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# FACTORIZATION METHOD IN ANNULAR PLATES VIBRATIONS

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In the present paper a general solution for the vibration of a class of annular plates, with quadratic thickness variation, and more general, whose thickness varies parabolically with the radius, is obtained by the factorization of the differential equation (more general than [7]). Poisson's ratio is taken $\upsilon = 1/3$, applicable to many materials. The paper gives the exact solutions of vibrations differential equations of the eigenfrequencies' problem of annular plates above.

## 1. INTRODUCTION

The annular plates are used in many structural applications. Therefore, the analysis of the annular plates' vibrations is of interest to many mechanical, aeronautical and civil engineers. We base our considerations on the classical plates' theory [see 8].

We consider the free axisymmetrical vibrations of a circular disk whose flexural rigidity varies with the radius . Such a disk is governed by the differential equation [9]

$$D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\frac{\partial D}{\partial r}\left(\frac{\partial^2 D}{\partial r^2}+\frac{\upsilon}{r}\frac{\partial w}{\partial r}\right)=-\frac{1}{r}\int_0^r \rho_0 h\frac{\partial^2 w}{\partial t^2}rdr \qquad (1.1)$$

where h(r) is the axial thickness of radius r, w(r,t)-the deflection, and D(r)-the flexural rigidity.

Multiplying (1.1) with r , the derivative of the equation is :

$$\frac{1}{hr}\frac{\partial^2}{\partial r^2}\left(rD\frac{\partial^2 w}{\partial r^2}\right)+\frac{1}{hr}\frac{\partial}{\partial r}\left[\left(-\frac{D}{r}+\upsilon\frac{\partial D}{\partial r}\right)\frac{\partial w}{\partial r}\right]=-\rho_0\frac{\partial^2 w}{\partial t^2} \qquad (1.2)$$

The particular case of free vibration of linearly tapered plates is solved in [6], for Poisson's ratio $\upsilon = 1/3$.

The paper [2] shows the factorization of the fourth-order differential operator

$$L_2 = \frac{1}{\rho(x)}\frac{d^2}{dx^2}\left[\rho(x)\beta^2(x)\frac{d^2}{dx^2}\right] \quad , \tag{1.3}$$

in the some particular conditions, into a pair of second order Sturm-Liouville operators

$$L_1 = \frac{1}{\rho(x)}\frac{d}{dx}\left[\rho(x)\beta(x)\frac{d}{dx}\right] \tag{1.4}$$

The relationship which exists between differential operators $L_1$ and $L_2$, in more general conditions than [6] and [7], will be specified (see [3], [4]).

Thus, if $\rho$ and $\beta$ are two functions (not necessarily polynomials), the function $\alpha$ is as follows (1.5a) and the condition (1.5b) is verified

$$\frac{\rho'(x)}{\rho(x)} = \frac{\alpha(x)}{\beta(x)} \quad , \quad \alpha'(x) + \beta''(x) = C = \text{const.} \tag{1.5}$$

then the operator $L_2$ can be factored :

$$L_2 = L_1\{L_1 - C\} \tag{1.6}$$

The factorization (1.6) will be proved calculating $L_1^2$:

$$L_1^2[Y] = L_2[Y] + (\alpha' + \beta'')\, L_1[Y] + \beta(\alpha'' + \beta''')\, Y \tag{1.7}$$

Using (1.5b) we obtain :

$$L_1^2 = L_2 + \left(\alpha'(x) + \beta''(x)\right)L_1 \tag{1.8}$$

We can see the factorization of fourth-order differential equation too. The eigenvalue problems of the $L_1$ and $L_2$ operators are

$$L_1[Y] + \lambda\, Y = 0 \quad , \quad L_2[Y] + \tau\, Y = 0 \tag{1.9}$$

If the conditions (1.5) are verified, then the equation (1.9b) can be factored :

$$\{L_1[Y] + \lambda_1 Y\}\,\{L_1[Y] + \lambda_2 Y\} = 0 \tag{1.10}$$

where the eigenvalues $\lambda_1$ and $\lambda_2$ have the expressions

$$\lambda_i = \frac{-C + (-1)^i\sqrt{C^2 - 4\tau}}{2} \quad , \quad i \in \{1,2\} \tag{1.11}$$

This factorization of fourth-order differential equation can be proved using the factorization of fourth-order differential operator ( see (1.6)).

