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POLE-ZERO ASSIGNMENT OF VIBRATORY SYSTEM

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Recent results associated with simultaneous assignment of poles and zeros by state feedback control are presented. It is shown that the poles and some zeros of a vibratory system may be assigned by choosing the position vector and the control force. This objective is achieved with partial knowledge modal date associated with the eigenpairs which are intended to be changed. An example demonstrating the results is given.

1. Introduction: We summarise in this paper the main results obtained in [1]. The system under consideration is modelled by the matrix differential equation

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{o} \qquad (1)$$

with positive definite symmetric mass ,$\mathbf{M} \in \Re^{n \times n}$, and stiffness, $\mathbf{K} \in \Re^{n \times n,}$ matrices. Separation of variables,

$$\mathbf{x}(t) = \phi e^{st}, \qquad (2)$$

ϕ a constant vector, leads to the problem of determining the eigenvalues and the eigenvectors of the *open-loop linear pencil*

$$P_o(\lambda) = \mathbf{K} - \lambda \mathbf{M}, \quad \lambda = -s^2 \qquad (3)$$

The pencil (3) has n real positive eigenvalues λ_i satisfying

$$P_o(\lambda_i)\mathbf{\phi_i} = \mathbf{o} \tag{4}$$

associated with the $2n$ imaginary poles s_i.

Denote by $\hat{\mathbf{M}}$ and $\hat{\mathbf{K}}$ the sub-submatrices obtained by deleting the last row and column of **M** and **K**, respectively. Then if the system is excited by the harmonic excitation

$$\mathbf{f}(t) = \mathbf{e_n} \sin\omega t, \tag{5}$$

$\mathbf{e}_n$ the n-th unit vector, then the response has a particular solution of the form

$$x_n(\omega, t) = \frac{\det(\hat{\mathbf{K}} - \omega^2 \hat{\mathbf{M}})}{\det(\mathbf{K} - \omega^2 \mathbf{M})} \sin\omega t. \tag{6}$$

Let μ_i , $i=1,2,\ldots,n-1$ be the eigenvalues of $\hat{\mathbf{K}} - \omega^2 \hat{\mathbf{M}}$. Then (6) gives the following amplitudes of vibration

$$\begin{cases} 0, & when \quad \omega^2 = \mu_i; \;\; i = 1,2,\ldots,n-1 \\ \to \infty, & when \quad \omega^2 = \lambda_i; \;\; i = 1,2,\ldots,n \end{cases} . \tag{7}$$

The imaginary values $\sqrt{-\mu_i}$ are called the *zeros* of $x_n(\omega,t)$. The motivation to assign some poles and zeros of the system thus follows from (7). Assigning the poles appropriately ensures that the overall response of the system is of small amplitude of vibrations. By assigning the zeros certain amplitudes of vibrations may be eliminated.

The assignment of poles and zeros may be achieved by applying a control force $\mathbf{b}u(t)$, where $\mathbf{b}$ is a constant vector and $u(t)$ is the control function. The dynamics of the controlled system is thus governed by

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{b}u(t). \tag{8}$$

If the control function is chosen to satisfy

$$u(t) = \mathbf{a}^{\mathrm{T}}\mathbf{v}(t), \tag{9}$$

a a constant vector, then the system is said to be controlled by *state feedback,* since the output state vector **v**(t) is returned to the input. The equations of motion (9) may be written in the form

$$\mathbf{M}\ddot{\mathbf{v}} + (\mathbf{K} - \mathbf{b}\mathbf{a}^{\mathrm{T}})\mathbf{v} = \mathbf{o}. \tag{10}$$

Denote the stiffness matrix of the *closed loop system* by

$$\mathbf{K}_{\mathrm{C}} \equiv \mathbf{K} - \mathbf{b}\mathbf{a}^{\mathrm{T}}. \tag{11}$$

