FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

SUPPORT CONDITIONS, THEIR EFFECT ON MEASURED MODAL PARAMETERS

Thomas G. Carne & Clark R. Dohrmann
Sandia National Laboratories
Albuquerque, NM, USA 87185-0557

ABSTRACT

During a modal test, a structure must be supported in some manner to the surrounding environment. If a model of the structure is to be reconciled with modal test data, then the support conditions must either be included in the model or assumed to have negligible effects. For example, supports can be ignored in a simulated free test if they do not affect the structure significantly. Frequently, a precise determination of the actual support conditions is not performed. Consequently, there is uncertainty in the conditions and how they affect both the measured modal frequencies and modal dampings. This study examines the effects of support conditions on the measured modal parameters, and discusses the proper design of support conditions for modal testing.

INTRODUCTION

Modal testing is frequently used to validate the accuracy of structural dynamic models. The modal tests are performed on a structure to measure the modal frequencies, damping ratios, and mode shapes. However during the modal test, a structure must be supported in some manner to the surrounding environment. The structure may even need to be constrained or preloaded in order to test it in some operating condition. For example, an aircraft or rocket launch vehicle may need to be preloaded in order to simulate the flight conditions, and constraints are added to the structure to load it. These supports, constraints, or boundary conditions, will affect the modal parameters of the structure.

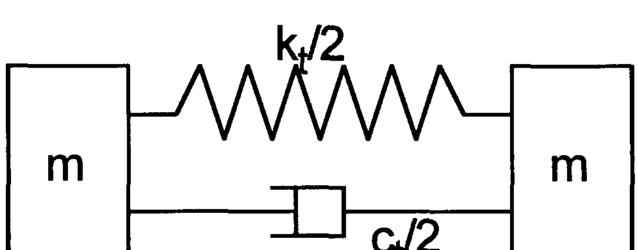
To validate the structural model, the desired modal parameters are those for the structure without the added supports. Thus, one must be concerned with the design of the support conditions because they will affect the modal parameters. The added stiffness, damping, and mass need to be considered when designing the support conditions. These additional conditions must be accounted for when experimental data is compared with predictions from the model.

Historically, there has been concern for support stiffness and its effect on measured modal frequencies. Bisplinghoff, Ashley and Halfman [1] discuss the effects of support stiffness and mass on the modal frequencies, based on results of Rayleigh [2]. Wolf [3] discusses the effects of support stiffness with regard to modal testing of automotive bodies. He reports that the rule of thumb to simulate free boundary conditions is to design the support system so that the rigid body modes, that is the modes that would be at zero frequency except for the support conditions, are no more than one-tenth the frequency of the lowest elastic mode. But, it is seldom possible to achieve this separation for vehicle tests. He states that test engineers use a 1:3 to 1:5 separation ratio between the rigid body modes and the lowest elastic mode. Wolf shows that these stiff supports can lead to significant errors in the measured modal frequencies.

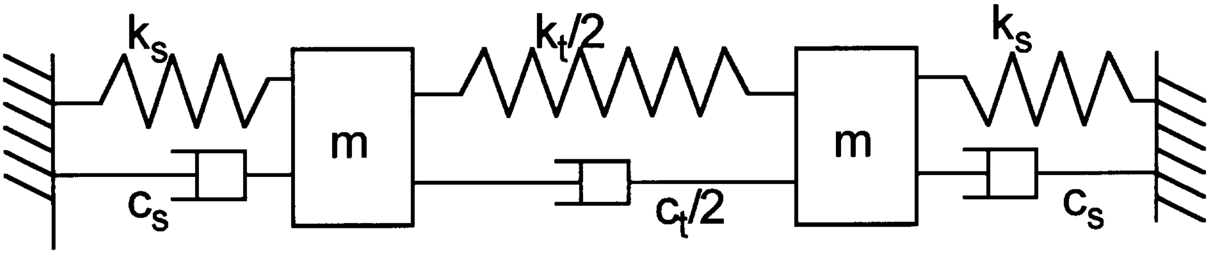
In this paper we examine the effect of support conditions on the modal frequencies, but our primary emphasis is their effect on modal damping ratios. Most finite element models could include the support stiffnesses and masses in the model, thus taking into account those effects. For validation purposes, the supported model could be compared with the supported test for a true comparison. Then the support conditions could be removed from the model. However, structural dynamic models often do not initially include damping, but then use the measured modal damping ratios from a test to create a model including damping. There is typically no validation of the damping model; it is taken directly from the test with the support conditions included. Consequently, one must be concerned with how the support conditions affect the measured damping. Formulas are derived in the following sections to predict the effect of support conditions on both the measured modal frequencies and damping ratios. Several examples are included to demonstrate their application.

