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DISCOVERING THE RECTILINEAR MODEL OF COMPLEX TORSIONAL VIBRATORY SYSTEMS WITH THE AID OF BOND GRAPHS

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ABSTRACT

The solution of *complex lumped parameter torsional vibratory systems*, via the Holzer's method or the Transfer Matrix method, is lengthy and time consuming. In addition to that, the presence of damping prohibits the application of the former and exceedingly prolongs the application of the latter. In this paper, *bond graphs* have been used as a medium to facilitate the rectilinear modelling of complex damped torsional-vibratory-systems. It has been demonstrated that through elimination of transformers, the bond graph of a torsional system will be converted to the bond graph of a rectilinear system on the basis of which the rectilinear model of the torsional system can be established. The model has been proven to follow a very simple pattern and to be rapidly constructible, irrespective of the complexity of the torsional system. This approach has converted the project-type problems (via the Holzer's or Transfer Matrix methods) to simple class-room tutorials.

NOMENCLATURE

c_i rectilinear damping coefficient of the damper applied upon mass i
c_{ij} rectilinear damping coefficient of the damper joining masses i and j
C_i torsional damping coefficient of the damper applied on torsional mass i
C_{ij} torsional damping coefficient of the damper joining torsional masses i and j
F_i force applied on rectilinear mass i
F'_{ij} tangential force applied on shaft j by shaft i
J_i moment of inertia of torsional mass i
$J_{i,j}$ moment of inertia of torsional mass i sitting on shaft j
k_{ij} rectilinear stiffness coefficient of the spring joining masses i and j
K_{ij} torsional stiffness coefficient of the shaft joining torsional masses i and j
m_i value of mass i in rectilinear model

M_i torque applied on rotational mass i
n_i rotational speed of shaft i
$N_{i.j}$ no of teeth of gear (rotational mass) i.j
$r_{i.j}$ radius of torsional mass i sitting on shaft j
s_i speed ratio of shaft i with respect to the trunk (shaft 1) = $s_{i.1}$
$s_{i.j}$ speed ratio of shaft i with respect to shaft j
x_i rectilinear displacement of rectilinear mass i
θ torsional displacement
ω angular velocity

INTRODUCTION

Complex lumped parameter rectilinear vibratory systems can be exactly formulated and then solved by widely available commercial packages, while the solution of *complex lumped parameter torsional vibratory systems*, via the Holzer's method or the Transfer Matrix method, is lengthy and time consuming. The former is unable to handle damped systems and the application of the latter upon damped systems is cumbersome. Both methods are numerical and the solution time dramatically increases by the number of branches of the torsional system

Rectilinear modelling of a *single-branch lumped parameter torsional vibratory* system was the only case cited in the literature, such as Church (1963), Crede (1965), Hartog (1956), Timoshenko et al (1974) and Tuplin (1966); until recently that Abhary (1991) approached complex multi-branch torsional systems via rectilinear modelling.

In this paper, *bond graphs* have been used as a medium to discover the pattern of the rectilinear model of a complex torsional system. This has been achieved by demonstrating that elimination of transformers of the bond graph model of a torsional system would convert it to the bond graph of the rectilinear model of the torsional system. This research has verified that the rectilinear model of a complex lumped parameter torsional vibratory system, irrespective of its complexity, follows a simple pattern.The rectilinear simulation method is first established for a two-branch lumped parameter torsional vibratory system and then generalised to cover the most general case of the problem.