## 2. THE FACTORIZATION OF THE DIFFERENTIAL EQUATION

Considering the factorization (1.6), used in [2], [3] and [4] , if we may write the differential equation, which results by separating variables from (1.2), as

$$\frac{1}{\rho(r)}\frac{d^2}{dr^2}\left[\rho(r)\beta^2(r)\frac{d^2Y(r)}{dr^2}\right] + \frac{1}{\rho(r)}\frac{d}{dr}\left[\rho(r)\beta(r)\frac{dY(r)}{dr}\right] + \tau Y(r) = 0 \tag{2.1}$$

in the following conditions:

$$\frac{\rho'(r)}{\rho(r)} = \frac{\alpha(r)}{\beta(r)} \quad , \quad \alpha'(r) + \beta''(r) = C = \text{const.} \tag{2.2}$$

then, the differential equation can be factored.

It means that:

$$\begin{cases} \rho(r) = a_1 \cdot h(r) r \\ \rho(r)\beta^2(r) = a_2 \cdot rD(r) \\ -\dfrac{D(r)}{r} + \upsilon \dfrac{\mathrm{d}D(r)}{\mathrm{d}r} = a_3 \cdot \rho(r)\beta(r) \end{cases} \quad , \quad a_1, a_2, a_3 \in R \tag{2.3}$$

Using (2.3) we obtain the differential equation

$$\upsilon \frac{\mathrm{d}D(r)}{\mathrm{d}r} - \frac{D(r)}{r} = C_1 \cdot r\sqrt{D(r)h(r)} \quad , \quad C_1 \in R \tag{2.4}$$

where

$$D(r) = D_0 h^3(r) \tag{2.5}$$

Then the relation (2.4) becomes

$$\frac{\mathrm{d}h(r)}{\mathrm{d}r} - \frac{1}{3\upsilon r} h(r) = C_1 \cdot r \quad , \tag{2.6}$$

a first-order linear ordinary differential equation.

The general solution of (2.6) is :

$$h(r) = C_1 \cdot r^2 + C_2 \cdot r^{\frac{1}{3\upsilon}} \quad , \quad C_1 , C_2 \in R \tag{2.7}$$

Using (2.2a) , (2.3) and (2.7), the condition (2.2b) becomes:

$$\begin{cases} 5 \cdot C_1 + C_2 \cdot \left(\dfrac{2}{3\upsilon} + 1\right)\left(\dfrac{1}{3\upsilon} - 1\right) r^{\frac{1}{3\upsilon} - 2} = \text{const.} \quad , \quad \text{if} \quad \upsilon \neq \dfrac{1}{3} \\ 5 \cdot C_1 = \text{const} \quad , \quad \textit{if} \quad \upsilon = \dfrac{1}{3} \end{cases} \tag{2.8}$$

In conclusion , only if $\upsilon \in \{1/6 , 1/3\}$ (see (2.8) and (2.1)), and relation (2.7) is verified , the differential equation (1.2) can be solved by the factorization method .

In the particular case of Poisson's ratio $\upsilon = 1/3$ , which is applicable to many materials , (2.7) becomes (the thickness varies parabolically with the radius) :

$$h(r) = C_1 \cdot r^2 + C_2 \cdot r \quad , \quad C_1 , C_2 \in R \tag{2.10}$$

## 3. THE DIFFERENTIAL EQUATION (QUADRATIC THICKNESS VARIATION)

Let us consider a disk with quadratic thickness variation with the radius r (see (2.10) and fig.1):

$$h(r) = h_0 r^2 \quad , \quad h_0 = \frac{H}{r_2^2} \quad , \quad D = D_0 r^6 \quad , \quad D_0 = \frac{E h_0^3}{12(1 - \upsilon^2)} \tag{3.1}$$

We denote

$$\xi = \frac{r}{r_2} \quad , \quad c = \frac{r_1}{r_2} < 1 \tag{3.2}$$

We write r instead of $\xi$. Poisson's ratio is taken $\upsilon = 1/3$, applicable to many materials.

