FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION

DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

A FORMULATION FOR THE FORCED VIBRATION OF A MULTI-SUPPORTED STRING

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This paper examines a simple model, namely the vibration of an infinite string resting on sets of periodic supports, which are modeled as linear spring-mass-damper systems. The q-th set of periodic supports is offset from a reference origin by a distance x_q. These offsets can thus be used to create spatially-distributed gratings along the length of the string. The objective here is to develop a means to model the vibration response of more complicated stiffened elastic structures.

INTRODUCTION

The objective of this study is to gain an understanding of the physical mechanisms that govern the vibrational and acoustic response of complicated stiffened elastic structures, such as fluid-loaded beams, plates, and cylindrical shells. The physics that control the vibrational and acoustic response of periodically stiffened structures (uniform stiffeners with equal spacing) are well understood. For arbitrarily stiffened structures, only qualitative characteristics of the vibrational and acoustic response are currently known. Presented here is a formulation which may eventually allow for quantitative mathematical models.

I. IMPLICIT FORMULATION OF WAVENUMBER SPECTRA

Elementary discussions of wave phenomena often begin with modeling transverse waves on a string. In this paper, the classical problem is reformulated to include sets of periodic supports, modeled as spring-mass-damper systems, which exert an applied force on the string proportional to the string's displacement.

A. Development of the multi-supported string model

A flexible string of infinite extent is assumed to have no resistance to bending or to shear, and to be of uniform density, ρ. For small string displacements, $w(x,t)$, the tension, T, within the string is assumed constant. Figure 1 provides an illustration of the multi-supported string.

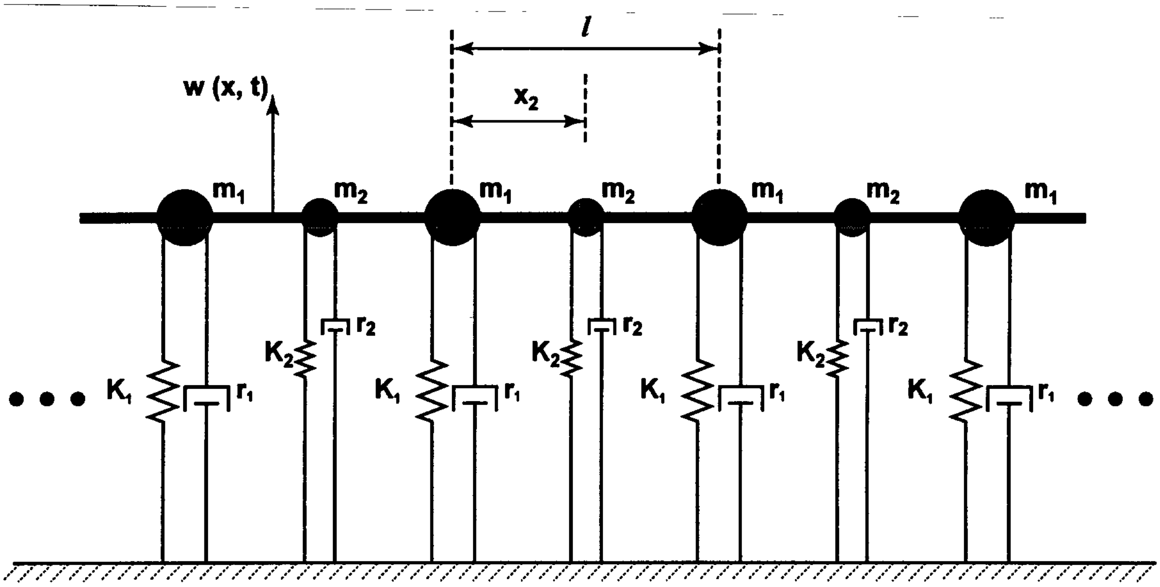

FIG. 1. Static equilibrium of an infinite string resting on multiple sets of periodic supports. Here there are two support sets, $Q = 2$, with the $q = 1$ set of supports having no offset, $x_1 = 0$

The displacements of the supports, which are modeled as single degree of freedom spring-mass systems, cause restoring forces to be exerted upon the string. Each set of supports, therefore, can be modeled as a set of applied forces acting on the string. The forced response of the supported string can be written as

$$T\frac{\partial^2 w(x,t)}{\partial x^2} - \rho\frac{\partial^2 w(x,t)}{\partial t^2} + f(x,t) = f_o(x,t) \tag{1}$$

where $f_o(x,t)$ denotes a steady-state excitation force applied to the string at $x = x_o$, namely $f_o(x,t) = \hat{f}\delta(x-x_o)e^{-i\omega t}$, while $f(x,t)$, represents the sum of all support forces. The free response is obtained by setting the amplitude of the excitation force to zero, $\hat{f} = 0$.

