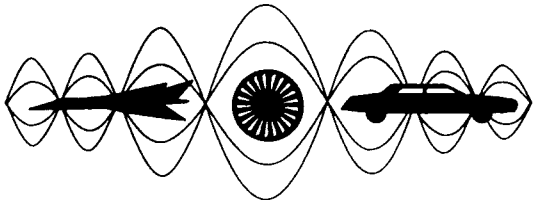

FIFTH INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
DECEMBER 15-18, 1997
ADELAIDE, SOUTH AUSTRALIA

Invited Paper

A CONVENIENT ACCURACY CRITERION FOR TIME DOMAIN FE-CALCULATIONS

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ABSTRACT

An accuracy criterion that is well suited to time domain finite element (FE) calculations is presented. It is then used to develop a method for selecting time steps and element meshes that produce accurate results without significantly overburdening the computer. Use of this method is illustrated with a simple example, where comparison with an analytical solution shows that results are sufficiently accurate, which is not always the case with more primitive methods for determining the discretisation.

INTRODUCTION

The finite element method is a numerical method that is particularly well suited for predicting acoustic responses at low frequencies, especially when the boundary conditions are irregular. In theory, its accuracy is limited only by the accuracy of the mathematical model on which it is based (e.g. the governing partial differential equations and modeling of boundary conditions), as one can always increase accuracy by refining the discretisation. Unfortunately, finer meshes require more memory and calculation time on the computer, so the user of the method is constantly in a dilemma: If too coarse a grid is used, results will be inaccurate, whereas a grid that is too fine will overburden the computer. There is therefore a need for a method to obtain good discretisations - i.e. discretisations that will achieve a specified degree of accuracy with the lowest possible burden on the computer. The aim of this paper is to present such a method for time domain calculations.

The first step is to find a good way to specify accuracy - i.e. a criterion that is simple and yet gives a sharp dividing line between what is acceptable and what is not. This criterion combined with an *a priori* error estimator can then be used to determine some bounds for the discretisation - in this case a maximum time step size and maximum element side length.

THE ACCURACY CRITERION

In reference [1], it is pointed out that the dominating error in time domain FE-calculations takes the form of a falsely predicted dispersion. The accuracy criterion presented here has therefore been constructed so that it reflect this error.

It is well known in the field of acoustics that the acoustic pressure field $p(x,y,z,t)$ in any enclosure can be split into eigenfunctions. If any coupling between eigenmodes is neglected and all damping is viscous, then the acoustic pressure is given by the following expression:

$$p(x,y,z,t)=Re\left\{ \sum_{j=1}^{\infty} \alpha_j.p_j(x,y,z).e^{i\omega_j t} \right\} + p_0(x,y,z,t) \qquad (1)$$

where the ω_j's are complex eigenfrequencies, the $p_j(x,y,z)$'s are the corresponding complex eigenmode shapes and $p_0(x,y,z,t)$ is a real function which arises from the excitation. The α_j's are complex constants expressing the phase and amplitude of excitation of the resonance modes at the time t=0.

Under the same assumptions as above, a time domain finite element solution can be decomposed in a similar way, the only difference being in the values of the α_j's, ω_j's, $p_j(x,y,z)$'s and $p_0(x,y,z,t)$ which only will be approximations of the correct values or functions. This is expressed as follows:

$$\tilde{p}(x,y,z,t)=Re\left\{ \sum_{j=1}^{\infty} \tilde{\alpha}_j.\tilde{p}_j(x,y,z).e^{i\tilde{\omega}_j t} \right\} + \tilde{p}_0(x,y,z,t) \qquad (2)$$

where a superposed ~ denotes the finite element approximation of the corresponding quantity in Equation (1).

