FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

OPTIMAL PULSE SHAPING IN REVERBERATION-LIMITED ENVIRONMENTS

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Abstract

In active sensing systems, such as radar and sonar, the transmitted waveform has an important influence on target detection (especially in reverberation limited environments), parameter estimation and the resolution capability of the system. A method of generating waveforms which maximizes the output signal-to-noise (ambient plus interference) ratio of a replica correlator receiver is given. These waveforms are optimal only for a particular realization of the noise. A large set of such waveforms can be generated by Monte Carlo simulation. From this set, the robust optimum waveform, i.e., the one least sensitive to different noise realizations, can be determined.

1. INTRODUCTION

In active sensing systems, such as radar and sonar, one of the main objectives is to make a detection by extracting the desired signal information in the presence of noise. The term noise here refers to a combination of ambient noise, whose signal properties are uncorrelated with the transmitted signal, and interference (reverberation in sonar or clutter in radar) which consists of any unwanted echoes from either a continuum and/or discrete set of scatterers.

It is known, under certain assumptions (Sec. 2.2), that the signal-to-interference ratio is maximized if the transmitted waveform is chosen so that the integral of the overlap between the signal ambiguity function and the reverberation scattering function is minimized. In this paper a method of finding optimal waveforms which complements this approach will be presented. For the reasons of simplicity and utility, the receiver structure will have the form of a replica correlator or its equivalent matched filter realization.[1]

[1]Note that the replica correlator is not the optimum receiver when reverberation is present although it is the optimum structure when the noise is stationary white Gaussian. The replica correlator is chosen because it is usually much simpler to implement than the optimum receiver and its performance is not too different from the optimum for many situations.

The performance of such a receiver structure can be optimized by an appropriate choice of waveform. The chosen waveform is optimal in the sense that the signal-to-noise (ambient plus reverberation) ratio at the output of the detector is maximized for a given realization of the reverberation. It will be seen that such waveforms are not unique and a further requirement of robustness against changes in the environment is required to select the optimum solution. In practice, a number of realizations of the reverberation (as characterized by its statistical properties) and its corresponding optimal waveforms can be generated by Monte Carlo simulation. The robust optimum waveform can then be found from this solution set.

The structure of the rest of the paper can be summarised as follows. Section 2 considers the detection problem in terms of the signal and detector models to be used. The optimal pulse shaping algorithm is given in Section 3. This is followed by some illustrative examples of the algorithm's use in Section 4. The paper then concludes with some brief comments about the algorithm and its implications.

2. DETECTION IN NOISE AND INTERFERENCE

2.1 Signal model

The received signal at the input of a receiver system is modelled as

$$r(t) = \frac{A_t}{\sqrt{a_t}}\, x\left(\frac{t-b_t}{a_t}\right) + \sum_{k=1}^{M} \frac{A_k}{\sqrt{a_k}}\, x\left(\frac{t-b_k}{a_k}\right) + n(t), \tag{1}$$

where the first term represents the desired target echo, the second term consists of all the unwanted echoes (of which there are M) and $n(t)$ is the ambient noise term assumed to be uncorrelated with the transmitted waveform. Note that the second term can either represent interfering sources and/or reverberation (given a sufficiently large number of scatterers). The range and Doppler spreading of the target echo can also be incorporated by modelling the target as a suitable set of scatterers, although this will not be attempted in this paper. $x(t)$ is the transmitted waveform, A_j is the amplitude of the echo from either the target ($j = t$) or a scatterer ($j = k$), a_j is the scaling factor associated with the radial motion of the scatterer and b_j is the round trip time delay of the scattered signal. The time delay and scale factor are related to the range and radial velocity of the scatterer by the following[1]:

$$b_j = \frac{2R}{c} \cdot \frac{1}{1-\beta_{js}} \cdot \frac{1}{1+\beta_{ss}} \quad \text{and} \quad a_j = \frac{1+\beta_{js}}{1-\beta_{js}} \cdot \frac{1-\beta_{ss}}{1+\beta_{ss}}, \tag{2}$$

$$\text{with} \qquad \beta_{js} = \frac{\vec{v}_j.\hat{n}}{c} \quad \text{and} \quad \beta_{ss} = \frac{\vec{v}_s.\hat{n}}{c}, \tag{3}$$

where R is the range of the scatterer, c is the speed of wave propagation in the medium, $\vec{v}_j$ and $\vec{v}_s$ are the velocities of the scatterer and source (relative to the medium) respectively and $\hat{n}$ is the unit vector in the direction of the scatterer relative to the source.