THE SINGLE DEGREE-OF-FREEDOM SYSTEM

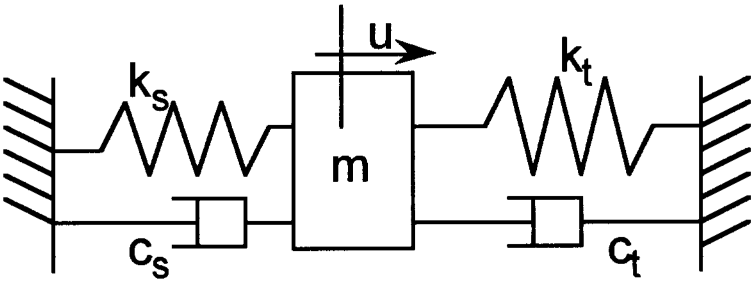
Perhaps the best way to develop an understanding of the effects of support conditions is to examine a single degree-of-freedom (dof) system. Wolf [3] also analyzed a single dof system, but we examine a somewhat different system, that also includes damping. Let us consider a simple model, pictured below, of a freely supported structure (free boundary conditions), consisting of two masses connected by a linear spring and a viscous damper with motion restricted to a single direction. This system has two modal frequencies: zero and $\sqrt{(k_t/m)}$. The zero frequency mode is called the rigid body mode as there is no elastic deformation, while the higher frequency mode is the elastic mode. For the elastic mode, the masses move the same anount but in opposite directions. Although this is just a two dof system, we can think of the dof's as two modal dof's of a more complex multi-dof system.



We could add support conditions in several ways, but let us add them symmetrically as diagrammed below. Here k_t and c_t designate the true stiffness and damping of the structure, while k_s and c_s designate the added support stiffness and damping.



This system is a two degree-of-freedom system, but we are really only concerned with the effects of the supports on the elastic mode. Since that mode shape is symmetric, we can simplify the equations by cutting the dynamic system in half, and imposing a fixed boundary condition, as diagrammed below.



We now have just a single dof system, but with this system we can examine the effects of a support system on the elastic mode of the original two dof system. This model is even more useful because it also models the effects of an added constraint on the single dof system with a fixed boundary condition. This is a trivial model, but we can gain insight from its solution. The homogeneous differential equation is

$$m\ddot{u} + (c_t + c_s)\dot{u} + (k_t + k_s)u = 0.$$

Solving, we find $$u(t) = u_o\, e^{-\zeta_m \omega_m t} \cos(\omega t),$$

where the measured frequency, $$\omega_m = \sqrt{(k_t + k_s)/m}\,, \tag{1}$$

the damped natural frequency, $$\omega = \omega_m \sqrt{1-\zeta^2}\,,$$

and the measured damping ratio, $$\zeta_m = (c_t + c_s)/(2m\omega_m). \tag{2}$$

Now, following Wolf's example [3], we define the true frequency of the structure as

$$\omega_t = \sqrt{k_t / m}\,, \tag{3}$$

and the rigid body frequency, due to the support stiffness as

$$\omega_s = \sqrt{k_s / m} \ . \tag{4}$$

Substituting (3) and (4) in (1), we find a simple expression for the true frequency in terms of the measured frequency and the support frequency.

$$\omega_t = \omega_m \left[1 - \frac{\omega_s{}^2}{\omega_m{}^2} \right]^{1/2} \tag{5}$$

Or if $(\omega_s/\omega_m)^2 \ll 1.0$, then

$$\Delta\omega = \omega_m - \omega_t \cong \frac{\omega_s{}^2}{2\omega_m}. \tag{5a}$$

From (5), it is easy to see the effect of added support stiffness on the measured frequency of the test item. If the support stiffness is such that the ratio of the rigid body frequency, ω_s, to the measured frequency, ω_m, is 1:10, then the true frequency would be less than one half of one percent different from the measured frequency. So the 1:10 ratio is a good rule of thumb for most applications with reasonable accuracy. However, if the ratio were 1:3 as referenced by Wolf, then the error would be over five percent which generally would be unacceptable. Wolf shows a case in which the error would be even as large as fifteen percent for a different dynamic system. However, as discussed earlier, the support stiffness could be included in the model, the supported model validated with the test data, and then the support conditions removed from the model.