Subtracting Equation (2) from Equation (1) will give an expression for the error on the FE-approximation of the exact solution. The expression which results (where x,y and z have been omitted for clarity) is:

$$p(t)-\tilde{p}(t)=\mathrm{Re}\left\{ \sum_{j=1}^{\infty} \left(\alpha_j p_j e^{i\omega_j t} - \tilde{\alpha}_j \tilde{p}_j e^{i\tilde{\omega}_j t}\right) \right\} + p_0(t)-\tilde{p}_0(t) \qquad (3)$$

The format of Equation (3) suggests that there are four possible sources of error - namely the errors on α_j, ω_j, $p_j(x,y,z)$ and $p_0(x,y,z,t)$. Of these, all but the error on the real part of ω_j are neglected, since this error is judged to have by far the most significant effect on the dispersion error. The reasoning behind this postulation is as follows:

The errors on α_j and $p_0(x,y,z,t)$ do not give rise to dispersion error because the quantities express the initial conditions and excitation whereas the predicted speed of sound in the model is independant of initial conditions and excitation. The imaginary part of ω_j influences only the decay rate of the response, so an error on it could not possibly give rise to dispersion.

The reason why the error on $p_j(x,y,z)$ is not judged to influence the dispersion error can be explained with the aid of the fundamental wave identity $c=f\lambda$ (where c is the speed of sound, f is the frequency and λ is the wave length). Using this it can be seen that any error on the speed

of sound can be expressed as a combination of an error on frequency and an error on wavelength. When looking at errors on the eigenmode $\boldsymbol{p}_j(x,y,z)$, they can not readily be interpreted as either of these. An example of an eigenmode predicted with FEM is shown in Figure 1.

On the whole, one can not really say that there is an error on the wave length in Figure 1, whereas the error on the real part of ω_j is over 20%.

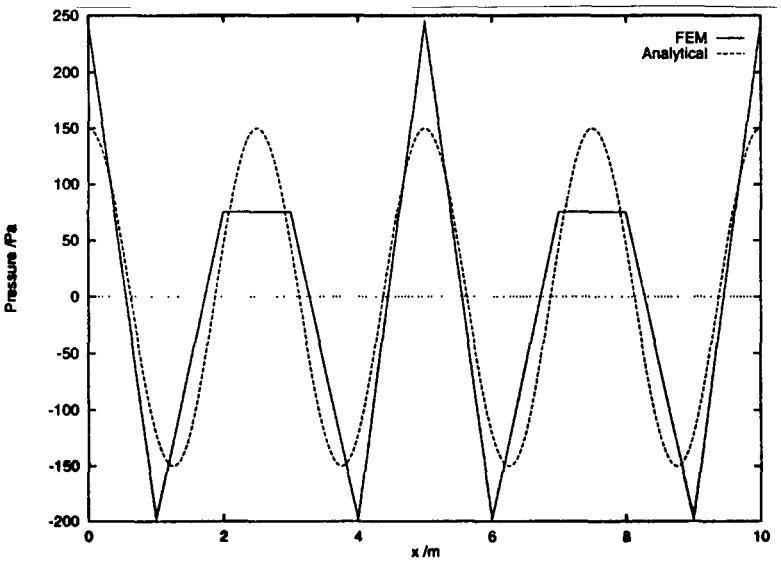

Figure 1: The eighth non-trivial eigen mode in a 10m long hard 1-dimensional cavity calculated analytically and using an FE-model consisting of ten linear elements. Note that the error cannot be described as an overall error on wavelength.

Neglecting all errors except the error on the real part of ω_j reduces Equation (3) to

$$p(t) - \tilde{p}(t) \simeq \mathrm{Re}\left\{ \sum_{j=1}^{\infty} \alpha_j \boldsymbol{p}_j \mathrm{e}^{-\mathrm{Im}\{\omega_j\}t} \left(\mathrm{e}^{i\omega_j t} - \mathrm{e}^{i\tilde{\omega}_j t} \right) \right\} \qquad (4)$$

where ω_j is the real part of ω_j.

Equation (4) shows that the error only can be large if the phase error $(\omega_j - \tilde{\omega}_j)t$ is considerable and if the corresponding modal participation $\alpha_j \boldsymbol{p}_j \mathrm{e}^{-\mathrm{Im}\{\omega_j\}t}$ is significant. Figure 2 illustrates this error by showing three functions that only differ in the frequency of the highest harmonic component.

Figure 2 clearly shows that the largest deviations occur at the times when the phase error is around π, but considerable errors are also present at other times, except when the phase error is close to zero. To get an acceptable solution, one must therefore demand that the phase error be less than a certain value for all frequency components that participate significantly in the solution.