2.2 Coherent detection

In order to extract the target information from the noise and interference, replica correlation processing will be used. The output of such a processor is given by

$$\Psi = \int r(t)x_r^*(t)dt, \tag{4}$$

where the replica waveform, parametrised by a_r and b_r, is

$$x_r(t) = \frac{1}{\sqrt{a_r}}\, x\left(\frac{t - b_r}{a_r}\right). \tag{5}$$

The performance measure of the receiver is given by its output signal-to-noise ratio,

$$SNR = \frac{E\left\{|\{\Psi|H_1\}|^2\right\}}{E\left\{|\{\Psi|H_0\}|^2\right\}}, \tag{6}$$

where $\{\Psi|H_1\}$ and $\{\Psi|H_0\}$ denote the output of the replica processor (Eq. (4)) conditioned on the presence of signal only and noise only, respectively. $E\{\cdot\}$ denotes ensemble averaging. It is convenient to work with the normalized correlator output of Eq.(4) in the form

$$\begin{aligned} \psi &= \frac{1}{A_t E}\sqrt{\frac{a_r}{a_t}}\Psi \\ &= \int u(t)u^*\left(\frac{t-\tau_t}{s_t}\right)dt + \sum_{k=1}^{M} r_k \int u(t)u^*\left(\frac{t-\tau_k}{s_k}\right)dt + \frac{1}{\sqrt{SNR_a}}N_R, \end{aligned} \tag{7}$$

where

$$u(t) = \frac{1}{\sqrt{E}}\, x(t), \tag{8}$$

($\sqrt{E}$ is the normalization factor such that $\int |x(t)|^2 dt = E$ is the energy of the waveform)

$$\tau_k = \frac{(b_r - b_k)}{a_k}, \qquad s_k = \frac{a_r}{a_k}, \tag{9}$$

$r_k = (A_k/A_t)\sqrt{a_k/a_t}$ is a measure of the interference amplitude relative to the target (the subscript $k = t$ for the target and $k = 1$ to M for the scatterers), $SNR_a = A_t^2 E/(\sigma_n^2 T)$ is a measure of the signal-to-ambient noise ratio at the input of the receiver (with σ_n^2 being the ambient noise power and T being the pulse length of the transmitted waveform) and

$$N_R = \frac{1}{\sigma_n\sqrt{a_t T}}\int n(t)\, u^*\left(\frac{t - b_r}{a_r}\right)dt \tag{10}$$

is a random variable describing the integrated ambient noise. The contribution to the return from the scatterers (second term in Eq.(7)) can be generalized to one of a dense (continuum) environment by

$$r_k \longrightarrow r(\tau, s)\,\Delta\tau\Delta s, \tag{11}$$

where $r(\tau, s)$ is the density of the scattering amplitude in the (τ, s) (or equivalently the range-Doppler) space. Therefore, the SNR can be evaluated in the usual way by making the following assumptions:

1. Echoes which result from scattering at different ranges and velocities are statistically independent, i.e., $E\{r(\tau, s)\, r^*(\tau', s')\} \equiv \sigma(\tau, s)\,\delta(\tau - \tau')\,\delta(s - s')$.

2. Ambient noise and reverberation are uncorrelated.

3. $r(\tau, s)$ is a zero mean random process.

4. $n(t)$ is a zero mean white Gaussian random process with $E\{n(t)n^*(t')\} \equiv \sigma_n^2 T \delta(t-t')$.