Then, separation of variable

$$\mathbf{v}(t) = \boldsymbol{\phi} \sin\omega t \tag{12}$$

leads to the problem of evaluating the eigenpairs of the *closed loop pencil*

$$P_C(\lambda) = \mathbf{K}_{\mathrm{C}} - \lambda\mathbf{M}, \quad \lambda = \omega^2. \tag{13}$$

If the controlled system is excited by the harmonic force (5) then the vibrations are governed by

$$\mathbf{M}\ddot{\mathbf{w}} + \mathbf{K}_{\mathrm{C}}\mathbf{w} = \mathbf{e}_{\mathbf{n}} \sin\omega t \tag{14}$$

and the frequency response function $y_n(\omega)$ of the closed loop pencil is now

$$y_n = \mathbf{e}_n^T \left(\mathbf{K}_{\mathrm{C}} - \omega^2\mathbf{M}\right)^{-1} \mathbf{e}_n. \tag{26}$$

Let $\hat{\mathbf{K}}_{\mathrm{C}}$ be the submatrix obtained by deleting the last row and column of $\mathbf{K}_{\mathrm{C}}$. Then by the Cramer's rule we have

$$y_n(\omega) = \frac{\det(\hat{\mathbf{K}}_{\mathrm{C}} - \omega^2\hat{\mathbf{M}})}{\det(\mathbf{K}_{\mathrm{C}} - \omega^2\mathbf{M})}. \tag{27}$$

Denote the eigenvalues of $\mathbf{K_C} - \lambda\mathbf{M}$ by τ_i, i=1,2,...,n, and the eigenvalues of $\hat{\mathbf{K}}_\mathbf{C} - \lambda\hat{\mathbf{M}}$ by θ_i, i=1,2,...,n-1. Since in general $\mathbf{K_C}$ is not symmetric the closed loop system (21) has poles $\pm\sqrt{-\tau_i}$ and zeros $\pm\sqrt{-\theta_i}$ which are generally complex numbers. To avoid instability, which may occur in an actively controlled system, we must assure that the poles of the system do not lie in the right hand side of the complex plane.

2. PROBLEM DEFINITION. Suppose **M**, **K** and a set of n positive real numbers, $\tau,\theta_1,\theta_2,\ldots,\theta_{n-1}$ are given. The problem under consideration is to obtain the vectors **a** and **b** such that

$$\det(\mathbf{K_C} - \lambda\mathbf{M}) = 0, \ \text{for } \lambda = \tau,\lambda_2,\lambda_3,\ldots,\lambda_n, \tag{30}$$

and

$$\det(\hat{\mathbf{K}}_\mathbf{C} - \lambda\hat{\mathbf{M}}) = 0, \ \text{for } \lambda = \theta_1,\theta_2,\ldots,\theta_{n-1}. \tag{31}$$

3. THE ALGORITHM. The following algorithm summarises the main results obtained in [1].

Input: $\mathbf{K}\in\Re^{nxn}$, $\mathbf{M}\in\Re^{nxn}$, two positive definite symmetric matrices; and a set of n positive numbers $\tau,\theta_1,\theta_2,\ldots,\theta_{n-1}$.

Algorithm:

(a) Determine the eigendecomposition

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda}$$

where $\Lambda = diag\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$,

$$\mathbf{\Phi} = [\boldsymbol{\phi}_1|\boldsymbol{\phi}_2|\ldots|\boldsymbol{\phi}_\mathbf{n}],$$

and partition

$$\mathbf{K} = \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{k}} \\ \hat{\mathbf{k}} & k_{nn} \end{bmatrix}, \ \hat{\mathbf{K}} \in\Re^{(n-1)\times(n-1)},$$

$$\mathbf{M} = \begin{bmatrix} \hat{\mathbf{M}} & \hat{\mathbf{m}} \\ \hat{\mathbf{m}} & m_{nn} \end{bmatrix}, \ \hat{\mathbf{M}} \in\Re^{(n-1)\times(n-1)}.$$