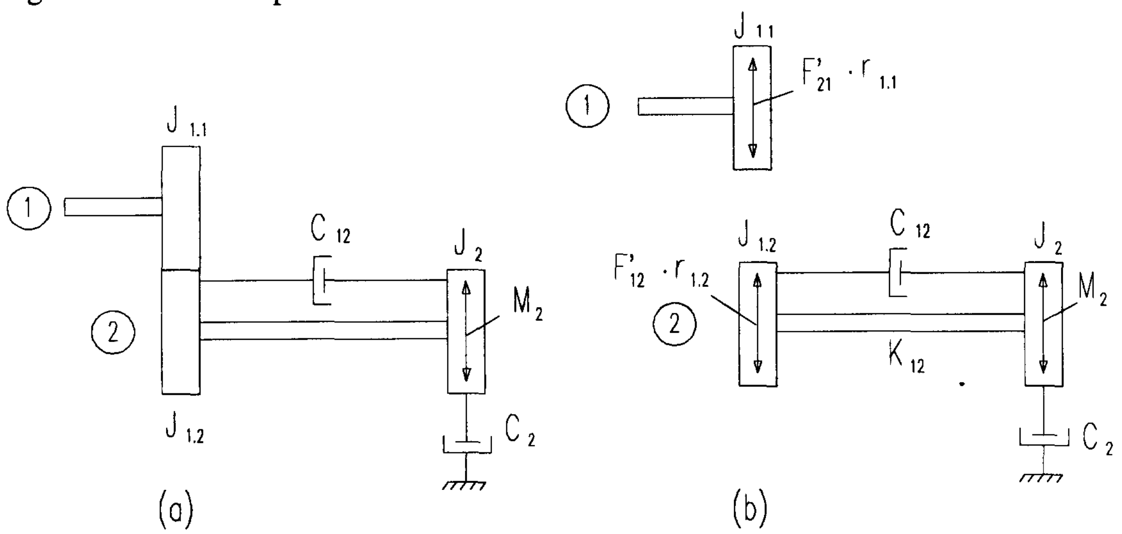

Fig. 1

ANALYTICAL APPROACH TO RECTILINEAR SIMULATION OF A TWO-BRANCH TORSIONAL VIBRATORY SYSTEM

Fig.1a illustrates an example of forced and damped vibration of a two-branch lumped parameter torsional vibratory system. Encircled numbers denote the shaft numbers and the reference branch could be arbitrarily nominated, say branch 1, but it is preferred to be the input shaft.

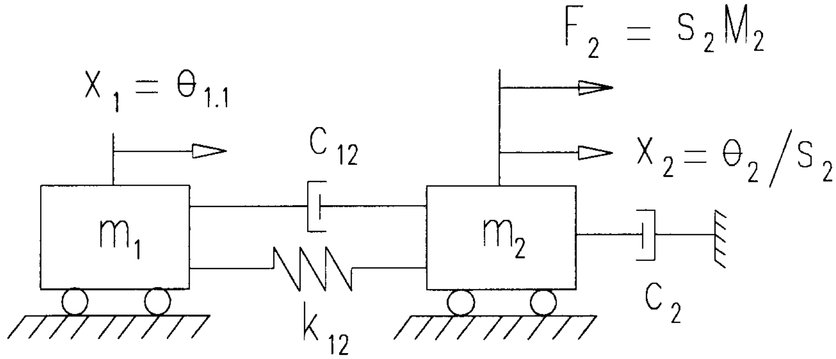

Fig. 2

The differential equations of motion of branches 1 and 2 in matrix form, considering their free-body diagrams in Fig.1b, are respectively as follows:

$$\begin{bmatrix} J_{1.1} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{1.1} \\ \ddot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} F'_{21} \cdot r_{1.1} \\ 0 \end{Bmatrix} \tag{1}$$

$$\begin{bmatrix} J_{1.2} & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{1.2} \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} C_{12} & -C_{12} \\ -C_{12} & C_{12}+C_2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{1.2} \\ \dot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} K_{12} & -K_{12} \\ -K_{12} & K_{12} \end{bmatrix} \begin{Bmatrix} \theta_{1.2} \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F'_{12} \cdot r_{1.2} \\ M_2 \end{Bmatrix} \tag{2}$$

and the speed ratio of branch 2 with respect to branch 1 is

$$s_2 = s_{2.1} = \frac{r_{1.1}}{r_{1.2}} = \frac{n_2}{n_1} = \frac{\theta_{1.2}}{\theta_{1.1}} = \frac{N_{1.1}}{N_{1.2}} \tag{3}$$