This then gives

$$SNR = |\chi_{uu}(\tau_t, s_t)|^2 \left[\frac{1}{SNR_a} \frac{a_r}{a_t} + \sigma_R^2 \int \int p(\tau, s) |\chi_{uu}(\tau, s)|^2 d\tau ds \right]^{-1}, \tag{12}$$

where

$$\chi_{uu}(\tau, s) = \int u(t) u^* \left(\frac{t-\tau}{s} \right) dt \quad \text{and} \quad p(\tau, s) = \frac{\sigma(\tau, s)}{\sigma_R^2} \tag{13}$$

with $\sigma_R^2 \equiv \int \int \sigma(\tau, s) d\tau ds$ so that $\int \int p(\tau, s) d\tau ds = 1$. The scattering function, $\sigma(\tau, s)$, is a combination of the average scattering strength and the joint probability density of finding a scatterer with a particular τ and s. The normalized scattering function, $p(\tau, s)$, can be interpreted as a probability density function which describes the statistical knowledge of the reverberation.

If the receiver is assumed to be matched to the target, and all velocities are small in comparison with the velocity of wave propagation, then Eq. (12) can be further simplified to

$$SNR \simeq \left[\frac{1}{SNR_a} + \sigma_R^2 \int \int p(\tau, \phi) |\chi_{uu}(\tau, \phi)|^2 d\tau d\phi \right]^{-1}, \tag{14}$$

where

$$\chi_{uu}(\tau, \phi) = \int a(t) a^*(t-\tau) e^{-j2\pi\phi t} dt \tag{15}$$

is the ambiguity function using the narrowband approximation, [2]

$$u(t) = a(t) \exp(j2\pi f_c t). \tag{16}$$

Note that the relationship between the Doppler shift and the scale factor, $\phi = (1/s - 1) f_c$, has been used. The normalized scattering function now becomes $p(\tau, \phi)$. The physical interpretation of SNR is now straightforward[2, 3, 4] in view of Eq.(14). The signal-to-noise ratio is essentially composed of the signal-to-ambient noise ratio, SNR_a, and the corresponding signal-to-reverberation term. By an appropriate choice of the transmitted waveform the reverberation component can be reduced by minimizing the overlap between the scattering function of the scatterers and the signal ambiguity function in the delay(range) – Doppler(velocity) space. Note that in the limit of zero reverberation the maximum value of SNR is SNR_a.

3. OPTIMAL PULSE SHAPING

As a means of pulse shaping, the transmitted waveform can be considered as a weighted sum of a set of basis waveforms, i.e.,

$$x(t) = \sqrt{E} \sum_{i=1}^{N} z_i l_i(t), \tag{17}$$

[2] The narrowband approximation is valid when the time-bandwidth product of the signal is much less than $c/(2v)$, where c is the speed of wave propagation in the medium and v is the relative speed of the target.

where $\{l_i(t)\}$ is some predetermined set of basis waveforms and $\{z_i\}$ is a set of adjustable weights used to control the shape of $x(t)$. By using the expansion in Eq.(17), the normalised output of the replica correlator (Eq.(7)) can be written as

$$\psi^* = Z^\dagger L Z + Z^\dagger R Z, \tag{18}$$

where $\dagger$ denotes hermitian conjugation, Z is a column vector of the weights z_i and

$$\begin{aligned} L_{ij} &= \chi_{ij}(\tau_t, s_t) && (19)\\ R_{ij} &= \sum_{k=1}^{M} r_k \chi_{ij}(\tau_k, s_k) + \frac{1}{\sqrt{SNR_a}} N_R^* \delta_{ij}. && (20) \end{aligned}$$

with

$$\chi_{ij}(\tau, s) = \int l_i^*(t) l_j\left(\frac{t-\tau}{s}\right) dt \tag{21}$$

being the un-normalized broadband ambiguity function (or wavelet transform). The last term of Eq. (20) assumes an orthonormal basis[3] and $Z^\dagger Z = 1$. Note also that this term depends on the weights so that the moments of N_R are dependent on the waveform. This will make the waveform optimization problem more difficult. In practice, since N_R will be treated as an input parameter, this difficulty can be avoided by simply ignoring the weight dependence.[4].