Considering the solution (3.3)

$$w(t, r) = W(r) \cos(\omega t + \varphi) \ , \tag{3.3}$$

the equation (2.2) becomes:

$$\frac{1}{r^3}\frac{d^2}{dr^2}\left(r^7\frac{d^2W}{dr^2}\right)+\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{dW}{dr}\right)+\left(-\frac{32\rho}{3Eh_0^2}\omega^2\right)W=0 \quad , \quad r\in[c,1] \ . \tag{3.4}$$

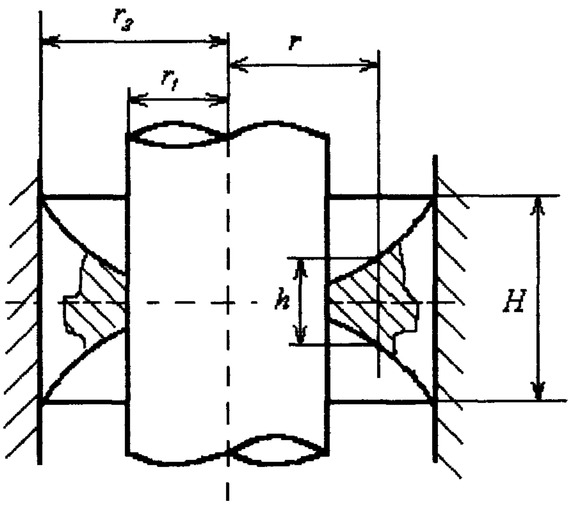


*Fig. 1*

The mode shape $W(r)$ and the natural frequency $\omega$ are determined by the fourth-order linear ordinary differential equation (3.4) with variable coefficients, and various boundary conditions.

## 4. THE GENERAL SOLUTION (QUADRATIC THICKNESS VARIATION)

Using the factorization of fourth order differential operator, we rewrite the differential equation (3.4):

$$\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{d}{dr}\right)\left[\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{dW}{dr}\right)-5W\right]+\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{dW}{dr}\right)+\tau W=0 \ , \tag{4.1}$$

where

$$\tau=-\frac{32\rho}{3Eh_0^2}\omega^2<0 \ . \tag{4.2}$$

Then, the differential equation (4.1) can be factored (see(1.10)):

$$\left\{\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{dW}{dr}\right)+\lambda_1 W\right\}\left\{\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{dW}{dr}\right)+\lambda_2 W\right\}=0 \ , \tag{4.3}$$

where

$$\lambda_i=\frac{-5+(-1)^i\sqrt{25-4\tau}}{2} \quad , \quad i\in\{1,2\} \ . \tag{4.4}$$

We rewrite (4.3):

$$\left\{r^2W'' + 5rW' + \lambda_1 W\right\}\left\{r^2W'' + 5rW' + \lambda_2 W\right\} = 0 \ . \tag{4.5}$$

The solutions of Euler differential equations (4.5), see [8], are the form:

$$W(r) = r^p \quad , \quad p \in R \ . \tag{4.6}$$

Calculating, it results

$$p_{ij} = -2 + (-1)^j \sqrt{\frac{13 + (-1)^i \sqrt{25 - 4\tau}}{2}} \quad , \quad i, j \in \{1,2\} \ . \tag{4.7}$$

Denoting

$$a = \left|\frac{13 - \sqrt{25 - 4\tau}}{2}\right| \quad , \quad b = \frac{13 + \sqrt{25 - 4\tau}}{2} \quad , \tag{4.8}$$

where $\tau < 0$ , we can write the solutions of (4.5) :

$$W_1(r) = \begin{cases} \dfrac{1}{r^2}\left(C_1 r^{\sqrt{a}} + C_2 r^{-\sqrt{a}}\right) & , \tau \in [-36, 0] \\ \dfrac{1}{r^2}\left[C_1 \cos\left(\sqrt{a}\ \ln r\right) + C_2 \sin\left(\sqrt{a}\ \ln r\right)\right] & , \tau \in (-\infty, -36) \end{cases} \ , \tag{4.9}$$

$$W_2(r) = \frac{1}{r^2}\left(C_3 r^{\sqrt{b}} + C_4 r^{-\sqrt{b}}\right) \ . \tag{4.10}$$