and also

$$F'_{12} = -F'_{21} \tag{4}$$

Multiplying both sides of Equation 2 by s_2 and employing Equations 3 and 4, it can be written as follows:

$$s_2^2 \begin{bmatrix} J_{2.1} & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{1.1} \\ \ddot{\theta}_2/s_2 \end{Bmatrix} + s_2^2 \begin{bmatrix} C_{12} & -C_{12} \\ -C_{12} & C_{12}+C_2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{1.1} \\ \dot{\theta}_2/s_2 \end{Bmatrix} + s_2^2 \begin{bmatrix} K_{12} & -K_{12} \\ -K_{12} & K_{12} \end{bmatrix} \begin{Bmatrix} \theta_{1.1} \\ \theta_2/s_2 \end{Bmatrix} = \begin{Bmatrix} -F'_{21} \cdot r_{1.1} \\ s_2 M_2 \end{Bmatrix} \tag{5}$$

Equations 1 and 5 can now be assembled together and presented as a single matrix equation:

$$\begin{bmatrix} J_{1.1}+s_2^2 J_{2.1} & 0 \\ 0 & s_2^2 J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{1.1} \\ \ddot{\theta}_2/s_2 \end{Bmatrix} + \begin{bmatrix} s_2^2 C_{12} & -s_2^2 C_{12} \\ -s_2^2 C_{12} & s_2^2 C_{12}+s_2^2 C_2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{1.1} \\ \dot{\theta}_2/s_2 \end{Bmatrix} + \begin{bmatrix} s_2^2 K_{12} & -s_2^2 K_{12} \\ -s_2^2 K_{12} & s_2^2 K_{12} \end{bmatrix} \begin{Bmatrix} \theta_{1.1} \\ \theta_2/s_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ s_2 M_2 \end{Bmatrix} \tag{6}$$

Equation 6 is the differential equation of motion of the rectilinear system illustrated in Fig. 2 which is, in fact, the rectilinear model of the torsional system shown in Fig.1a. In Fig.2

$$\begin{array}{llll} x_1 = \theta_{1.1} & x_2 = \theta_2/s_2 & m_1 = J_{1.1} + s_2^2 \cdot J_{1.2} & m_2 = s_2^2 \cdot J_2 \\ k_{12} = s_2^2 \cdot K_{12} & c_{12} = s_2^2 \cdot C_{12} & c_2 = s_2^2 \cdot C_2 & F_2 = s_2 \cdot M_2 \end{array} \tag{7}$$

BOND GRAPH MODEL OF THE TWO-BRANCH TORSIONAL VIBRATORY SYSTEM

Fig.3a illustrates the bond graph model of the two-branch vibratory system shown in Fig.1a, in which s_2 is the *modulus of the transformer.* Fig.3b which contains no transformers, depicts the bond graph model of the rectilinear model of the torsional vibratory system shown in Fig. 2. Comparison of these two equivalent bond graphs reveals that **elimination of transformers from the bond graph model of a torsional vibratory system leads to the bond graph model of its rectilinear model.**

Close inspection and cross check of the two bond graph models in Fig.3 reveals that a transformer may be eliminated from the bond graph of a torsional system provided that the elements on the output side of the transformer are changed as follows:

(i) *All angular velocities are divided by the modulus of the transformer.*

(ii) *All I, C and R parameters, i.e. moments of inertias, stiffness and damping coefficients respectively, are multiplied by the square of the modulus of the transformer.*

(iii) *The value of all effort sources, i.e. torques, are multiplied by the modulus of the transformer.*

The above rules can help verify the fourth rule:

(iv) *If transformer* $s_{i.j}$ *is followed, on its output side, by another transformer* $s_{j.k}$, *then due to the elimination of the former, the modulus of the latter will be changed to the product of the two moduli, i.e.*

$$s_{i.j} \cdot s_{j.k} = s_{i.k} \tag{8}$$

while all elements on the output side of the latter remain intact.