It can be seen from Eqs.(18)-(21) that the signal component can be enhanced by choosing a waveform so that $\chi_{ij}(\tau, s)$ is maximized for the target and minimized for the unwanted echoes. The waveshape of the transmitted signal can be optimized with respect to the environment (ambient noise and interference) by choosing the weights, z_i, such that the performance of the replica correlator is maximized. One reasonable measure of the detector's performance is its output signal-to-noise ratio (SNR). The term noise here refers to the ambient noise and interference collectively. It should be noted that the following formulation is based on a non-statistical approach in that the noise power (ambient noise and reverberation) is not averaged but taken as "instantaneous" from a given time series of the received waveform. The output SNR is given by

$$SNR = \left|\frac{\{\psi|H_1\}}{\{\psi|H_0\}}\right|^2 = \frac{1}{|\Delta|^2}, \tag{22}$$

where

$$\Delta = \frac{Z^\dagger R Z}{Z^\dagger L Z}. \tag{23}$$

Instead of maximizing the SNR, it turns out to be more convenient mathematically to minimize $|\Delta|^2$. For simplicity, all quantities hereafter will be taken to be real without a loss of generality. The problem becomes one of minimizing

$$\Delta^2 = \left(\frac{Z^T R Z}{Z^T L Z}\right)^2, \tag{24}$$

[3]An orthonormal basis also ensures that the energy of the waveform will be independent of the weights. This is not the case for an arbitrary basis in general.

[4]This assumes that the statistics of N_R do not change significantly with the weights. When the noise power spectral density is white, the moments of N_R will be independent of the waveform.

where the superscript T denotes transpose.

The procedure for constructing the optimal waveform is summarised by the following algorithm.

- Choose a set of basis waveforms, $\{l_i(t)\}$.
- Compute the signal matrix L using Eq.(19).
- Input the range, Doppler and amplitude distributions of scatterers.
- Input the ambient noise.
- Compute the noise matrix R using Eq.(20).
- Solve the eigenvalue problem $(R + R^T)Z = \Delta(L + L^T)Z$.
- If there is no sign change between eigenvalues then the optimal weight is given by the eigenvector corresponding to the minimum value of $|\Delta|$.
- If there is a sign change between eigenvalues then the optimal weights are given by the vectors which correspond to the zeros of Δ.

Two important implications of the pulse shaping algorithm should be noted: (i) the properties of the scatterers are assumed to be deterministic and (ii) the existence of multiple solutions.

(i) Echoes which result from a dense environment of multiple targets, such as a target complex or reverberation, are more realistically modelled as random processes. The pulse shaping algorthm should be seen as a mechanism for generating the optimal waveforms for a given realization of the scatterers. To simulate the random nature of reverberation it is expected that some sort of averaging over a number of realizations is required.

(ii) A priori, any solution generated by the algorithm is as good as any other since they all minimize the overlap between the scattering function and the signal ambiguity function for a given realization of the scatterers. To break this impasse, another criterion is needed to select a unique solution. Clearly, an important consideration about the acceptability of a waveform, aside from its practical implementation, is its robustness against changes in the environment and other system parameters. By imposing a robustness requirement the "robust optimum" waveform can be selected from the solution set.

Monte Carlo simulation provides a powerful and tractable means of addressing both of these problems. In the context of a Monte Carlo simulation the robust optimum waveform can be selected as follows:

1. For each trial of the simulation, generate a realization of the scatterers from a specified set of statistical parameters (mean, variance, etc) which describe the amplitude, range and range rate of the scatterers.
2. Use the pulse shaping algorithm to generate a set of waveforms corresponding to each realization of scattering function.
3. Compute the overlap integral, $\int\int p(\tau,\phi)\,|\chi_{uu}(\tau,\phi)|^2\,d\tau d\phi$, or its discrete equivalent for all waveforms.

4. Find the waveform which corresponds to the smallest overlap integral and hence the maximum signal to interference ratio. This is then the robust optimum waveform from a sample of N trials. Ideally, the resulting SNR should approach SNR_a.