A single <u>set</u> of periodic supports, $Q = 1$, produces a combination of inertial, damping, and spring forces equal to an applied force, $F_{1,n}(t)$, at the n-th support. For Q sets of periodic supports (each set having different support properties and offsets), the total force applied to the string is

$$f(x,t) = -\sum_{q=1}^{Q}\sum_{n} F_{q,n}(t) = -\sum_{q=1}^{Q}\sum_{n}\left\{m_q\frac{d^2u_{q,n}(t)}{dt^2} + r_q\frac{du_{q,n}(t)}{dt} + K_q u_{q,n}(t)\right\}\delta\left(x-(nl+x_q)\right) \tag{2}$$

where $u_{q,n}(t)$ is the q,n-th support displacement and m_q, r_q, K_q denote, respectively, the mass, damping, and spring coefficients of support set q. Each set of supports may possess a different offset, x_q, provided $|x_q| \leq l$, where l is the fundamental periodicity of every support set in x. The single index n under the summation symbol implies an infinite sum over all n.

The transverse displacement of the string is assumed to have a harmonic time dependence, given as $e^{-i\omega t}$. Therefore, upon substitution of Eq. (2) into Eq. (1), with $w(x,t) = w(x)e^{-i\omega t}$, there follows

$$T\frac{d^2 w(x)}{dx^2} + \rho\omega^2 w(x) + \sum_{q=1}^{Q}\sum_{n}(m_q\omega^2 + ir_q\omega - K_q)u_{q,n}\delta\big(x-(nl+x_q)\big) = f_o(x). \tag{3}$$

The solution $w(x)$ is assumed bounded and to have a Fourier transform pair defined as

$$W(k) = \int_{-\infty}^{\infty} w(x)e^{-ikx}dx \qquad w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(k)e^{ikx}dk \,. \tag{4a, 4b}$$

The transform of Eq. (3) into the wavenumber domain, k, yields

$$a(k,\omega)W(k) - \frac{1}{T}\sum_{q=1}^{Q}\int_{-\infty}^{\infty}\{\sum_{n}(m_q\omega^2 + ir_q\omega - K_q)u_{q,n}\delta\big(x-(nl+x_q)\big)\}e^{-ikx}dx = F_o(k) \tag{5}$$

where $a(k,\omega) = k^2 - k_o^2$ and $k_o = \omega/c_o$,with $c_o = \sqrt{T/\rho}$, the free wavenumber of the unsupported string. $F_o(k) = -\frac{\hat{f}e^{-ikx_o}}{T}$ is the transform of the applied excitation, normalized by the uniform string tension. Since the displacements of the supports and string are assumed to be continuous at positions, $x = nl + x_q$, it then follows that $u_{q,n} = w(nl + x_q)$. By the use of Poisson's Summation theorem in the form

$$\sum_{n} e^{\pm inz} = 2\pi\sum_{n}\delta(z - n2\pi) = \sum_{n}\delta(\frac{z}{2\pi} - n) \tag{6}$$

and substitution of $w(nl + x_q)$ into Eq. (5), an implicit expression for the string's response in the wavenumber domain is obtained. Similar manipulations are presented in detail in Mace[1] and Cray.[2] There follows

$$a(k,\omega)W(k) + \sum_{q=1}^{Q} B_q(\omega)\underline{W}_q(k) = F_o(k) \tag{7}$$

where we have defined

$$B_q(\omega) = \frac{m_q}{Tl}(\omega_q^2 - \omega^2 - 2i\omega\omega_q\zeta_q) \tag{8}$$

$$\underline{W}_q(k) = \sum_{n} W(k + nk_l)e^{in\alpha_q} \tag{9}$$

with $\omega_q = \sqrt{K_q / m_q}$, the natural frequency, and $\zeta_q = r_q/(2m_q\omega_q)$, the viscous damping factor, of the q-th support set. The Bloch[3] wavenumber, $k_l = 2\pi/l$, corresponding to the system's fundamental periodicity, is frequency independent. For notational convenience, the phase associated with the Bloch wavenumber and the q-th offset is denoted, $\alpha_q = k_l x_q$.