The above four rules can be applied to any complex bond graph containing any number of transformers. This will be illustrated in the next section.

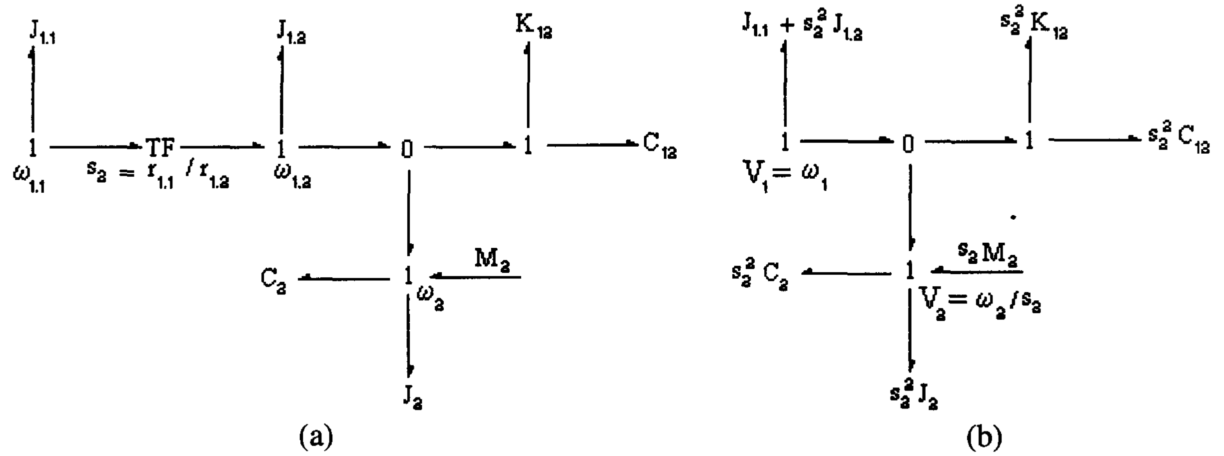

Fig. 3

RECTILINEAR SIMULATION OF A COMPLEX TORSIONAL VIBRATORY SYSTEM VIA BOND GRAPHS

Fig.4 shows a most general case of a complex multi-branch torsional vibratory system under forced damped vibration. The rectilinear model of this system can be achieved as follows:

- Establish the bond graph model of the system, Fig. 5.
- Eliminate the transformers of the bond graph.
- Construct the linear model of the system according to the transformerless bond graph.