(b) Obtain

$$\mathbf{a} = \frac{\lambda_1 - \tau}{\lambda_1} \mathbf{K}\phi_1$$

and partition

$$\mathbf{a} = \begin{pmatrix} \hat{\mathbf{a}} \\ a_n \end{pmatrix}.$$

(c) Solve the eigenvalue problem $(\hat{\mathbf{K}} - \mu_i \hat{\mathbf{M}})\psi_{\mathbf{i}} = \mathbf{o}$ and find

$\mu_i, \psi_{\mathbf{i}}$, for $i=1,2,...,n-1$. Denote

$$\Psi = [\psi_{\mathbf{1}} | \psi_{\mathbf{2}} | \ldots | \psi_{\mathbf{n}}].$$

(d) For $k=1,2,...,n-1$ calculate

$$r_k = \frac{1}{\hat{\mathbf{a}}^{\mathbf{T}} \psi_{\mathbf{k}}} \frac{\theta_k - \mu_k}{\mu_k} \prod_{\substack{i=1 \\ i \neq k}}^{n-1} \frac{\theta_i - \mu_k}{\mu_i - \mu_k}$$

and define

$$\mathbf{r} = (r_1, r_2, \ldots, r_{n-1})^T.$$

(e) Determine

$$\hat{\mathbf{b}} = -\hat{\mathbf{K}} \Psi \mathbf{r}.$$

(f) Obtain

$$b_n = \frac{1}{a_n}(1 - \hat{\mathbf{a}}^{\mathbf{T}} \hat{\mathbf{b}})$$

and define

$$\mathbf{b} = \begin{pmatrix} \hat{\mathbf{b}} \\ b_n \end{pmatrix}.$$

Output: Two vectors $\mathbf{a} \in \Re^{\mathrm{n}}$ and $\mathbf{b} \in \Re^{\mathrm{n}}$.

The closed loop pencil $(\mathbf{K} - \mathbf{b}\mathbf{a}^{\mathbf{T}}) - \lambda\mathbf{M}$ has eigenvalues $\tau, \lambda_2, \lambda_3, \ldots, \lambda_n$, where $\lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of the open loop pencil $\mathbf{K} - \lambda\mathbf{M}$. The *(n-1)×(n-1)* subpencil $(\hat{\mathbf{K}} - \hat{\mathbf{b}}\hat{\mathbf{a}}^{\mathbf{T}}) - \lambda\hat{\mathbf{M}}$ has eigenvalues $\theta_1, \theta_2, \ldots, \theta_{n-1}$.

4. AN EXAMPLE. An harmonic force

$$f(t) = 1.5\sin t + 2.2\sin 2.04t + 1.7\sin 3t$$

is applied on the mass-spring system shown in Figure 1(a). The forced vibration of the system are governed by the differential equation

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{e}_4 f(t)$$

where

$$\mathbf{M} = diag\{1 \quad 2 \quad 2 \quad 1\}$$

$$\mathbf{K} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -2 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

and $\mathbf{e}_4 = (0\ 0\ 0\ 1)^{\mathrm{T}}$. Suppose that the initial conditions are given by

$$\mathbf{v}(0) = \dot{\mathbf{v}}(0) = \mathbf{o}.$$

The response of the system consists of two components

$$\mathbf{v} = \mathbf{g} + \mathbf{p}$$

where $\mathbf{g}$ is the general solution of

$$\mathbf{M}\ddot{\mathbf{g}} + \mathbf{K}\mathbf{g} = \mathbf{o}$$

and $\mathbf{p}$ is a particular solution of

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{d}f(t).$$

Solving these equations gives the vibration response of the mass m_4

$$v_4 = g_4 + p_4$$

where

$$p_4 = 0.2143\sin t + 45.2724\sin 2.04t - 0.4939\sin 3t$$

and

$$g_4 = 2.1215\sin 0.3366t + 0.1505\sin 1.3599t$$
$$-45.1812\sin 2.0412t + 0.0835\sin 2.6212t$$

The large amplitude of vibration is due to the forced excitation near the resonance frequency 2.0412.