II. EXPLICIT SOLUTION FOR THE WAVENUMBER SPECTRA

The summation defined by Eq. (9) has the useful periodicity property [4]

$$\underline{W}_q(k + k_\ell m) = \underline{W}_q(k)e^{-im\alpha_q}. \tag{10}$$

This property will shortly prove essential in solving Eq. (7)explicitly for $W(k)$. The zeros of the continuous function $a(k,\omega)$ for real k are at $\pm k_o$ and are simple. Thus, $a(\pm k_o,\omega) = 0$, and we are led to define

$$Y(k) = \begin{cases} 1\big/a(k,\omega) & \text{for } k \neq \pm k_o \\ 0 & \text{for } k = \pm k_o \end{cases}. \tag{11}$$

Then, Eq. (7) yields

$$W(k) = Y(k)F_o(k) - Y(k)\sum_{q=1}^{Q} B_q(\omega)\underline{W}_q(k) + a_1\delta(k - k_o) + a_2\delta(k + k_o) \tag{12}$$

for all k, where the Dirac delta functions must be included to allow for possible solutions at the unsupported string's free wavenumber. However, for *non-zero $B_q(\omega)$*, it may be shown that the areas associated with the impulses at $\pm k_o$ must be equal to zero, $a_1 = a_2 = 0$, for <u>bounded</u> solutions of $w(x)$. Then, it follows from (12) that

$$W(k + k_l m) = Y(k + k_l m)F_o(k + k_l m) - Y(k + k_l m)\sum_{q=1}^{Q} B_q(\omega)\underline{W}_q(k)e^{-im\alpha_q} \tag{13}$$

for all m, where the periodicity property, Eq. (10), has been used. Therefore, multiplying Eq. (13) through by $e^{im\alpha_p}$ and summing on m yields

$$\underline{W}_p(k) \equiv \sum_m W(k + k_l m)e^{im\alpha_p} = \underline{F}_p(k) - \sum_{q=1}^{Q} B_q(\omega)\underline{W}_q(k)\underline{Y}_{pq}(k) \tag{14}$$

for $1 \leq p \leq Q$, where we defined

$$\underline{F}_q(k) = \sum_n Y(k + k_l n)F(k + k_l n)e^{in\alpha_q} \qquad for\ 1 \le q \le Q,$$

$$\underline{Y}_{pq}(k) = \sum_n Y(k + k_l n)e^{in(\alpha_p - \alpha_q)} \qquad for\ 1 \le p,q \le Q. \tag{15}$$

Then, re-arranging Eq. (14),

$$\sum_{q=1}^{Q}\left[\delta_{pq} + B_q(\omega)\underline{Y}_{pq}(k)\right]\underline{W}_q(k) = \underline{F}_p(k) \tag{16}$$

for $1 \le p \le Q$, where δ_{pq} denotes the Kronecker delta. Define the quantities

$$C_{pq}(k) = \delta_{pq} + B_q(\omega)\underline{Y}_{pq}(k) \qquad for\ 1 \le p,q \le Q, \tag{17}$$

and complex QxQ matrix

$$C(k) = \left[C_{pq}(k)\right]_1^Q . \tag{18}$$

Then, in matrix form, Eq. (16) can be rewritten as

$$C(k)\tilde{W}(k) = \tilde{F}(k) \tag{19}$$

where the Qx1 column vectors are defined as

$$\tilde{W}(k) \equiv \left[\underline{W}_1(k)\cdots\underline{W}_Q(k)\right]^T, \tilde{F}(k) \equiv \left[\underline{F}_1(k)\cdots\underline{F}_Q(k)\right]^T. \tag{20}$$

The wavevector spectra of the supported string is solved explicitly with substitution of the solution of $\underline{W}(k)$ from Eq. (19) ($\tilde{W}(k) = C^{-1}(k)\tilde{F}(k)$) into Eq. (12).