Let us now turn our attention to the measured damping ratio. Following the example of the frequency analysis above, let us define damping ratios for the true system and for the support system.

$$\zeta_t = c_t / (2m\,\omega_t) \qquad \textit{and} \qquad \zeta_s = c_s / (2m\,\omega_s) \tag{6}$$

Substituting (6) into (2), we find an expression which relates the three damping ratios.

$$\zeta_m\,\omega_m \;=\; \zeta_t\,\omega_t \;+\; \zeta_s\,\omega_s$$

The above expression can now be solved for the true damping ratio in terms of the other damping ratios.

$$\zeta_t = \zeta_m \frac{\omega_m}{\omega_t} \left[1 - \frac{\omega_s}{\omega_m} \frac{\zeta_s}{\zeta_m} \right] \tag{7}$$

This expression has similarities to that for the frequencies, equation (5), except that the frequency ratio inside the brackets is no longer squared and it is also multiplied by the ratio of the damping ratios. So if we have a frequency ratio of 1:10, as the rule of thumb suggests, and if the support and measured damping ratios are equal, then there would be a ten percent error if the true damping was assumed equal to the measured damping.

However, suppose now we are testing a lightly damped structure and that the frequency ratio is still 1:10, but the support damping is five percent and the measured damping is one percent. Now the ratio of dampings in the brackets has a large effect, and the true damping is only 0.5 percent. So one would have a hundred percent error if one assumed the measured damping was the true damping. Let us now consider the case in which the frequency ratio is 1:3. If the true damping ratio is again 0.5 percent and the support damping ratio is five percent, then the measured damping ratio would be 2.14 percent, resulting in three hundred percent error if one assumed the measured damping was the true damping.

From these examples and equation (7), one can see that the situation for the measured damping ratios is different from that for the measured frequencies. Assuming the true damping ratio is the same as the measured damping ratio is much more likely to result in huge errors than for the frequencies. Unfortunately, most finite element models do not include damping, so one cannot validate a damping model with test data, and then remove the support damping. Frequently, test-derived modal damping is used in the model to create the damping model. In the next section, approximate formulas will be derived for the frequency and damping corrections for a general multi-dof structural dynamic system, similar to those for the single dof system.

THE MULTI-DEGREE-OF-FREEDOM SYSTEM

Let us first examine the real eigenvalue problem for the multi-dof system which yields the eigen or modal frequencies. As we did for the singe dof system, we will separately designate the true stiffness and the support stiffness matrices, and we can write the eigenvalue equation for the true system as,

$$\left[K_t - \omega_t^2 M\right] \phi = 0, \tag{8}$$

where K_t is the stiffness matrix of the unsupported system, ω_t is the eigenvalue associated with the eigenvector ϕ, and M is the mass matrix. For the supported system, we add the stiffness matrix K_s for the support stiffness, and its eignenvalue equation is

$$\left[K_t + K_s - \omega_m^2 M\right] \psi = 0, \tag{9}$$

where ω_m is the measured eigenvalue associated with the eigenvector ψ.

Now let us take the K_s to be small, that is, if we express $\omega_m = \omega_t + \Delta\omega$ and $\psi=\phi+\Delta\phi$, then $\Delta\omega/\omega_m = \varepsilon$ is small compared to unity, and $\Delta\phi = O(\varepsilon)$. Now, we can substitute these expressions into (9) to yield

$$\left[K_t + K_s - \left(\omega_t^2 + 2\Delta\omega\ \omega_m - \Delta\omega^2\right)M\right](\phi + \Delta\phi) = 0. \tag{10}$$

Expanding (9) into separate terms, using (8), premultiplying by ψ', and dropping terms which are $O(\varepsilon^2)$, we have

$$\phi'\left[K_t - \omega_t^2 M\right]\Delta\phi + \psi'\left[K_s - 2\Delta\omega\ \omega_m M\right]\psi \cong 0. \qquad (11)$$

However, if we take the transpose of the first term in (11), use the fact that the stiffness and mass matrices are symmetric, we see from equation (8) that the first term is zero. So (11) reduces further, and we can solve for $\Delta\omega$, yielding

$$\Delta\omega \cong \frac{1}{2\omega_m}\ \frac{\psi' K_s \psi}{\psi' M \psi}.$$

Further, if the mode shape ψ is mass normalized to unity modal mass, then the change is frequency is simply

$$\Delta\omega \cong \frac{1}{2\omega_m}\ \psi' K_s \psi. \qquad (12)$$

This formula for the change in frequency is quite simple, and it is straightforward to evaluate, using the finite element analysis of the supported structure. The matrix K_S will have very few non-zero terms, so the triple product $\psi' K_S \psi$ will be easy to evaluate. Comparing (12) with (5a), one can see that the result for the multi-dof case reduces exactly to that of the single dof case when K_S is just one by one and equal to k_S. Also note that ω_m is in the denominator on the right hand side of (12), so at higher modal frequencies $\Delta\omega/\omega_m$ varies proportionally as $(1/\omega_m)^2$.

