FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION

DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

TRAPPED MODES IN ELASTIC CONSTRUCTIONS LYING ON THE BOTTOM OF CHANNEL OF NONCOMPRESSIBLE WEIGHT LIQUID

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The phenomenon of trapped modes (standing waves) near the underwater constructions has been studied. The influence of construction dynamic on the process of trapped modes formation has been studied. It has been also shown that existence and a number of resonance frequencies essentially depend on elastic construction and channel parameters. The influence of liquid nonhomogenity on the initiation of trapped modes has been investigated separately.

1 Introduction

A fundamental problem in the theory of surface waves is the determination of the characteristic standing modes and frequencies of a system oscillating under gravity [1], [2], [3]. If the free surface extends to infinity, the modes of vibration will be expected to form a continuous spectrum, with an infinite amount of energy in each mode. Ursell F. showed that the theory of surface waves leads to a phenomenon essentially new in a classical mechanics, i.e., to a discrete as well as to a continuous spectrum. A part from continuous spectrum, that corresponds to travelling waves, the real discrete spectrum corresponds to standing waves, call in [1] trapped modes.

The peculiarities of mixed spectrum formation in elastic systems with inclusions have been investigate [4], [5].

This paper are devoted to the possibility of the mixed spectrum existence in the problem on the standing wave localized in the area of dynamical inclusion (thin plate, membrane) on the bottom of a channel. The first problem is plate oscillation on the bottom of trench in the channel with liquid. The second problem is plate oscillation on bottom of trench in the channel with liquid. The second problem is plate oscillation on the top of the projection in the channel. The third problem is membrane oscillation on the bottom of channel. As a result we find the influence of the parameters of the channel, and the bottom relief for existence inclusions trapped modes.

2 Statement of the problem

Consider an elastic construction on the bottom of a three-dimensional channel filled with an ideal noncompressible liquid. Cartesian axes are chosen so that y is directed vertically upwards and x and z in the plane of the unperturbed bottom. The motion of the liquid is described by velocity potential $\Phi(x, y, z, t)$ in the linear theory. The velocity potential is found from the following boundary problem:

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \text{ in the domain occupied by liquid,} \quad (1)$$

$$\Phi_{tt} + g\Phi_y = 0, \text{ on the free surface,} \quad (2)$$

$$\frac{\partial \Phi}{\partial n} = \begin{cases} w_t & \text{on the moving part of boundary} \\ 0 & \text{on the rigid part} \end{cases} \quad (3)$$

where g is the gravity acceleration, w is the small construction displacement determined by the equation

$$Lw + Mw_{tt} = p, \quad p = \rho(\Phi_t + gw) \quad (4)$$

on the moving part of the bottom. We suppose that bottom topography is given stepped configuration. The submerged elastic constriction can be modeled by the thin plane or the membrane. Then

$$Lw = \left[\begin{array}{l} Dw_{zzzz} + kw, \\ -T(w_{xx} + w_{zz}) + kw, \end{array} \right. \quad (5)$$

where M is elastic construction mass per unit length, D is cylindrical rigidity, T is membrane tension force, k is elastic foundation rigidity.

Then solutions to (1)-(4), corresponding of frequency ω and wavenumber m travelling along the elastic construction can be sought in the form:

$$w(x, z, t) = \text{Re } \{w_0(x)e^{\text{i}(mz-\omega t)}\}, \quad (6)$$
$$\Phi(x, y, z, t) = \text{Re } \{\varphi(x, y)e^{\text{i}(mz-\omega t)}\}.$$

Substituting (6) into (1)-(4), that the function $\varphi(x, y)$ defined on a two-dimensional domain W which is a cross-section of fluid orthogonal to the z-axis, satisfies the following boundary value problem:

$$\begin{cases} \varphi_{xx} + \varphi_{yy} = m^2\varphi \text{ in } W, \\ \varphi_y = \frac{\omega^2}{g} \text{ on } F \\ \varphi_y = \left[\begin{array}{ll} -\text{i}\,\omega w_0, & \text{on } S_m \\ 0, & \text{on } S_r \end{array} \right. \end{cases} \quad (7)$$

$$(L\omega - M\omega^2)w_0 = p. \quad (8)$$

Here F and $S = S_m \cup S_r$ denote parts of the boundary ∂W, lying in the free surface, in the moving surface and in the solid surface respectively.

The problem (7)-(8) is a spectral problem, in a sense that one of the parameters of the mechanical system involved should be treated as spectral parameter, which is to be found with the corresponding non-trivial solution.