The accuracy criterion is therefore as follows:
For all modes that participate significantly in the solution, it must be demanded that

$$\left| (\omega_j - \tilde{\omega}_j)t \right| \leq \varepsilon \qquad (5)$$

where ε is the maximum acceptable phase error.

Equation (5) is remarkably simple, which has the advantage of giving the engineer using it the opportunity to understand it. Another advantage of Equation (5) is that it is easy to evaluate *a*

priori (see the next section). Before using the error estimator however, one should be aware that it requires a certain amount of engineering judgement to use it - Identifying which modes are significant and selecting the value of ε cannot be done objectively. The authors short experience with this formula indicates that setting $\varepsilon=\pi/10$ gives a high degree of accuracy, whereas selecting $\varepsilon=\pi/3$ gives a modest degree of accuracy. The example later in the paper demonstrates how to identify significant eigenmodes.

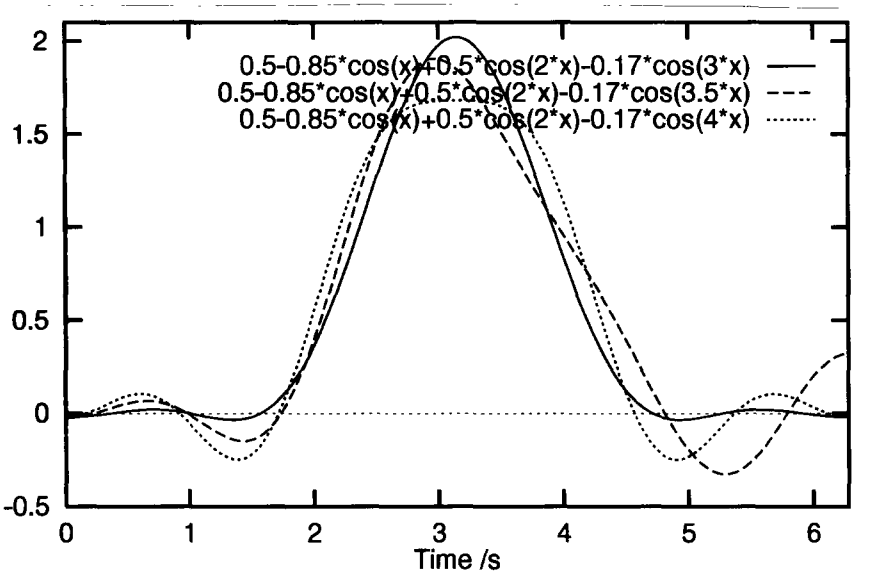

Figure 2: A function with different degrees of distortion on the highest frequency component. Illustrates the effect of phase errors from error on frequency.

APPLICATION TO FEM

The accuracy criterion (Equation (5)) can be used on many time domain FE-calculation strategies. Two of the most useful of these shall be treated here: The first is ordinary finite elements (see e.g. Huebner [2]) with consistent mass matrix using the trapezoidal rule to discretise the time domain, the second is special Lagrange elements with lumped mass matrix (see Jensen [3]) using the central difference method to discretise the time domain. In both cases, the total error is approximately equal to the sum of the error due to the time discretisation and the error due to the spatial FE-discretisation. Good reliable expressions for both error types are very common in the literature (e.g. Hughes [4] and Strang & Fix [5]). The following two expressions are given in Jensen [1]:

$$\omega_j - \tilde{\omega}_j \simeq k\left(\frac{h}{2\pi c/\omega_j}\right)^{2p}\omega_j \tag{6}$$

$$\omega_j - \tilde{\omega}_j \simeq \omega_j \frac{12\beta - 1}{24}(\omega_j \Delta t)^2 \tag{7}$$

where h is the element side length, c is the speed of sound, Δt is the time step size and p is the order of complete polynomial in the element shape function. k is a constant that depends on element type (partially tabulated in reference [1]) and β is equal to ¼ for the trapezoidal rule and 0 for the central difference method. Naturally, Equation (6) is the error due to the spatial discretisation whereas Equation (7) is the error due to the time discretisation.

To test whether the accuracy criterion is fulfilled, it would seem logical to insert Equations (6) and (7) into Equation (5) for all eigenmodes. This would however be too cumbersome, and because the two errors tend to cancel one another, it would also give unreliable answers. It is therefore a better idea to test whether the two errors meet the criterion individually - This would certainly be on the safe side and would only need to be done for the highest significant frequency component.