We can now turn to the issue of damping in a supported structure. For the support damping, the situation is more complicated than for the modal frequency because one will typically not have an analytical model of the damping in the structure. But the issue is the same as for the stiffness, given a measurement of a modal damping, how can we determine the modal damping of the structure without the support system. We will show in the following analysis that we can determine the modal damping in the structure if the support system makes a small change to the modes of the structure, and we have the mode shape components at the support connections and a damping model for the support system. Let us now look at the complex eigenvalue equation for a particular mode of the system.

$$\psi'\left[K + i\,\omega_m\left(C_t + C_s\right) - \omega_m^2 M\right]\psi = 0, \qquad (13)$$

where we have pre-multiplied by the transpose of the real mode shape ψ, which is the real eigenvector resulting from the undamped eigenvalue problem. K is the total stiffness matrix, ω_m is the measured frequency, i is $\sqrt{-1}$, C_t is the damping matrix for the true structure, and C_S is that for the support system. We expand this equation and assume that $\psi'(C_t + C_S)\,\psi$ is an adequate measure of the damping even though ψ is the real mode shape from the undamped equation. The damping ratio is now defined conventionally, within the assumptions above, as

$$\zeta_m = \frac{1}{2\omega_m}\ \frac{\psi'\left(C_t + C_s\right)\psi}{\psi' M \psi}. \qquad (14)$$

Expanding (14), and defining
$$\zeta_t = \frac{1}{2\omega_t}\,\frac{\psi' C_t \psi}{\psi' M \psi}, \tag{15}$$

and taking the mode shape to be mass normalized, we have

$$\zeta_m = \frac{\omega_t}{\omega_m}\zeta_t + \frac{\psi' C_s \psi}{2\,\omega_m}.$$

Solving for ζ_t, we find
$$\zeta_t = \zeta_m \frac{\omega_m}{\omega_t}\left[1 - \frac{\psi' C_s \psi}{2\,\omega_m \zeta_m}\right]. \tag{16}$$

This formula for the true damping ratio, as the frequency formula, is a fairly simple expression. Given the measured modal damping, the measured modal frequency, the mode shape components at the support dof's, and the damping matrix for the supports dof's, the true damping ratio of the unsupported structure can be calculated. Equation (16) can also be compared to (7) for the single dof case. Again, these equations are very similar, and (16) reduces to (7) for the single dof case.

Equation (16) reveals some important features, just as (7) did. Because the quantity in the brackets is the difference between unity and a positive ratio, the difference between the true damping ratio and the measure damping ratio can be significant if the last term in the brackets is not close to zero. As an example, a two dof system was created with support damping such that the rigid body mode has a 5.0 % damping, true damping set to 0.5 %, and the ratio of the rigid body mode frequency to the elastic mode frequency set at 1:2, This example produced a measured damping ratio of 2.61 %, over five times the true damping.

CONCLUSIONS

In this paper we have examined the effects of support stiffness and damping on measured modal frequencies and damping ratios. The single dof system revealed interesting results which produced insight for more complex multi-dof systems. The increase in the measured frequency of the supported system was related to the square of the ratio of the frequencies of the rigid body mode and the elastic mode. The damping was much more sensitive as the correction involved both the ratio of frequencies and the ratio the dampings. Consequently, even for softly a supported structure, the measured damping could be far from the true damping ratio. Approximate formula were derived for the multi-dof case which showed the change in modal frequency and modal damping for added support stiffness and damping. These formulas were similar to the single dof formulas and reduce to them for the single dof case. These formulas can be used to aid in the design of a support system for modal testing of free or constrained structures.

ACKNOWLEDGMENTS

The authors wish to acknowledge discussions and helpful suggestions made by Randy Mayes and Todd Simmermacher of Sandia National Laboratories.

REFERENCES

1. Bisplinghoff, R. L., Ashley, H., & Halfman, R.L., Aeroelasticity, Addison-Wesley Publishing Company, Inc., Cambridge, MA, pp. 771-779, 1955

2. Strutt, J.W. (Lord Rayleigh), The Theory of Sound, vol.1, 2nd ed., Dover Publications, Inc., New York, 1945.

3. Wolf Jr., J.A., "The Influence of Mounting Stiffness on Frequencies Measured in a Vibration Test", SAE Paper 840480, Society of Automotive Engineers, Inc., 1984.