For $\varphi(x,y)$ we require the "localization property"

$$\int_W |\nabla\varphi|^2 dxdy + \int_F \varphi^2 dx < \infty. \qquad (9)$$

In this case the energy of the wave mode is finite per unit length of the z-axis, and the mode (or wave) is said to be trapped.

Remark. Without the elastic construction and for the constant liquid depth the problem (7)-(8) leads only to continuous spectrum starting with $\omega_t = \sqrt{gm \tanh mH}$ which is the cut-off frequency, H is the depth of the liquid in the channel.

3 Liquid oscillation in a channel with uneven bottom. Case of the rectangular trench with an elastic bottom.

Let us decompose W as (see Fig. 1)

$$W = W^{(+)} \cup W^{(-)}$$

and $\varphi = \varphi^{(+)}$ in $W^{(+)}$ $\varphi = \varphi^{(-)}$ in $W^{(-)}$. These potentials must satisfy the following relations:

$$\begin{cases} \varphi^{(+)}_{xx} + \varphi^{(+)}_{yy} = m^2\varphi^{(+)} \text{ in } W^{(+)} \\ \varphi^{(+)}_y = \frac{\omega^2}{g}\varphi^{(+)}_y \quad y = H \\ \varphi^{(+)}_y = 0, \quad |x| > a, \quad y = h \end{cases} \qquad \begin{cases} \varphi^{(-)}_{xx} + \varphi^{(-)}_{yy} = m^2\varphi^{(-)} \text{ in } W^{(-)} \\ \varphi^{(-)}_y = -\mathrm{i}\omega w_0, \ |x| < a, \quad y = 0 \\ \varphi^{(-)}_x = 0, \ x = \pm a, \end{cases} \qquad (10)$$

$$\begin{cases} \varphi^{(+)} = \varphi^{(-)}, \quad |x| < a, \ y = h \\ \varphi^{(+)}_y = \varphi^{(-)}_y, \quad |x| < a, \ y = h \end{cases} \qquad (11)$$

$$(k_m - M\omega^2)w_0 = -\mathrm{i}\rho\omega \int_{-a}^{a} \varphi(x,y)dx + 2a\rho g, \quad y = 0. \qquad (12)$$

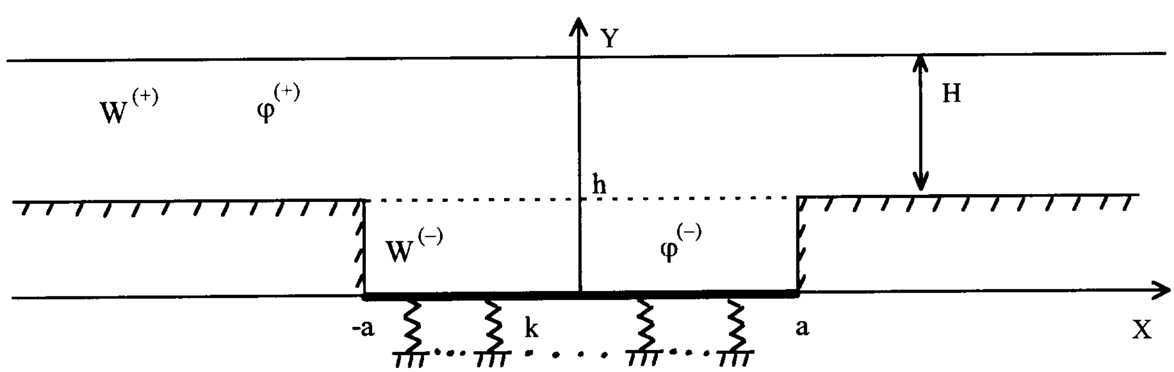

Figure 1:

Where $k_m = Dm^4 + k$ is generalized rigidity of the plate, ρ is the liquid density, w_0 is the unknown constant. The problem (10)-(12) is a spectral problem and the parameters ω, m, D, C, M are spectral parameters. The Green function allows us to express the solution of problem (10.1)-(11) in the form [5]

$$\varphi^{(+)}(x,y) = \int_{-a}^{a} \varphi_y^{(-)}(\eta,h) G(x-\eta,y,\omega) d\eta. \tag{13}$$