Inserting Equations (6) and (7) individually into Equation (5) and solving for Δt and h yields:

$$h \le \frac{c}{f}\left(\frac{\varepsilon}{|k|T2\pi f}\right)^{1/2p}$$

$$\Delta t \le \sqrt{\frac{3\varepsilon}{|12\beta-1|\pi^3 T f^3}}, \qquad \beta \neq \frac{1}{12} \tag{8}$$

where T is the total duration of the solution and $f=\omega_j/2\pi$ for the eigenmode with the highest frequency that still is significant.

NUMERICAL EXAMPLE

The geometry of the problem is shown in Figure 3. It is a cubic cavity ($0<x<9$m, $0<y<9$m and $0<z<9$m) with a linear acoustic medium with density $\rho=1.2$ kg/m^3 and speed of sound $c=360$ m/s. All boundaries are hard. The excitation is a stiff square piston ($x=0$, $0<y<3$m and $0<z<3$m) with the acceleration given in Equation (9). A plot of the function is shown in Figure 4 (it is one period of a sine function weighted with a Hanning function). Figure 5 shows its amplitude spectrum.

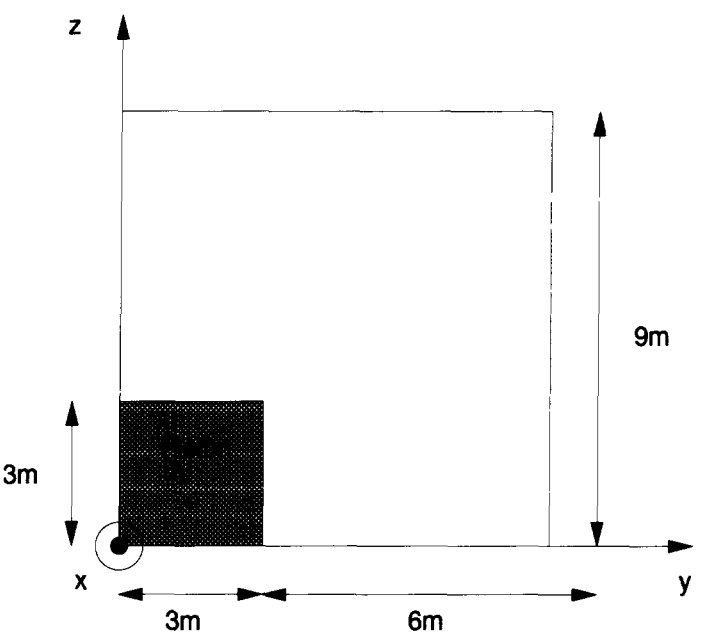

Figure 3: The geometry of the example problem.

$$a(t) = \begin{cases} \sin^2\left(\pi\frac{t}{t_A}\right)\sin\left(2\pi\frac{t}{t_A}\right) & t\in[0,t_A] \\ 0 & t\notin[0,t_A] \end{cases}, \quad t_A=\frac{1}{45}\text{s} \tag{9}$$

Dividing the piston into 64 squares of equal size, approximating each by a point source at its centre and calculating the response using image sources gives a solution that is visually indistinguishable from the exact mathematical solution. This shall be referred to as the analytical

solution.

Two finite element solutions based on the accuracy criterion are calculated: The first uses 20-noded serendipity elements with consistent mass matrix and the trapezoidal rule, whereas the second uses 27-noded Lagrange elements with lumped mass matrix and the central difference method.