Thus the general solution of (4.1) is:

$$W(r) = W_1 + \frac{1}{r^2}\left(C_3 r^{\sqrt{b}} + C_4 r^{-\sqrt{b}}\right) \ . \tag{4.11}$$

## 5. THE DIFFERENTIAL EQUATION (PARABOLIC THICKNESS VARIATION)

Let us consider a disk whose thickness varies parabolically with the radius r (see (2.10))

$$h(r) = h_0 r(r + C) \quad , h_0 > 0 \, , C > 0 \, . \tag{5.1}$$

The flexural rigidity is

$$D(r) = D_0 r^3 (r + C)^3 \quad , \quad D_0 = \frac{E h_0^3}{12\left(1 - \upsilon^2\right)} \ . \tag{5.2}$$

The governing differential equation , for the case $\upsilon = 1/3$, applicable to many materials , becomes :

$$\frac{1}{r^2(r+C)}\frac{\partial^2}{\partial r^2}\left[r^4(r+C)^3\frac{\partial^2 w}{\partial r^2}\right] + \frac{1}{r^2(r+C)}\frac{\partial}{\partial r}\left[r^3(r+C)^2\frac{\partial w}{\partial r}\right] + \frac{32\rho_0}{3Eh_0}\frac{\partial^2 w}{\partial t^2} = 0 \tag{5.3}$$

By separating variables

$$w(t, r) = W(r)\cos(\omega t + \varphi) \ , \tag{5.4}$$

the mode shape $W(r)$ and the natural frequency $\omega$ are determined by the fourth-order linear ordinary differential equation (5.5) with variable coefficients, and various boundary conditions.

$$\frac{1}{r^2(r+C)}\frac{d^2}{dr^2}\left[r^4(r+C)^3\frac{d^2W}{dr^2}\right] + \frac{1}{r^2(r+C)}\frac{d}{dr}\left[r^3(r+C)^2\frac{dW}{dr}\right] - \frac{32\rho_0}{3Eh_0}\omega^2 W = 0 \, , \tag{5.5}$$

We consider that $r \in [r_1, r_2]$.

Replacing the variable(see fig.1)

$$r = \xi r_2 \ , \tag{5.6}$$

the equation (5.5) becomes

$$\frac{1}{\xi^2(\xi+b)}\frac{d^2}{d\xi^2}\left[\xi^4(\xi+b)^3\frac{d^2W}{d\xi^2}\right]+\frac{1}{\xi^2(\xi+b)}\frac{d}{d\xi}\left[\xi^3(\xi+b)^2\frac{dW}{d\xi}\right]-\frac{32\rho_0}{3Eh_0}\omega^2W=0\,, \tag{5.7}$$

where $\xi\in[c,1]$, and (see fig.1).

$$c=\frac{r_1}{r_2}<1\quad,\quad b=\frac{C}{r_2}\,. \tag{5.8}$$

## 6. THE GENERAL SOLUTION (PARABOLIC THICKNESS VARIATION)

Based on the form used in [3] and [4], see (1.3) and (1.4), we can write the Sturm-Liouville operator, respectivelly the fourth order differential operator , as

$$L_1=\frac{1}{\rho(\xi)}\frac{d}{d\xi}\left(\rho(\xi)\beta(\xi)\frac{d}{d\xi}\right)\quad,\quad L_2=\frac{1}{\rho(\xi)}\frac{d^2}{d\xi^2}\left(\rho(\xi)\beta^2(\xi)\frac{d^2}{d\xi^2}\right) \tag{6.1}$$

where we admit that

$$\rho(\xi)=\xi^2(\xi+b)\quad,\quad\beta(\xi)=\xi(\xi+b)\,. \tag{6.2}$$

The equation (5.7) becomes

$$L_2[W]+L_1[W]+\tau Y=0\,, \tag{6.3}$$

where we denoted

$$\tau=-\frac{32\rho_0}{3Eh_0^2}\omega^2<0\,. \tag{6.4}$$

Using the factorization (1.6) of the fourth order differential operator , the relation (6.3) can be rewritten in the form:

$$L_1\left[L_1[W]-\left(\frac{d\alpha}{d\xi}+\frac{d^2\beta}{d\xi^2}\right)W\right]+L_1[W]+\tau W=0\,. \tag{6.5}$$