The corresponding Green function $G(x,y,\omega)$ is

$$G(x,y,\omega) = \begin{cases} A_0(\omega)e^{-\zeta_0|x|} + \sum_{k=1}^{\infty} A_k(\omega)e^{-\zeta_k|x|}, & \omega < \omega_b \\ \mathrm{i}A_0(\omega)e^{\mathrm{i}\zeta_0|x|} + \sum_{k=1}^{\infty} A_k(\omega)e^{-\zeta_k|x|}, & \omega > \omega_b, \end{cases} \tag{14}$$

where

$$A_0 = -\frac{2\xi_0 \cosh y\xi_0}{\zeta_0(\sinh 2H\xi_0 + 2H\xi_0)}, \quad A_k = -\frac{2\xi_k \cos y\xi_k}{\zeta_k(\sin 2H\xi_k + 2H\xi_k)};$$

ξ_0 is the positive root of the below equation

$$g\xi \tanh H\xi = \omega^2, \quad \zeta_0 = \begin{cases} \sqrt{m^2 - \xi_0^2}, & \omega < \omega_b \\ \sqrt{\xi_0^2 - m^2}, & \omega > \omega_b, \end{cases}$$

ξ_k is the positive root of the below equation

$$g\xi \tan H\xi = -\omega^2, \quad \zeta_k = \sqrt{\xi_k^2 + m^2}, \quad k = 1,2,\dots .$$

Thus, the problem for $\varphi^{(-)}$ includes

$$\begin{aligned} &\varphi_{xx}^{(-)} + \varphi_{yy}^{(-)} = m^2\varphi^{(-)} \ \text{ in } \ W^{(-)}, \\ &\varphi^{(-)} = \int_{-a}^{a} \varphi_y^{(+)}(\eta) G(x-\eta,y,\omega) d\eta, \ \ y = h \\ &\varphi_y^{(-)} = -\mathrm{i}\omega w_0, \ |x| < a, \quad y = 0 \\ &\varphi_x^{(-)} = 0, \ x = \pm a, \end{aligned} \tag{15}$$

If $w_0 = 0$ (the bottom of the trench is solid) for the spectrum parameter m we can have the following formulation

$$m^2 = \frac{-\int_{W^{(-)}} (\mathrm{grad}\varphi^{(-)})^2 dxdy + \int_{-a}^{a}\int_{-a}^{a} f(x)G(x-\eta,\omega)f(\eta)d\eta dx}{\int_{W^{(-)}} {\varphi^{(-)}}^2 dxdy} < 0.$$

Below the cut-off frequency ($\omega < \omega_b$) the Green function (14) is negatively determined function. In this case, we have not real discrete spectrum. $0 < \omega < \omega_b$

For $w_0 \neq 0$, we can show that the problem (15) can have only one discrete frequency lying bellow ω_b if the following inequality is true

$$k_m > 2a\rho g$$

Consider the case of the long waves $ma >> 1$, $mH << 1$. The approximation of the Green function gives

$$G(x, H, \omega) = \frac{2}{a}\sum_{0}^{\infty}\frac{A_k}{\zeta_k}\delta(x)$$

Where $\delta(x)$ is Dirac delta-function. By the solution of the (15), the spectral parameter M is given by

$$M = 2\rho a \frac{\omega^2}{(\Omega_0^2 - \omega^2)} \frac{Bm - \tanh mH}{m(1 - Bm \tanh mH)}$$

where $\Omega_0^2 = \frac{k_m - 2\rho a}{M}$, $\quad B = \frac{g}{\omega_b^2 - \omega^2} + \frac{1}{4m}\left[\frac{1}{\tanh mH} - \frac{1}{mH}\right]$

Let ω be an cut-off frequency ($\omega = \omega_b$), then the liquid occupying domain $W^{(+)}$ is at rest. In this case, Green function has the form

$$G(x, \omega_b) = A_0|x| + \sum_{k=1}^{\infty} A_k e^{-\zeta_k|x|}$$

and we must consider

$$\int_{-a}^{a} \varphi_y^{(+)}(x, y)dx = 0, \quad y = h$$

We have free oscillation of liquid only in the trench $W^{(-)}$. The necessary condition for parameters of the plate are so

$$\omega_b^2 = \frac{k_m - 2a\rho g}{M + \rho(m\tanh mh)^{-1}\omega_b^2}, \quad k_m > 2a\rho g.$$

