}(x)}{\partial x^3} - \frac{\partial^5 \Phi(x)}{\partial x^5} \frac{\partial^2 \overline{\Phi}(x)}{\partial x^2} \right\} + H\rho \frac{\omega}{2} \operatorname{Im} \frac{\partial \Phi}{\partial x} \overline{\Phi} dz \quad (17).$$

If h=0 (there is no ice) we have $D = 0$ and the full energy flow is equal to the energy flow in reservoir with free surface: $\langle \pi \rangle = H\rho \frac{\omega}{2} \operatorname{Im} \frac{\partial \Phi}{\partial x} \overline{\Phi} dz$

5.3 SATISFYING TO THE PRINCIPLE OF THE ENERGY CONSERVATION

a) ANALYTICAL :

An energy flow must be equal in every cross section of the reservoir in his different parts: with free surface or covered with ice. Let us consider an energy flow through any cross section of reservoir with free surface (inside the crack). Energy flow is given as (12). If the potential $\Phi = \Phi_0 e^{\alpha x} e^{-i\omega t}$, then the energy flow $\langle \pi \rangle$ doesn't depend on the X and consequently it is the same in any reservoir cross section inside the crack. Now let us consider an energy flow in any cross section of the reservoir covered with ice (on the left and on the right of the crack), which is given as (17). It is the same in any reservoir cross section outside the crack, because the energy flow doesn't depend on X. So it remains only to check a value of the energy flow at the crack boundaries by $x = \pm a$. We must get an equality: $\int_{-H}^{0} \rho \frac{\omega}{2} \operatorname{Im} \frac{\partial \Phi_3}{\partial x} \overline{\Phi}_3 dz =$

$= \frac{DH^2}{2\omega} \operatorname{Im}\left\{ \frac{\partial^4 \Phi_1}{\partial x^4} \frac{\partial^3 \overline{\Phi}_1}{\partial x^3} - \frac{\partial^5 \Phi_1}{\partial x^5} \frac{\partial^2 \overline{\Phi}_1}{\partial x^2} \right\} + \int_{-H}^{0} \rho \frac{\omega}{2} \operatorname{Im} \frac{\partial \Phi_1}{\partial x} \overline{\Phi}_1 dz$, where Φ_3 is the potential in the crack, Φ_1 - the potential outside the crack (to the left or to the right). Boundary conditions $\frac{\partial^4 \Phi_1}{\partial x^4} = 0$ and $\frac{\partial^5 \Phi_1}{\partial x^5} = 0$ bring to zero the first item. Thus we get: $\int_{-H}^{0} \rho \frac{\omega}{2} \operatorname{Im} \frac{\partial \Phi_3}{\partial x} \overline{\Phi}_3 dz = \int_{-H}^{0} \rho \frac{\omega}{2} \operatorname{Im} \frac{\partial \Phi_1}{\partial x} \overline{\Phi}_1 dz$ .This equality is always true, because of boundary conditions: $\Phi_1 = \Phi_3$ and $\frac{\partial \Phi_1}{\partial x} = \frac{\partial \Phi_3}{\partial x}$. Thus our simulation satisfys to the principle of energy conservation.

b) PRACTICALLY :

Let us estimate a value of energy flow through any cross section of reservoir inside the crack. There are 2 travelling waves inside the crack. So we have:

$\Phi = \Phi_{31} e^{ik^* x} + \Phi_{32} e^{-ik^* x}$. We get from (12): $\langle \pi \rangle = \frac{\rho\omega}{2} k^* (|\Phi_{31}|^2 - |\Phi_{32}|^2) H$.

For a value of energy flow outside the crack (for example to the right) we get from (17): $\langle \pi \rangle = \frac{\rho\omega}{2} k(|T|^2) H + \frac{DH^2}{2\omega} \operatorname{Im}\{ (ik)^4 T(-ik)^3 \overline{T} - (ik)^5 T(-ik)^2 \overline{T} \} =$

$= \frac{\rho\omega}{2} k(|T|^2) H + \frac{DH^2}{\omega} k^7 |T|^2$. Using the computed values of coefficients K and T we make sure, that our simulation satisfys to the principle of energy conservation.

CONCLUSIONS

1. The reflection coefficient depends periodically on the crack width;
2. The value of this period depends on the length of the gravitation wave at the free liquid surface;
3. The reservoir depth and the wave frequency have an influence on the period by means of influence on the length of the gravitation wave at the free liquid surface;
4. The ice thickness has an influence only on the value of the reflection coefficient (and on the amplitude), and also on the displacement of the graph along X-axis;
5. The value of period is equal to the half of the length of the gravitation wave at the free liquid surface;
6. The reflection decreases with increasing of the reservoir depth;
7. The reflection increases with increasing of the ice thickness;
8. By shot waves an ice plate carries the greater part of the energy than by long waves;
9. By shot waves an ice plate carries the greater part of the energy than the water in reservoir.

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