From relations (1.5a) and (6.2) we obtain

$$\alpha(\xi)=3\xi+2b\,, \tag{6.6}$$

Then it results

$$L_1\big[L_1[W]-4W\big]+\tau W=0\,. \tag{6.7}$$

The solutions of the equation (6.7) are the solutions of the equations (see (1.10)) :

$$L_1[W]+\lambda_iW=0\quad,\quad\lambda_i=\frac{-5+(-1)^i\sqrt{25-4\tau}}{2}\quad,\quad i\in\{1,2\} \tag{6.8}$$

Using (6.1) , the equation (6.8a) becomes

$$\frac{1}{\xi^2(\xi+b)}\frac{d}{d\xi}\left[\xi^3(\xi+b)^2\frac{dW}{d\xi}\right]+\lambda_iW=0\quad,\quad i\in\{1,2\}\,. \tag{6.9}$$

Performing the calculation, we obtain

$$\xi(\xi+b)\frac{d^2W}{d\xi^2}+(5\xi+3b)\frac{dW}{d\xi}+\lambda_iW=0\quad,\quad i\in\{1,2\}\,. \tag{6.10}$$

The change of variable

$$u=\frac{\xi+b}{b}\,, \tag{6.11}$$

leads to a Gauss equation (see [8] )

$$u(u-1)\frac{d^2W}{du^2}+(5u-2)\frac{dW}{du}+\lambda_i W=0 \quad , \quad i\in\{1,2\} . \tag{6.12}$$

Its canonical form is

$$z(z-1)\frac{d^2W}{du^2}+\left[(p+q+1)z-s\right]\frac{dW}{du}+pqW=0 . \tag{6.13}$$

Using (6.12) , (6.13) and (6.8b), it follows that

$$s_i=2 , \quad p_i+q_i=4 , \quad p_iq_i=\frac{-5+(-1)^i\sqrt{25-4\tau}}{2} , \quad i\in\{1,2\}. \tag{6.14}$$

In conclusion , the general solution of (5.7) is :

$$W(\xi)=A\,{}_2F_1\left(p_1,q_1,2,\frac{\xi+b}{b}\right)+C\,{}_2F_1\left(p_2,q_2,2,\frac{\xi+b}{b}\right)+Bv_{12}\left(\frac{\xi+b}{b}\right)+Dv_{22}\left(\frac{\xi+b}{b}\right) \quad , \tag{6.15}$$

where

$$v_{i2}(x)={}_2F_1(p_i,q_i,2,x)\ln x+\sum_{n=1}^{\infty}\frac{(p_i)_n(q_i)_n}{n!(2)_n}x^n\left[\psi(p_i+n)-\psi(p_i)+\psi(q_i+n)-\right.$$

$$\left.-\psi(q_i)-\psi(n+2)+\psi(2)-\psi(n+1)+\psi(1)\right]+\frac{1}{x}\frac{1}{(1-p_i)(1-q_i)} \tag{6.16}$$

We use the hypergeometric function ${}_2F_1(a,b,c;x)$ and $\psi$, the logarythmic derivative of $\Gamma$ function.

## 7. CONCLUSION

The paper involves factoring the fourth-order linear differential operator, which appears in the equation of motion, into a pair of second order operators (more general than [6]).

For the annular plates with variable thickness, the conditions in which the factorization is possible are presented. In this way, all the annular plates whose thickness varies with the radius for which this method is applicable are found.

The paper determines the general solution of vibrations differential equation of the eigenfrequencies' problem of the free axisymmetrical vibrations of annular plates with quadratic thickness variation, and more general, whose thickness varies parabolically with the radius.

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