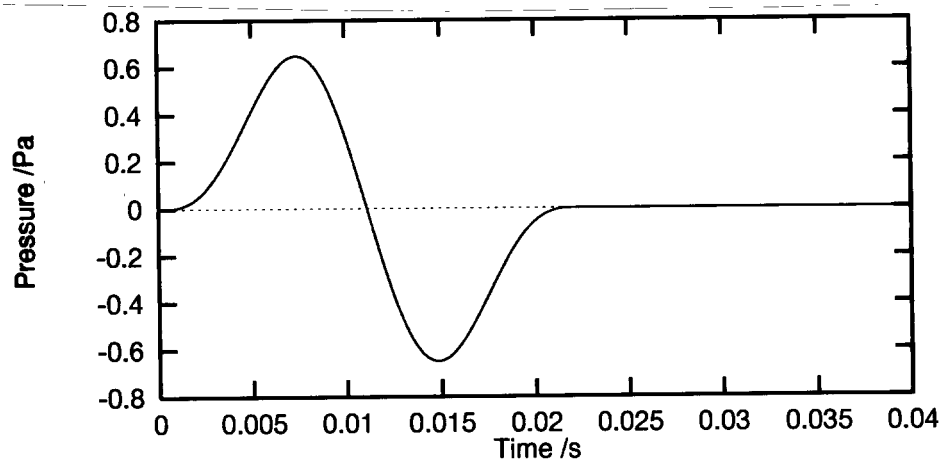

Figure 4: Excitation - Acceleration as a function of time.

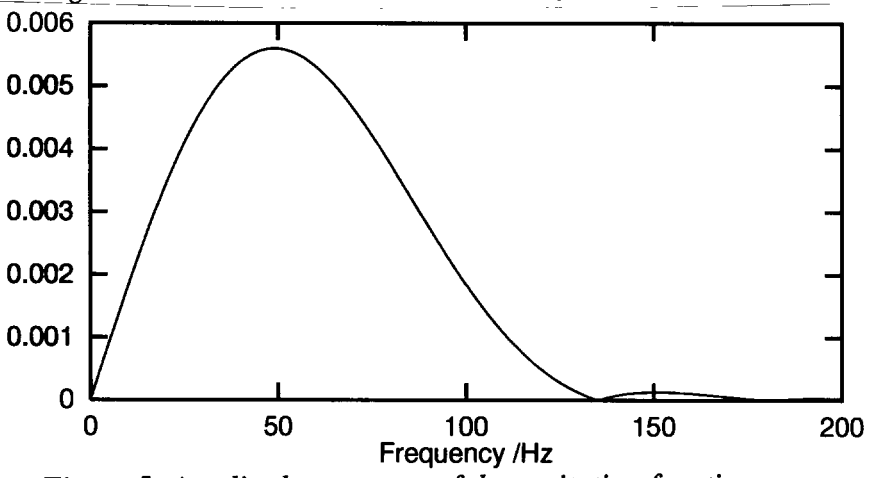

Figure 5: Amplitude spectrum of the excitation function.

The maximum frequency is obtained by assuming that, for the purposes of estimating accuracy, the amplitude spectrum at any receiver position can be approximated by the amplitude spectrum of the excitation (Figure 5). On the basis of this, a value of f_{max}=95 Hz seems reasonable. If one furthermore concentrates on calculating the solution up to T=0.08s and uses $\varepsilon=\pi/10$, Equation (8) yields $h \leq 1.01$ m and $\Delta t \leq 0.47$ ms for the trapezoidal rule and $h \leq 0.789$ m and $\Delta t \leq 0.66$ ms for the central difference method. In order to fit the geometry, the element side lengths are rounded down to h=1 m and h=0.75 m respectively. (The constants p and k were obtained from