4. ILLUSTRATIVE EXAMPLES

Three examples[5] are given to illustrate the above procedure. Firstly, a waveform basis needs to be chosen. To keep the computational load at a reasonable level a basis is chosen so that the L and R matrices can be computed in closed form. One such basis is the truncated Fourier cosine series modulating a carrier such that

$$x(t) = \sqrt{\frac{T}{2}} \sum_{i=1}^{N} z_i l_i(t), \tag{25}$$

where

$$\begin{aligned} l_1(t) &= \frac{1}{\sqrt{T}} rect\left(\frac{t}{T}\right) \cos\left(2\pi f_c t\right), \\ l_i(t) &= \sqrt{\frac{2}{T}} \cos\left[2\pi(i-1)\frac{t}{T}\right] rect\left(\frac{t}{T}\right) \cos\left(2\pi f_c t\right), \quad i \geq 2 \end{aligned} \tag{26}$$

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \leq T/2 \\ 0, & |t| \geq T/2 \end{cases}, \tag{27}$$

T is the pulse length and f_c is carrier frequency. The matrix elements of L and R require the evaluation of the ambiguity function (Eq.(21)) for Eq.(26).

The first example involves a stationary target in a background of reverberation with the following properties. The scatterers are assumed to be

(a) uniformly distributed in range in the vicinity of the target, i.e., $\tau = \sqrt{3}\sigma_\tau (2R_n - 1) + \mu_\tau$, where $\mu_\tau = 0$ and $\sigma_\tau = T/\sqrt{3}$ are the mean and standard deviation of the delay parameter, respectively. R_n is a uniformly distributed random variate lying in the interval $[0, 1]$,

(b) Gaussian distributed in Doppler difference with mean and standard deviation $\mu_\phi = 0$ and $\sigma_\phi = 1/T$ respectively,

(c) and Gaussian distributed in scattering amplitude with mean and standard deviation, $\mu_r = 0$ and $\sigma_r = \sigma_k = 0.1$ respectively.

The simulation is run over 100 trials using 100 scatterers. The waveform is assumed to have a basis with 5 weightings, a carrier frequency of 1 kHz and a pulse length of 1 second. The robust optimum waveform is shown in Fig. 1(a).

The second example is that of a moving target. The simulation is run with the same parameters as in the first case except with $\mu_\phi = 5/T$. The result is shown in Fig. 1(b).

[5] In the following examples, the ambient noise term is set to zero when generating the optimal waveforms. The ambient noise is then included separately afterwards in any performance calculations. This approximation becomes exact for the limiting case of white noise.

The third example examines the case of a combination of fixed and randomly distributed group of scatterers. Out of a total of 100 scatterers, the fixed set (which could, for example, represent a sea mount) consists of 81 scatterers located in the range, $\tau/T = 0.3$ to 0.7, with zero Doppler relative to the target. The remaining scatterers are assumed to have the same range and Doppler distributions as in the first example. Furthermore, the fixed scatterers have a Gaussian distributed amplitude with zero mean and standard deviation of 0.5, while the random set has a mean and standard deviation of zero and 0.1, respectively. From 100 Monte Carlo trials, the resulting waveform is shown in Fig. 1(c).

The performance of the optimal waveforms can also be compared as a function of target velocity. For a background of random scatterers with a given reverberation to noise ratio, the stationary target waveform does better at low target velocities, as expected, but not as well as the reference (Fig. 1(d)) and the moving target waveforms for higher velocity targets. The result is shown in Fig. 2(a). For the case of a background of fixed and random scatterers, Fig. 2(b) shows a significant improvement in the SNR of the optimum waveform over the reference for low Doppler targets. The poor performance of the reference waveform, for low Doppler, is mainly caused by the interference of the fixed scatterers coming through the sidelobes of the reference ambiguity function. In contrast, the optimum waveform's ambiguity function is concentrated away from the strongly reflecting fixed scatterers.

5. CONCLUDING REMARKS

The pulse shaping algorithm presented here provides a means of generating waveforms which optimize the replica correlator output in the presence of interfering targets. The algorithm, however, has nothing to say about the type of waveform basis. It is desirable to have a basis which is computationally simple and yet flexible enough to generate a wide class of waveforms. Having decided upon a basis, there remains the question of its dimensionality. A larger basis will generate a greater number of solutions and hence a better estimate of the optimum robust waveform. This, nevertheless, comes at the expense of a greater number of computations.