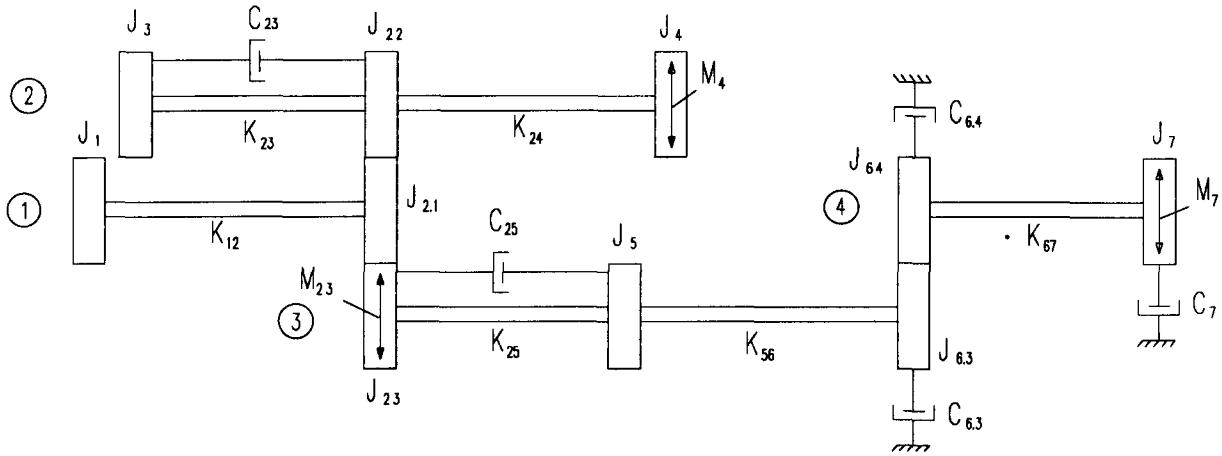

Fig. 4

The process of elimination of a transformer is carried out according to the rules (i) to (iv) explained in the previous section. The transformers in Fig.5 may be eliminated in any order, but it can be easily shown that *in a bond graph, the nomination of the innermost transformer for the next elimination would lead to the least labour intensive process*. Thus, the innermost transformer s_3 is eliminated first, Fig.6, which changes the modulus of transformer $s_{4.3}$ to s_4 according to Equation 5 and therefore next candidates for elimination are the innermost transformers s_2 and s_4 in any arbitrary order.