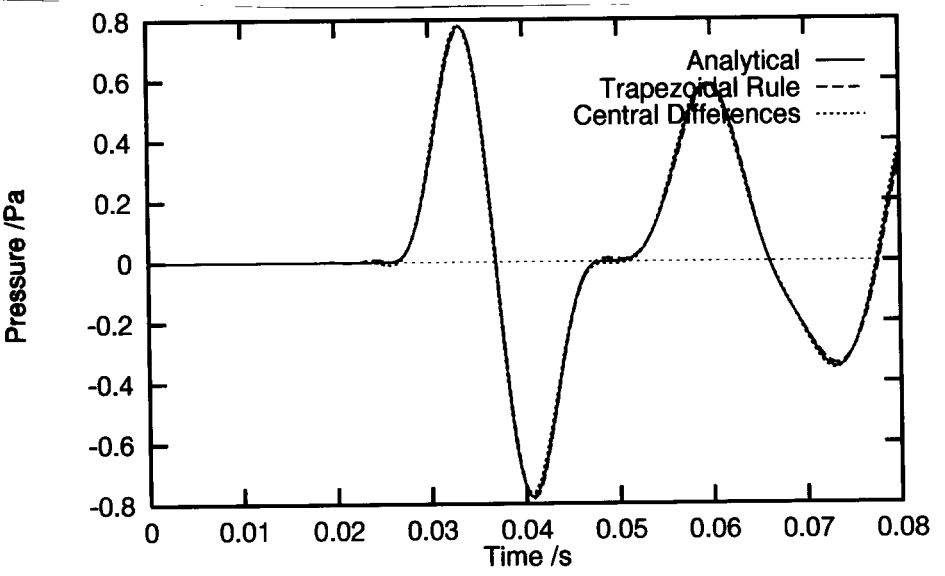

Figure 6: The response at (9,0,0) calculated using an analytical method and FEM using two different time integration algorithms.

Jensen [1] - k=-1.3 for the 20-noded serendipity element, k=3.5 for the 27-noded Lagrange element and p=2 in both cases)

The response at (9,0,0) is shown in Figure 6, while the response at (9,9,0) is shown in Figure 7.

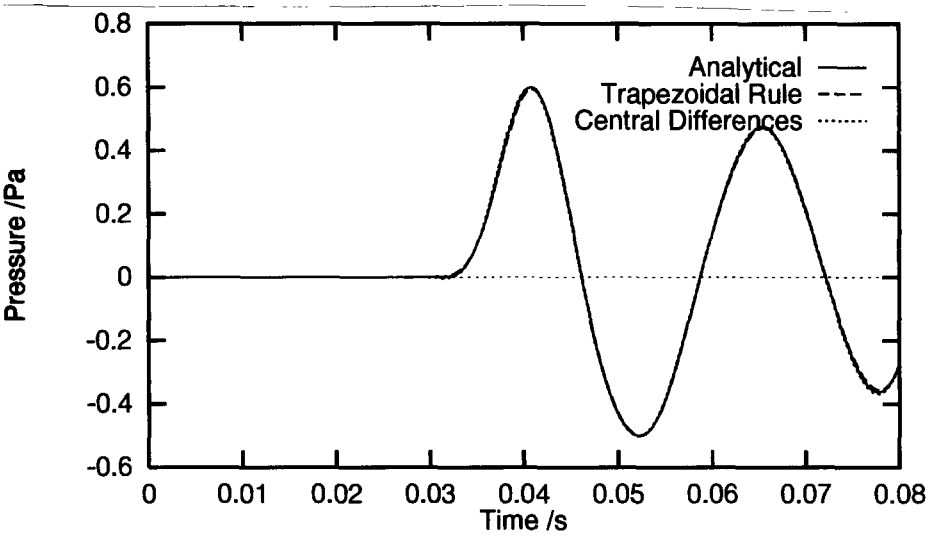

Figure 7: The response at (9,9,0) predicted with an analytical method and using FEM with two different time integration algorithms.

Figures 6 and 7 both show that both FEM strategies give good results and this is also the case for every receiver position that has been tried. The amplitude spectra of the curves in Figures 6 and 7 are shown in Figure 8.

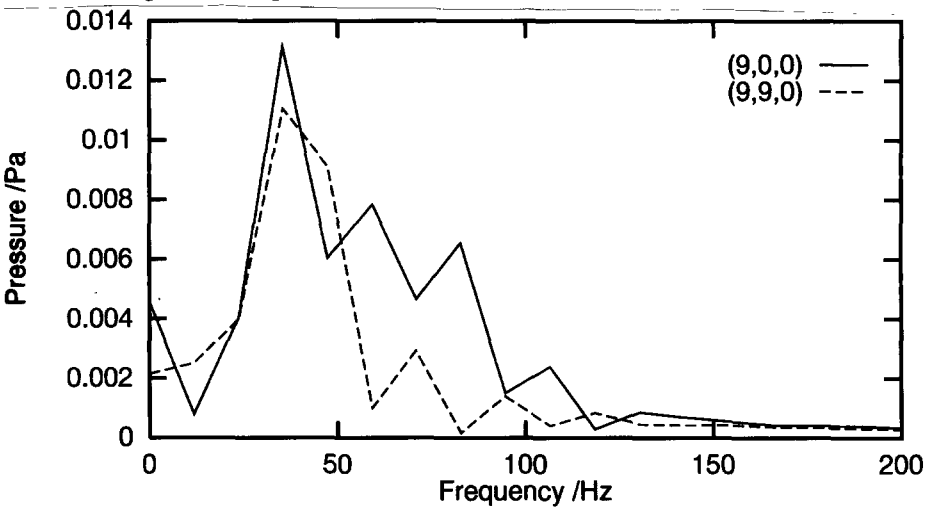

Figure 8: Amplitude spectrum of the first part of the response predicted with the central difference method at two different points.