Aside from the questions of computational efficiency, accuracy and robustness, is the underlying problem of estimating the environmental parameters, i.e., the amplitude, range and range rate distributions of the scatterers. Reliable estimation of the environment is essential to a useful application of the pulse shaping algorithm in realistic situations. These considerations naturally lead to the question of whether the pulse shaping algorithm can be implemented in an adaptive situation where the transmitted waveform can be continually adjusted to counter the changing environment.

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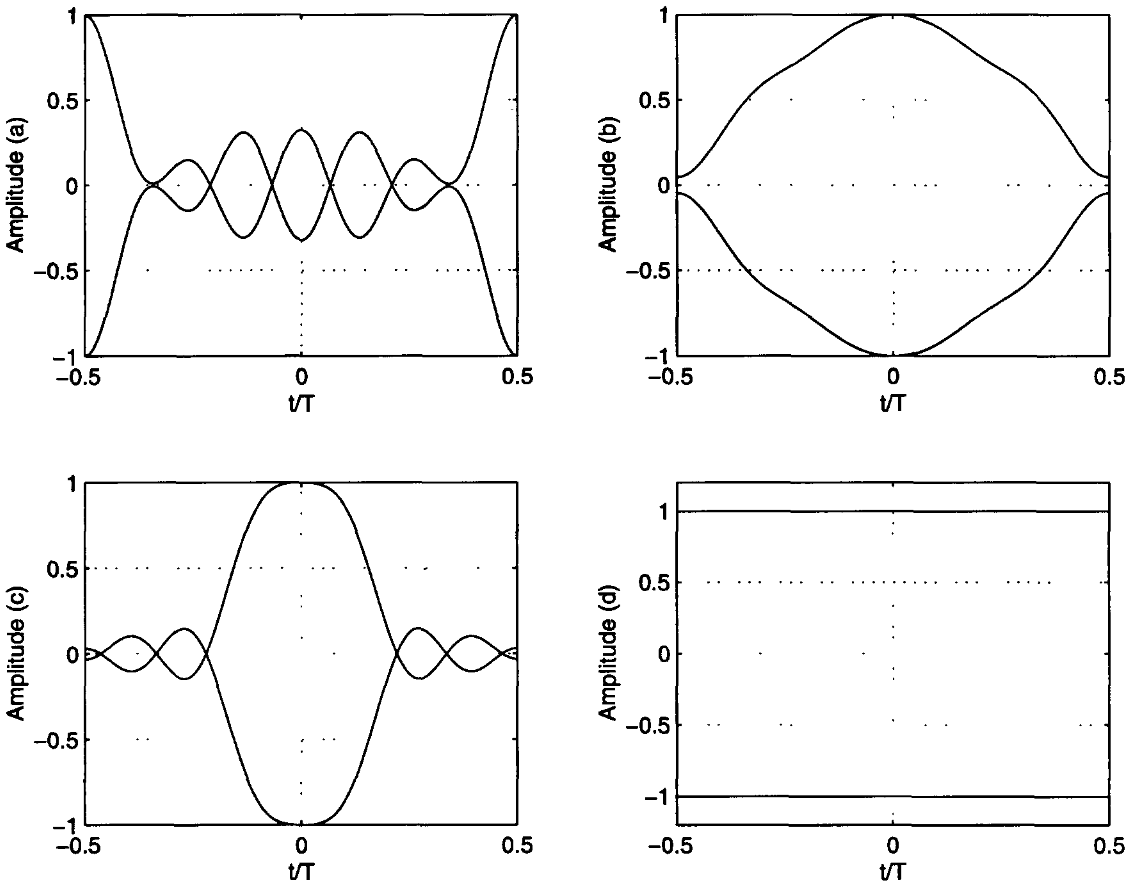

Figure 1: The envelopes of optimal waveforms using a basis of 5 weights for (a) stationary and (b) moving targets in a background of random scatterers. (c) The case of a stationary target in a background of fixed and random scatterers. (d) The rectangular envelope is used as the reference waveform.