4 Liquid oscillation in a channel with uneven bottom. Case of the rectangular projection with an elastic top.

For this case, we can write the following problem

$$\left\{\begin{array}{ll} \varphi_{xx}^{(+)} + \varphi_{yy}^{(+)} = m^2\varphi^{(+)} & \text{in} \quad W^{(+)}, \\ \varphi_y^{(+)} = \frac{\omega^2}{g}\varphi_y^{(+)} & y = H \\ \varphi_y^{(+)} = -\mathrm{i}\omega w_0, & |x| > a, \quad y = 0 \end{array}\right. \qquad \left\{\begin{array}{ll} \varphi_{xx}^{(-)} + \varphi_{yy}^{(-)} = m^2\varphi^{(-)} & \text{in} \quad W^{(-)}, \\ \varphi_y^{(-)} = 0, \ |x| > a, & y = -h \\ \varphi_x^{(-)} = 0, \ x = \pm a, & \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{ll} \varphi^{(+)} = \varphi^{(-)}, & |x| < a, \ y = h \\ \varphi_y^{(+)} = \varphi_y^{(-)} = f(x), & |x| < a, \ y = h \end{array}\right. \tag{17}$$

$$(k_m - M\omega^2)w_0 = -\mathrm{i}\rho\omega\int_{-a}^{a}\varphi dx + 2a\rho g. \tag{18}$$

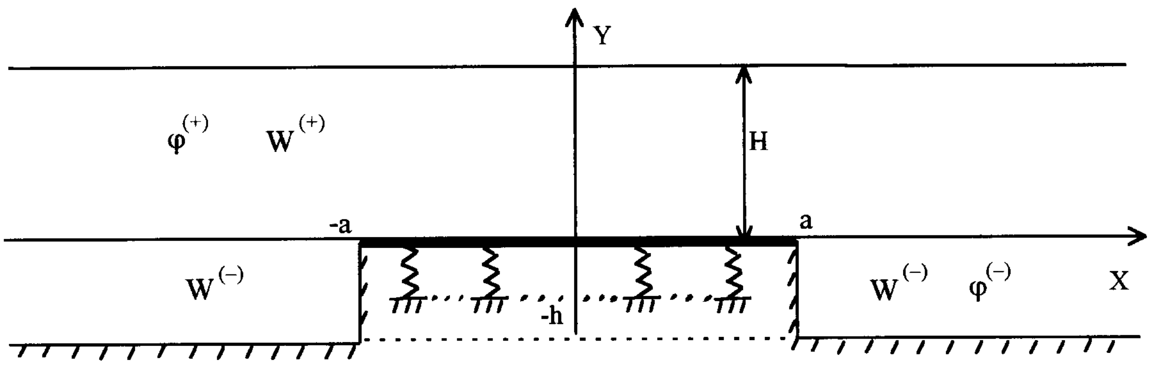

Figure 2:

The spectral problem (16) -(18) is equivalent the system of integral equations

$$\omega w_0 \int_{-a}^{a} G^{(+)}(x,\eta,\omega)d\eta + \int_{a}^{\infty} f(\eta)G_{\Sigma}^{(+)}(x,\eta,\omega)d\eta =$$

$$\int_{a}^{\infty} f(\eta)G^{(-)}(x,\eta,\omega)d\eta, \quad \text{for } |x|>a,\ y=0 \tag{19}$$

$$\tilde{C} - M\omega^2 = -\rho\omega^2 \int_{-a}^{a}\int_{-a}^{a} G^{(+)}(x,\eta,\omega)d\eta dx + \int_{-a}^{a}\int_{-a}^{a} f(\eta)G_{\Sigma}^{(+)}(x,\eta,\omega)d\eta dx. \tag{20}$$

Where

$$G^{(-)}(x,\omega) = \frac{1}{h}\left[\frac{e^{-m|x|}}{2m} + \sum_{k=1}^{\infty}\frac{e^{-\zeta_k^{(-)}|x|}}{\zeta_k}\right], \quad \zeta_k^{(-)} = \sqrt{m^2 + \frac{\pi^2 k^2}{h^2}},$$

$$G_{\Sigma}^{(+)} = G^{(+)}(x+\eta,\omega) + G^{(+)}(x-\eta,\omega).$$

If $w_0 = 0$ and $ma >> 1$, $mH << 1$ we can give the simple equation for ω

$$\tan a\sqrt{\frac{\omega^2}{gH} - m^2} = \sqrt{\frac{H+h}{H}}\sqrt{\frac{m^2 - \frac{\omega^2}{g(H+h)}}{\frac{\omega^2}{gH} - m^2}}.$$

Introduce the following notations $\omega_{b1}^2 = gm\tan H$ and $\omega_{b1}^2 = gm\tan(H+h)$. As a result the equation has only one root ω such that

$$\omega_{b1} < \omega < \omega_{b2}.$$

The statment confirms the result obtained in [3].