Figure 8 clearly shows that the frequency content in general is lower than in the excitation (Figure 5). This means that the FEM predictions are even more accurate than the *a priori* estimate predicted and one might therefore even be able to trust the prediction for T>80 ms. Figure 8 also shows that the frequency at (9,9,0) is lower than at (9,0,0), which explains why the FEM prediction at (9,9,0) is more accurate than at (9,0,0).

In Figure 9, the solution at (9,0,0) with and without use of the accuracy criterion are compared. The solution without use of the accuracy criterion was calculated using linear elements with consistent mass matrix and the trapezoidal rule. The element size was six elements per wavelength (a common rule of thumb), while ten time steps were used per oscillation period. Using a frequency of 95 Hz, this yields $h \leq 0.63$m and $\Delta t \leq 1$ ms, allowing a grid of 15x15x15 elements.

Figure 9 shows that the solution obtained without the accuracy criterion also is reasonably

accurate, but is seen to deteriorate with time. The new discretisation method does not suffer from this weakness. Both methods produce more accurate results than normaly can expected because of the error cancelation mentioned in the previous section and because (as can be seen from Figure 8) the maximum significant frequency really is 75 Hz rather than the 95 Hz from the initial estimate.

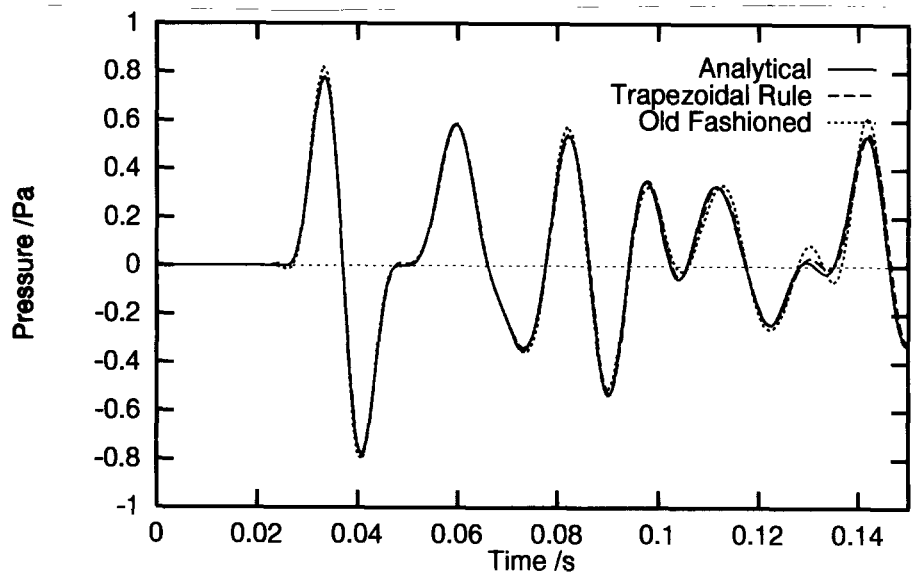

Figure 9: The response at (9,0,0) predicted by an analytical method and using FE-models based on two different discretisation principles.

CONCLUSION

An accuracy criterion that is based on sound scientific reasoning has been presented. This has then been combined with an *a priori* error estimator yielding a method for discretising time domain FE-calculations. The use of this method has been demonstrated in a simple example and the results compared with analytical calculations. The method performed well on this example, as the solution calculated on the basis of it had acceptable accuracy. The same example was also calculated using an FE-model based on commonly used rules of thumb and this produced significantly poorer results.

The new error estimator is also fairly easy to use, so if further experience shows that this example is typical of the difference between the two methods of mesh design, then the new method should be used for mesh design when high accuracy is important.

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