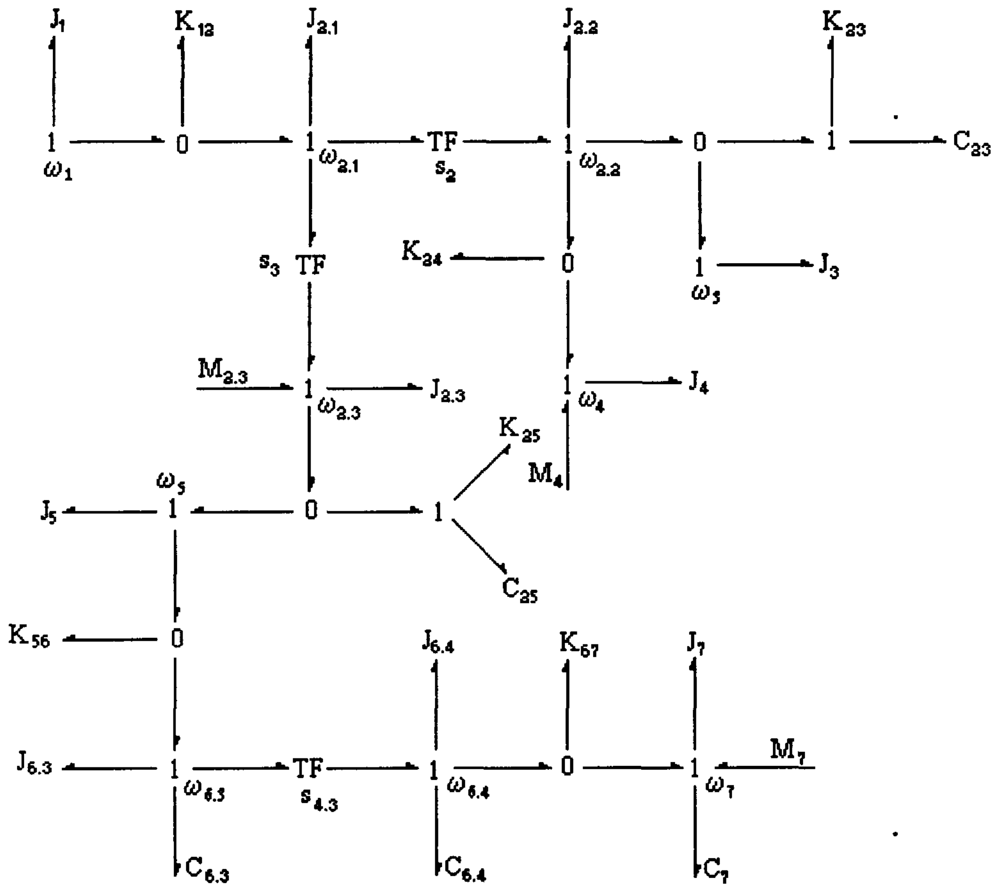

Fig. 5

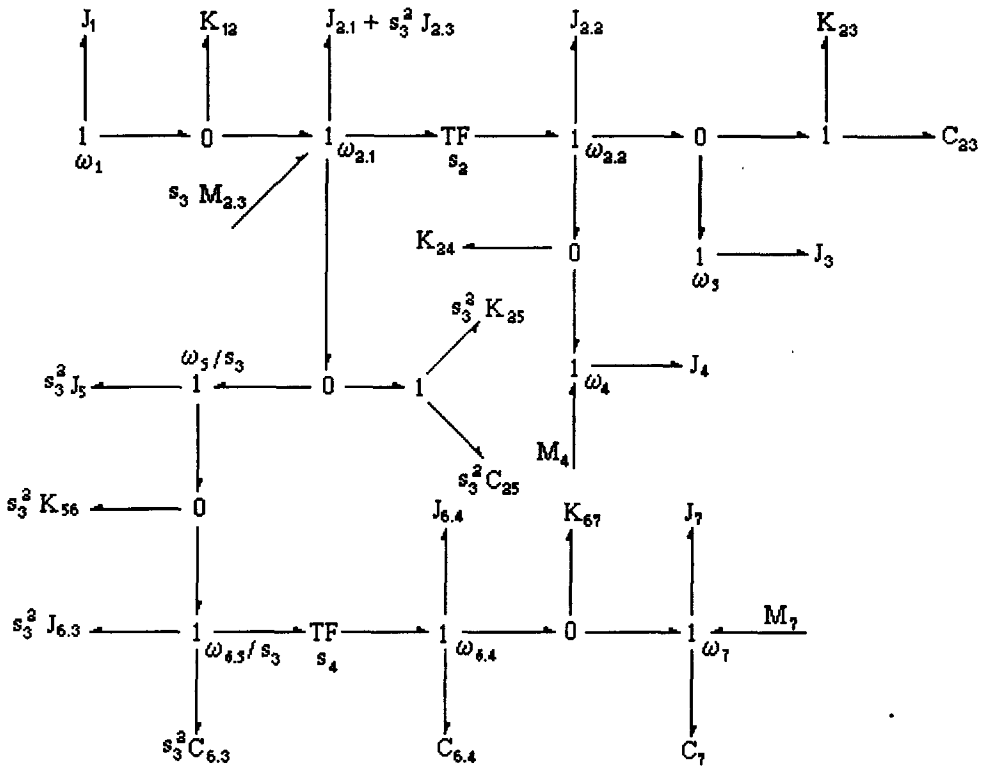

Fig. 6

This process results in Fig.7 which shows the transformerless bond graph, on the basis of which the rectilinear model of the system, Fig. 8, is constructed.