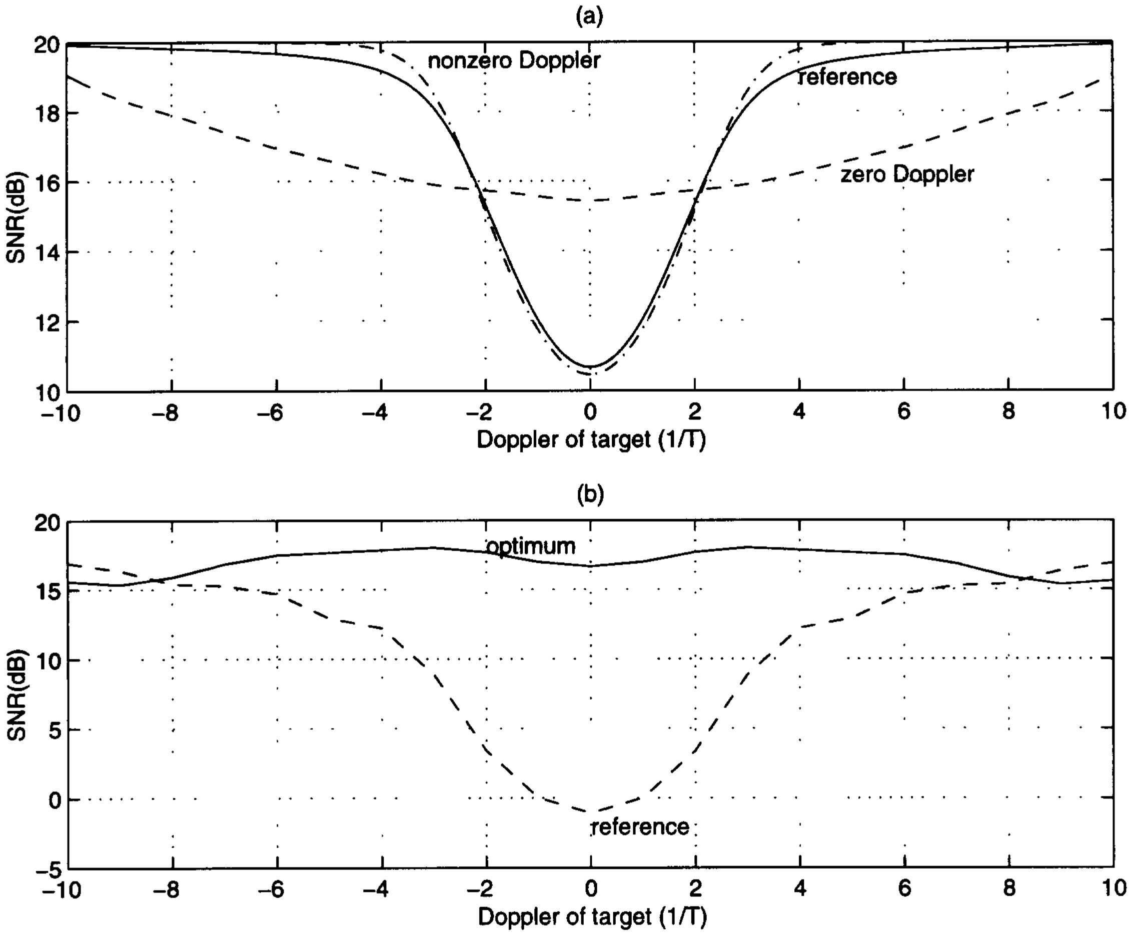

Figure 2: (a) The signal-to-noise ratios of waveforms optimized (i) for a zero Doppler target (dashed line) and (ii) a moving target (dash-dot line) as a function of the target's velocity. The reference waveform (solid line) is also shown for comparison. (b) The signal-to-noise ratios of the optimum (solid line) and reference (dashed line) waveforms as a function of target velocity for the combination of fixed and randomly distributed scatterers whose scattering strengths are $\sigma_f = 0.5$ and $\sigma_r = 0.1$ respectively. The signal-to-ambient noise ratio is assumed to be 20 dB in all cases.