We wish to assign the zeros of the control system shown in Figure 1(b) to the exciting frequencies, ie. $\theta_1 = 1$, $\theta_2 = (2.04)^2 = 4.1616$ and $\theta_3 = 9$. To avoid near-resonance-excitation it is also required to shift the eigenvalue λ_3=4.1667. We choose to shift λ_3 to the value τ=25. To avoid *spill-over,* the other eigenvalues required to remain unchanged. The problem under consideration is to obtain the coefficients a_i of the control function

$$u(t) = \mathbf{a}^{\mathbf{T}} w = \sum_{i=1}^{4} a_i w_i$$

and the gains b_i at the various masses such that the above design objectives are met. Following the algorithm of section 3 we obtain

$$\mathbf{a} = (-16.8017 \quad 19.6025 \quad 0.6213 \quad -7.4545)^T$$

$$\mathbf{b} = (0.0804 \quad -0.3820 \quad -0.2125 \quad 2.1023)^T$$

The state feedback control applied on the system of Figure 1(b) is therefore

$$\mathbf{b}u(t) = \begin{pmatrix} 0.0804 \\ -0.3820 \\ -0.2125 \\ 2.1023 \end{pmatrix} u(t)$$

where

$$u(t) = -16.8017w_1 + 19.6025w_2 + 0.6213w_3 - 7.4545w_4.$$

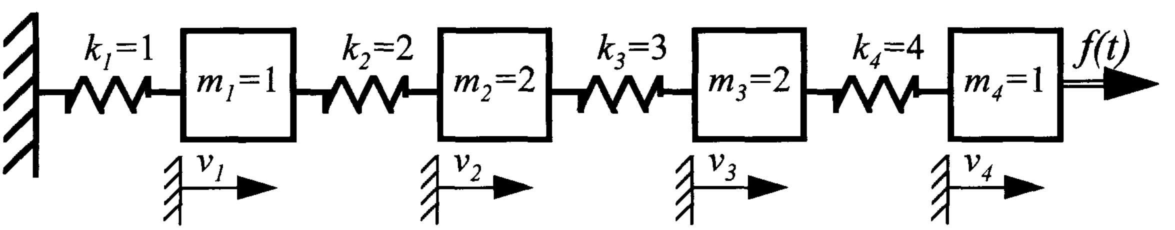
With this control the response of m_4 is given

$$\begin{aligned} w_4 &= -0.0914\sin 5t + 1.0952\sin 0.3366t \\ &+0.0454\sin 1.3599t + 0.0101\sin 2.6212t \end{aligned}.$$

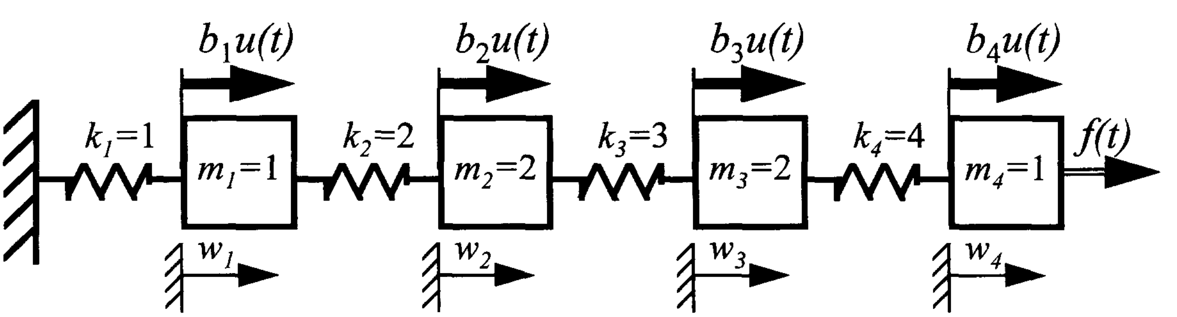
A significant reduction in the amplitude of vibration has been achieved by using the pole-zero assignment technique.

REFERENCES

Y.M. Ram, 1997, Pole-zero assignment of vibratory systems by state feedback control, Journal of Vibration and Control, (To appear).



(a) Forced system



(b) Forced controlled system

Figure 1