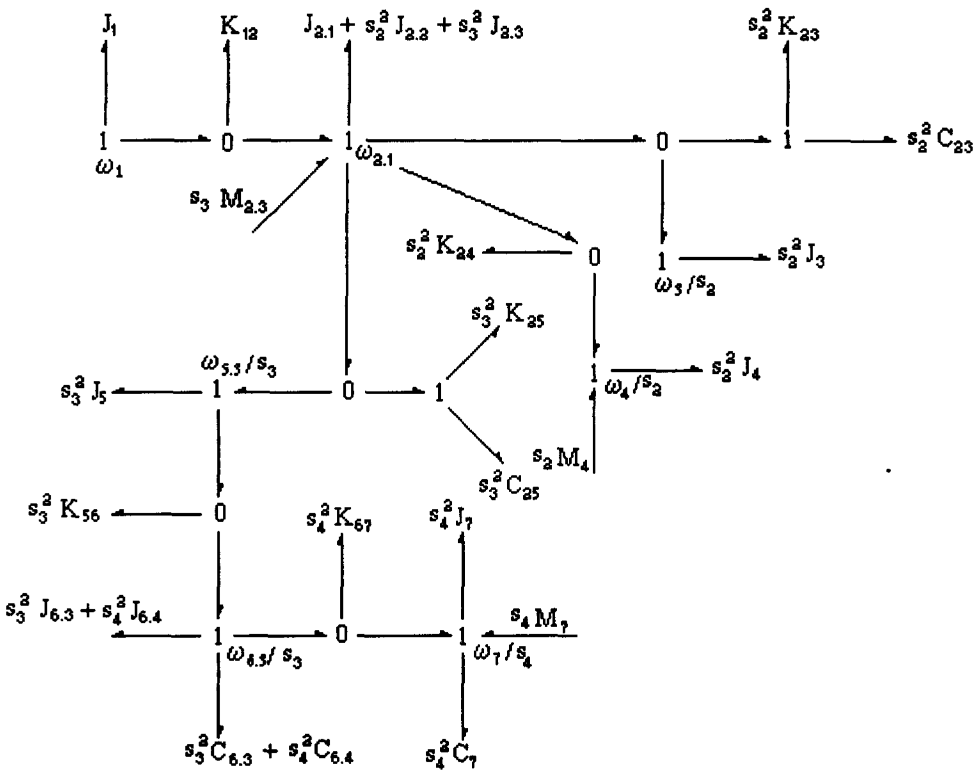

Fig. 7

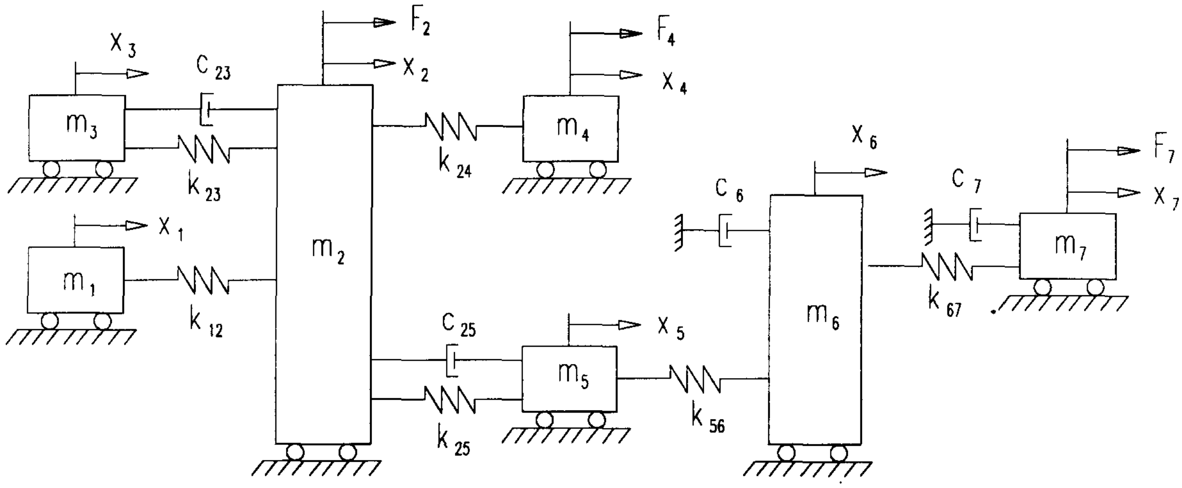

Fig. 8

In Fig. 8

$$\begin{cases} m_1 = J_1 \quad & m_2 = J_2 = J_{2.1} + s_2^2 \cdot J_{2.2} + s_3^2 \cdot J_{2.3} \\ m_3 = s_2^2 \cdot J_3 & m_4 = s_2^2 \cdot J_4 \qquad m_5 = s_3^2 \cdot J_5 \\ m_6 = J_6 = s_3^2 \cdot J_{6.3} + s_4^2 \cdot J_{6.4} & m_7 = s_4^2 \cdot J_7 \end{cases} \tag{9}$$