If $w_0 \neq 0$ we obtain the following results. The problem (16) - (18) has the real frequency lying in the region $0 < \omega < \omega_{b1}$ and can have two and more real frequency in the region $\omega_{b1} < \omega < \omega_{b2}$.

5 A membrane on the even bottom

In this case, the spectral problem (1)-(4) may be wright by the following integro-differential equation for $w_0(x)$:

$$\frac{d^2 w_0}{dx^2} + \lambda w_0 = -\mu \int_{-1}^{1} w_0(\eta)G(x-\eta,\omega)d\eta, \tag{21}$$

$$w_0(x) = 0, \quad x = \pm 1.$$

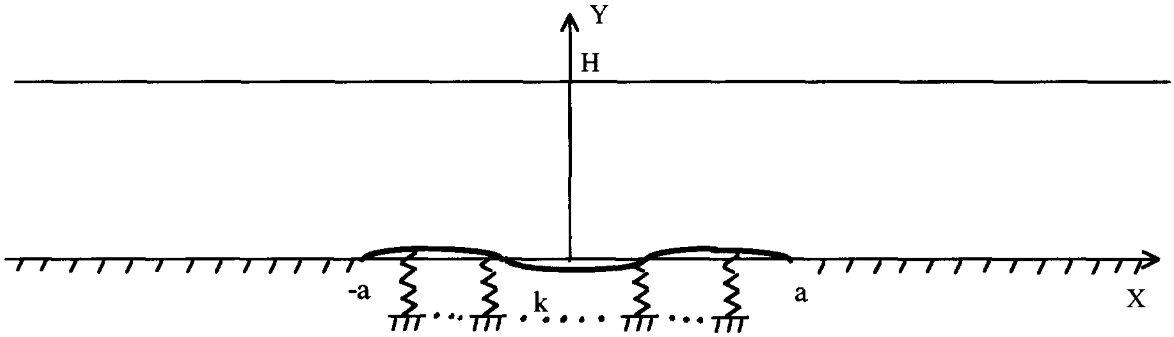

Figure 3:

Where $\lambda = \frac{a^2}{T}(M\omega^2 - k - mT + \rho g)$, $\mu = \frac{\rho a^3 \omega^3}{T}$. For long waves ($ma >> 1$, $mH << 1$), the Green function may be simplified as follows

$$G(x) = \frac{1}{2H\zeta_0} e^{-a\zeta_0 |x|} + \frac{H}{3a}\delta(x).$$

Differentiation of equation (21) with respect to x gives

$$\begin{aligned}
&\frac{d^4 w_0}{dx^4} + B_1 \frac{d^2 w_0}{dx^2} - B_2 w_0 = 0, \\
&w_0(x) = 0, \quad x = \pm 1, \\
&w_{0xx}(x) = -A \int_{-1}^{1} w_0(\eta) e^{-a\zeta_0|x-\eta|} d\eta, x = \pm 1.
\end{aligned}$$

As the result of the decision of this problem we get the following frequency equation

$$\frac{a\zeta_0 \tanh a\zeta_0 + S_2 \tan S_2}{(a\zeta_0)^2 + S_2^2} - \frac{a\zeta_0 \tanh a\zeta_0 - S_1 \tanh S_1}{(a\zeta_0)^2 - S_1^2} = \frac{S_1^2 + S_2^2}{2A_0 \cosh a\zeta}\omega^2. \tag{22}$$

Where A, B_1, B_2, S_1, S_2 are the functions of the mechanical system parameters and frequency. It is satisfies the condition

$$Tm^2 + k - \rho g > 0,$$

the number of the roots (22) is finite, which is equal to the existence trapped wave discrete spectrum bellow the cut-off frequency.

References

[1] Ursell, F.: Trapping modes in the theory of surface waves; Proc. Comb.Soc., 47 (1951), 347–358.

[2] McIver, P., Evans, D.V.: The trapping modes of surface waves above a submerged horizontal cylinder; J. Fluid Mech. 151 (1985), 243–255.

[3] Bonnet-Ben Dhia, A-S., Joly, P.: Mathematical analysis of guided water waves; SIAM J. Appl. Math. 53 No. 6 (1993), 1507–1550.

[4] Abramyan, A.K. Andreev, V.L. Indeitsev, D.A.: Feature of oscilation of dynamic mecanical systems of infinite length; Model in mechanics, Novosibirsk, RAS, 6(23) (1995). 2–23 (in Russian).

[5] Indeitsev, D.A., Osipova E.V.: Trapped modes in the cannal with liquid with a massive rigid die on the bottom; J. of Technical Physics. V.66, No 8 (1996), p.124–132.