The properties of $C^{-1}(k)$ will be discussed in a subsequent paper, for reference, however the following is provided.

For $Q = 2$, the $C(k)$ matrix becomes

$$C(k) = \begin{bmatrix} 1 + B_1\underline{Y}_{11}(k) & B_2\underline{Y}_{12}(k) \\ B_1\underline{Y}_{21}(k) & 1 + B_2\underline{Y}_{22}(k) \end{bmatrix} = \begin{bmatrix} 1 + B_1S(k,0) & B_2S(k,\alpha_1 - \alpha_2) \\ B_1S(k,\alpha_2 - \alpha_1) & 1 + B_2S(k,0) \end{bmatrix} \tag{21}$$

where $S(k,\Delta)$ is defined as the sum of the series

$$S(k,\Delta) \equiv \sum_n Y(k + k_l n)e^{in\Delta}. \tag{22}$$

Recalling that $a(k,\omega) = k^2 - k_o^2$ and Eqs. (11) and (15), the function $S(k,0)$ is

$$S(k,0)=\sum_n \frac{1}{(k+k_l n)^2-k_o^2}=\frac{l\sin(k_o l)}{2k_o\{\cos(kl)-\cos(k_o l)\}}, \tag{23}$$

where the sum of a series provided in Hansen[5] has been used. This function $S(k,0)$ has period k_l in k.

More generally, from Eq. (22) and Hansen [page 222, (14.3.3)],

$$S(k,\Delta)=\frac{l}{4k_o}\left(\frac{\exp(i\mathrm{L}(k+k_o)/k_l)}{\sin(l(k+k_o)/2)}-\frac{\exp(i\mathrm{L}(k-k_o)/k_l)}{\sin(l(k-k_o)/2)}\right), \tag{24}$$

where L = π(2*M*+1) - Δ, M = int(Δ/2π); here, int(t) means the integer strictly less than t. When Δ = 0, then M = -1, L = -π, Eq. (24) reduces to Eq. (23).

The determinant of $C(k)$ is

$$\det C(k)=1+(B_1+B_2)S(k,0)+B_1B_2\left[S^2(k,0)-S(k,\alpha_1-\alpha_2)S(k,\alpha_2-\alpha_1)\right]. \tag{25}$$

Wavenumbers that allow the $\det C(k_j)=0$ may be examined using weighted impulses or Dirac delta functions evaluated at $k=k_j$. However, for the forced response given by Eq. (19), damping may also be introduced into the parameter $\{B_q\}$, thus shifting the real and simple roots, k_j, from the real k axis.

Returning to the matrix $C(k)$, for $Q = 3$,

$$C(k)=\begin{bmatrix}1+B_1\underline{Y}_{11}(k) & B_2\underline{Y}_{12}(k) & B_3\underline{Y}_{13}(k)\\ B_1\underline{Y}_{21}(k) & 1+B_2\underline{Y}_{22}(k) & B_3\underline{Y}_{23}(k)\\ B_1\underline{Y}_{31}(k) & B_2\underline{Y}_{32}(k) & 1+B_3\underline{Y}_{33}(k)\end{bmatrix} \tag{26}$$

Suppressing the k dependence temporarily, the determinant is

$$\begin{aligned}\det C(k)&=(1+B_1\underline{Y}_{11})(1+B_2\underline{Y}_{22})(1+B_3\underline{Y}_{33})\\ &\quad+B_1B_2B_3(\underline{Y}_{12}\underline{Y}_{23}\underline{Y}_{31}+\underline{Y}_{13}\underline{Y}_{32}\underline{Y}_{21})-B_2B_3(1+B_1\underline{Y}_{11})\underline{Y}_{23}\underline{Y}_{32}\\ &\quad-B_1B_3(1+B_2\underline{Y}_{22})\underline{Y}_{13}\underline{Y}_{31}-B_1B_2(1+B_3\underline{Y}_{33})\underline{Y}_{12}\underline{Y}_{21}\end{aligned} \tag{27}$$

This determinant is also periodic with period k_l in k.

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