$$\begin{cases} k_{12} = K_{12} \quad k_{23} = s_2^2 \cdot K_{23} \quad k_{24} = s_2^2 \cdot K_{24} \\ k_{25} = s_3^2 \cdot K_{25} \quad k_{56} = s_3^2 \cdot K_{56} \quad k_{67} = s_4^2 \cdot K_{67} \end{cases} \tag{10}$$

$$c_{23} = s_2^2 \cdot C_{23} \quad c_{25} = s_3^2 \cdot C_{25} \quad c_6 = s_3^2 \cdot C_{6.3} + s_4^2 \cdot C_{6.4} \quad c_7 = s_4^2 \cdot C_7 \tag{11}$$

$$F_2 = s_3 \cdot M_{2.3} \quad F_4 = s_2 \cdot M_4 \quad F_7 = s_4 \cdot M_7 \tag{12}$$

$$\begin{cases} x_1 = \theta_1 \quad x_2 = \theta_{2.1} = \theta_{2.2}/s_2 = \theta_{2.3}/s_3 \quad x_3 = \theta_3/s_2 \quad x_4 = \theta_4/s_2 \\ x_5 = \theta_5/s_3 \quad x_6 = \theta_{6.3}/s_3 = \theta_{6.4}/s_4 \quad x_7 = \theta_7/s_4 \end{cases} \tag{13}$$

where

$$s_2 = N_{2.1}/N_{2.2} \quad s_3 = N_{2.1}/N_{2.3} \quad s_4 = s_{4.3} \cdot s_{3.1} = (N_{6.3}/N_{6.4}) \cdot (N_{2.1}/N_{2.3}) \tag{14}$$

It must be noted that

$$s_i = s_{i.1}$$
$$s_1 = s_{1.1} = 1$$

PATTERN OF RECTILINEAR MODEL

Close inspection of the rectilinear model, Fig.8, and cross checking it against the original vibratory system, Fig.4, considering Equations 9 to 14, reveals that the structure of the rectilinear model of a complex multi-branch lumped parameter torsional vibratory system follows a simple pattern which facilitates the direct and easy construction of the rectilinear model without the aid of the differential equations of motion or bond graphs (bond graphs were used herein only as a medium to obtain this pattern). The pattern is discovered to be as follows:

1. To obtain the geometry of the rectilinear model, which is similar to that of the torsional system, redraw the vibratory system with the following changes:
   (i) Replace torsional elements (inertias J, springs K and dashpots C) with their rectilinear counterparts (m, k and c respectively); replace torques M with forces F, and angular displacements θ with linear displacements x.
   (ii) Merge the rectilinear counterparts of meshed gears into a lumped mass.

2. Set the value of each rectilinear element to the product of its rotational counterpart and the square of the speed ratio of the shaft upon which the rotational counterpart is sitting, e.g.

$$m_3 = s_2^2 \cdot J_3 \qquad k_{25} = s_3^2 \cdot K_{25}$$
$$c_6 = s_3^2 \cdot C_{6.3} + s_4^2 \cdot C_{6.4}$$

3. Set the value of each force in the rectilinear model to the product of its torsional counterpart, torque, and the speed ratio of the shaft on which the torque applies, e.g.

$$F_4 = s_2 \cdot M_4$$

4. Set the linear displacement of each mass in the rectilinear model to the ratio of the angular displacement of its torsional counterpart (rotational mass) and the speed ratio of the shaft on which the rotational mass is sitting, e.g.

$$x_2 = \theta_{2.1} = \theta_{2.2}/s_2 = \theta_{2.3}/s_3 \quad x_4 = \theta_4/s_2$$

CONCLUSION

It was proven that a torsional lumped parameter vibratory system, irrespective of its complexity, can be converted to, and solved via, a corresponding rectilinear model which follows a simple pattern and it can be constructed very easily and rapidly. This conversion was facilitated with the aid of bond graphs which served as a powerful tool for this purpose. *This rectilinear simulation of torsional systems allows all lumped parameter vibratory systems, rectilinear and torsional, to be treated with the same mathematical tools and* ***leaves no useful and sensible application for the Holzer's method any more.***

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